Coherent states for four-mode systems in quantum optics

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We introduce and study the properties of a class of coherent states for the group SU(1,1)×SU(1,1) and derive explicit expressions for these using the Clebsch-Gordan algebra for the SU(1,1) group. We also derive a phase space representation based on pair coherent states rather than the standard harmonic-oscillator coherent states. We discuss the utility of the resulting "bi-pair coherent states" in the context of four-mode interactions in quantum optics.

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I. INTRODUCTION

Multimode radiation fields admit a variety of coherent states [1]. The utility of these states depends on the context in which they are to be used. Consider, for example, a two-mode radiation field with modes denoted by the annihilation and creation operators a, b, $a^\dagger$, and $b^\dagger$. The simplest type of coherent states that can be considered for this system will be the product state $|\alpha,\beta\rangle$, where

$$a|\alpha,\beta\rangle=\alpha|\alpha,\beta\rangle ,\quad b|\alpha,\beta\rangle=\beta|\alpha,\beta\rangle . \tag{1.1}$$

However, depending on the context, one may not need the full set $|\alpha,\beta\rangle$ and it might be more useful to have a new set of coherent states. Let us give three examples:

(i) Consider the radiation field of frequency ω with arbitrary polarization and its transformations on the Poincaré sphere. Here $a^\dagger a+b^\dagger b$ is conserved. In this case, it would be more appropriate to consider the underlying SU(2) group (obtained from $a^\dagger b$, $ab^\dagger$, and $a^\dagger a-b^\dagger b$) with the associated coherent states [2]

$$|\zeta,N,m\rangle=\exp(\zeta a^\dagger b-\zeta^* ab^\dagger)|N-m,m\rangle ,\quad (a^\dagger a-b^\dagger b)|\zeta,N,m\rangle=N|\zeta,N,m\rangle . \tag{1.2}$$

(ii) For the class of problems involving four-wave mixing, where the pump field is treated classically and undepleted, the relevant set of states, called the Caves-Schumaker states, defined by the relation

$$|\zeta\rangle=\exp(\zeta a^\dagger b^\dagger-\zeta^* ab)|0,0\rangle , \tag{1.3}$$

have been used in the context of the phenomena of squeezing [3].

(iii) Consider the down-conversion of photons of frequency Ω into two photons of frequency ω_a and ω_b. Assuming that one is dealing with situations where the two photons are created or destroyed together, the operator $Q_1=a^\dagger a-b^\dagger b$ is conserved. In such a situation it was found to be more convenient to introduce "pair coherent states," which were simultaneous eigenstates of ab and Q_1 [4],

$$ab|\zeta,q\rangle=\zeta|\zeta,q\rangle ,\quad Q_1|\zeta,q\rangle=q_1|\zeta,q\rangle . \tag{1.4}$$

The relevant operators here are

$$K_1^+=a^\dagger b^\dagger ,\quad K_1^-=ab ,\quad K_1^z=\tfrac{1}{2}(a^\dagger a+b^\dagger b+1) , \tag{1.5}$$

which form an SU(1,1) algebra with the commutation relations

$$[K_1^+,K_1^-]=-2K_1^z ,\quad [K_1^z,K_1^\pm]=\pm K_1^\pm . \tag{1.6}$$

The conservation law for Q_1 is related to the Casimir operator C for the SU(1,1) group,

$$C=\frac{(K_1^+K_1^-+K_1^-K_1^+)}{2}-K_z^2 , \tag{1.7}$$

which can be written as

$$C=\tfrac{1}{4}[1-(a^\dagger a-b^\dagger b)^2]=\tfrac{1}{4}(1-Q_1^2) . \tag{1.8}$$

Thus the eigenstate of Q_1 is also an eigenstate of C and the pair coherent state is related to the eigenstate of K_1^- introduced in Ref. [5].

It is thus clear from the foregoing that multimode systems will be characterized by a variety of coherent states and every set of coherent states would describe altogether different physical phenomena. It is also important to note that states like (1.2) and (1.3) can be produced by evolution under the effect of an appropriate Hamiltonian. States like (1.4) have, however, been shown to be produced by a dissipative process [4].

In this paper, we consider processes involving four modes of the radiation field, as there has been a considerable amount of recent work on physical systems requiring four modes of the field [6–8]. Clearly, such systems would admit a large number of different types of coherent states. In view of this, we consider processes where the direct product of SU(1,1) groups will be relevant. This would be the case in problems where one studies the emission of two photons of frequencies ω_1 and ω_2 and where each of them is emitted in either of the two modes of polarization, as in the case of a calcium cascade [6].

Other examples of processes where four modes of the radiation field are important involve phase conjugate resonators [9] and the process of down-conversion in the field of a standing pump wave [7]. In the latter case, the forward wave will produce the modes a and b and the backward pump will give the modes c and d. The Hamiltonian for such interactions will have the form

$$H=\epsilon_f^* ab+\epsilon_b^* cd+\text{c.c.} , \tag{1.9}$$

where ϵ_f and ϵ_b are the forward and backward fields. In such a context, it would be interesting to study the coherent states of the operator

$$K^-=(ab+cd)=K_1^-+K_2^- . \tag{1.10}$$

These states will be degenerate, and we will discuss in Sec. II the origin of this degeneracy.

Another system in which these states would play a role is in double beam polarization-dependent two-photon absorption processes, which have been recently considered [8]. In general, such processes involve four modes: two modes of polarization and two colors. In an atomic system involving the $J=0\rightarrow 1\rightarrow 0$ cascade, the absorption and emission will involve photon pairs, with each pair consisting of photons of the same polarization but different colors. The effective two-photon Hamiltonian for this process can be written in the form

$$H_1=\hbar[G(ab+cd)|e\rangle\langle g|+\text{c.c.}] , \tag{1.11}$$

where G represents the two-photon matrix element, and the excited state $|e\rangle$ is connected to the ground state $|g\rangle$ via a two-photon transition. If, in addition, the two-photon transition is driven by an external coherent pump of frequency ω_1, such that the condition of two-photon resonance is satisfied, then Eq. (1.11) will be supplemented by

$$H_{\text{ext}}=(\hbar G_0\epsilon^2|e\rangle\langle g|e^{-2i\omega_1 t}+\text{H.c.}) . \tag{1.12}$$

A detailed analysis following Ref. [10] shows that the radiation field is in a state that is the eigenstate of the operator $O=ab+cd$.

From the foregoing it is clear that many other physical systems will require the coherent states of the group SU(1,1)×SU(1,1), which we call "bi-pair coherent states." The plan of the paper is as follows: In Sec. II we use the Clebsch-Gordan algebra of the Kronecker product of SU(1,1)×SU(1,1) to derive an expression for the bi-pair coherent states (BPCS) in terms of the individual pair coherent states. In Sec. III we introduce a projection operator which projects out the BPCS onto the product of pair coherent states. We conclude with some possible applications of the BPCS to other fields. It is worth noting here that the states associated with the related product group SU(2)×SU(2) have been the subject of extensive investigation in the context of the semiclassical (Rydberg) states of the hydrogen atom [11].

II. CONSTRUCTION OF BI-PAIR COHERENT STATES

We construct the "bi-pair coherent states," which are coherent states of SU(1,1)×SU(1,1) in terms of the individual pair coherent states of each SU(1,1) group by using the Clebsch-Gordan (CG) coefficients of SU(1,1)×SU(1,1) [12,13]. We consider pairs of photon operators

$$\begin{aligned} &K_1^+=a^\dagger b^\dagger , \quad K_1^-=ab , \quad K_1^z=\tfrac{1}{2}(a^\dagger a+b^\dagger b+1) , \\ &K_2^+=c^\dagger d^\dagger , \quad K_2^-=cd , \quad K_2^z=\tfrac{1}{2}(c^\dagger c+d^\dagger d+1) , \end{aligned} \tag{2.1}$$

which satisfy the algebra

$$\begin{aligned} &[K_i^+,K_j^-]=-2\delta_{ij}K_j^z , \\ &[K_i^z,K_j^\pm]=\pm\delta_{ij}K_j^\pm . \end{aligned} \tag{2.2}$$

The corresponding Casimir operators are

$$C_i=\frac{(K_i^+K_i^-+K_i^-K_i^+)}{2}-K_{z_i}^2 \quad (i=1,2) , \tag{2.3}$$

which can be written in terms of photon operators as

$$\begin{aligned} &C_1=\tfrac{1}{4}[1-(a^\dagger a-b^\dagger b)^2] , \\ &C_2=\tfrac{1}{4}[1-(c^\dagger c-d^\dagger d)^2] . \end{aligned} \tag{2.4}$$

In the photon number basis the action of the C_i is

$$C_i|n_i+q_i,n_i\rangle=\left[\frac{1-q_i^2}{4}\right]|n_i+q_i,n_i\rangle . \tag{2.5}$$

These generate a representation $D_{m_1}^{j_1}$ and $D_{m_2}^{j_2}$ of SU(1,1) corresponding to the basis functions $|j_i,m_i\rangle$, labeled by the eigenvalues j_i and m_i of the Casimir operator C_i and K_i^z,

$$C_i|j_i,m_i\rangle=-j_i(j_i+1)|j_i,m_i\rangle \quad (i=1,2) , \tag{2.6}$$

$$K_i^z|j_i,m_i\rangle=m_i|j_i,m_i\rangle . \tag{2.7}$$

The states $|j_i,m_i\rangle$, which are relevant in the context of this paper, are the ones that correspond to the positive discrete series representation of SU(1,1), with the lowest state being $|j_i,-j_i\rangle$. m_i takes on the values $m_i=-j_i,-j_i+1,\ldots$. In what follows we shall be using both of the bases, as the quantum-optical applications involve the photon operators, whereas many of the standard group-theoretical treatments use the $|j_i,m_i\rangle$ basis [5,12]. The state $|j_i,m_i\rangle$ can be written as

$$|j_1,m_1\rangle=\frac{(a^\dagger)^{m_i-j_i-1}(b^\dagger)^{m_i+j_i}}{[(m_i+j_i)!(m_i-j_i-1)!]^{1/2}}|0,0\rangle \quad (i=1) . \tag{2.8}$$

with $|j_2,m_2\rangle$ defined by replacing $a\leftrightarrow c$ and $b\leftrightarrow d$. In the conventionally used number-state basis, this corresponds to $|n_1+q_1,n_1\rangle$, where

$$|n_1+q_1,n_1\rangle=\frac{(a^\dagger)^{n_1+q_1}(b^\dagger)^{n_i}}{[(n_i!)(n_i+q_i)!]^{1/2}}|0,0\rangle \quad (i=1) . \tag{2.9}$$

and state $|n_2+8_2,n_2\rangle$ is defined similarly with $a\leftrightarrow c$ and $b\leftrightarrow d$. Furthermore,

$$\begin{aligned} &b^\dagger b|n_1+q_1,n_1\rangle=n_1|n_1+q_1,n_1\rangle , \\ &d^\dagger d|n_2+q_2,n_2\rangle=n_2|n_2+q_2,n_2\rangle . \end{aligned} \tag{2.10}$$

Thus we can establish the connection between the two bases by the identification

$$m_i - j_i - 1 = n_i + q_i \ , \quad m_i + j_i = n_i \quad (i=1,2) \ , \tag{2.11}$$

and

$$q_i = -(1+2j_i) \ , \tag{2.12}$$

with $q_i = 0 \Longrightarrow j_i = -\frac{1}{2}$.

The pair coherent state corresponds to the simultaneous eigenstate of the operator K_i^- and C, i.e.,

$$K_i^- |\zeta_i, j_i\rangle = \zeta_i |\zeta_i, j_i\rangle \ , \quad C|\zeta_i, j_i\rangle = -j_i(j_i+1)|\zeta_i, j_i\rangle \ . \tag{2.13}$$

In terms of the basis states $|j_i, m_i\rangle$, $|\zeta_i, j_i\rangle$ can be written as [5]

$$|\zeta_i, j_i\rangle = N_{j_i} \sum_{m_i=-j_i}^{\infty} \frac{(\zeta_i)^{m_i+j_i}}{[(m_i+j_i)!(m_i-j_i-1)!]^{1/2}} |j_i, m_i\rangle \quad (i=1,2) \ . \tag{2.14}$$

The normalization of these states N_{j_i} is given by

$$N_{j_i} = [(|\zeta_i|)^{2j_i+1} I_{-2j_i-1}(2|\zeta_i|)]^{-1/2} \ . \tag{2.15}$$

In the number-state basis $|n_i+q_i, n_i\rangle$ the state labeled $|\zeta_i, q_i\rangle$ can be written as [4]

$$|\zeta_i, q_i\rangle = N_{q_i} \sum_{n_i=0}^{\infty} \frac{\zeta_i^{n_i}}{\sqrt{n_i!(n_i+q_i)!}} |n_i+q_i, n_i\rangle \quad (i=1,2) \ , \tag{2.16}$$

with

$$N_{q_i} = [(|\zeta_i|)^{-q_i} I_{q_i}(2|\zeta_i|)]^{-1/2} \ . \tag{2.17}$$

These states constitute a complete set in each sector q_i, and the completeness relation in each sector is given by

$$\int d^2\zeta_i \frac{2}{\pi} N_{q_i}^{-2} |\zeta_i|^{q_i} K_{q_i}(2|\zeta_i|) |\zeta_i, q_i\rangle\langle \zeta_i, q_i| = 1 \ , \tag{2.18}$$

where the identity refers to the subspace characterized by a fixed value of q_i. This can be written as

$$\int d^2\zeta_i \frac{2}{\pi} I_{q_i}(2|\zeta_i|) K_{q_i}(2|\zeta_i|) |\zeta_i, q_i\rangle\langle \zeta_i, q_i| = 1 \tag{2.19}$$

for the normalized states (2.16).

We now consider the group obtained by the addition of these SU(1,1) generators, i.e.,

$$K^+ = a^\dagger b^\dagger + c^\dagger d^\dagger = K_1^+ + K_2^+ \ ,$$
$$K^- = ab + cd = K_1^+ + K_2^- \ , \tag{2.20}$$
$$K^z = \tfrac{1}{2}(a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d + 2) = K_1^z + K_2^z \ .$$

The corresponding Casimir operator will be

$$C = \frac{(K^+K^- + K^-K^+)}{2} - K_z^2 \ . \tag{2.21}$$

The "bi-pair coherent states" are now the eigenstates of K^-, C_i, and C. If we restrict ourselves to the positive discrete series representations of SU(1,1), then the Kronecker product $D_{m_1}^{j_1} \times D_{m_2}^{j_2}$ reduces to the sum of irreducible representations, i.e., the Clebsch-Gordan series for SU(1,1) given by [12]

$$D^{j_i} \times D^{j_2} = \sum_{-J=-j_1-j_2}^{\infty} D^J \ , \tag{2.22}$$

where the allowed values of J are given by $-J = -j_1 - j_2 + n$. In the number-state basis we label the irreducible representations by D^q, where $q = -2J-1$, and we may write the Kronecker product as

$$D^{q_1} \times D^{q_2} = \sum_{q=q_1+q_2+1}^{\infty} D^q \ . \tag{2.23}$$

Thus a given representation in the Kronecker product is fixed by J, j_1, j_2 or q, q_1, q_2 and we have two alternative ways of labeling the BPCS, i.e, $|\zeta, q_1, q_2, q\rangle$ and $|\zeta, j, j_1, j_2, J\rangle$. There is an added degeneracy parametrized by

$$n = -J + j_1 + j_2 = \frac{q-(q_1+q_2+1)}{2} \quad (n=0,1,2\ldots) \ . \tag{2.24}$$

The eigenvalue problem that we wish to solve is

$$K^- |\zeta, q\rangle = \zeta |\zeta, q\rangle \ , \quad C|\zeta, q\rangle = -J(J+1)|\zeta, q\rangle = (\tfrac{1}{4} - q^2/4)|\zeta, q\rangle \ . \tag{2.25}$$

Labeling the states $|\zeta, q\rangle$ by the degeneracy parameter n and q_1 and q_2, the eigenvalue equations (2.25) are

$$K^- |\zeta, q_1, q_2, n\rangle = \zeta |\zeta, q_1, q_2, n\rangle \ ,$$
$$C|\zeta, q_1, q_2, n\rangle = \left[\frac{2n+q_1+q_2}{2}\right] \times \left[\frac{2n+q_1+q_2+1}{2} + 1\right] |\zeta, q_1, q_2, n\rangle \ . \tag{2.26}$$

The basis states for the representation D^J are labeled by the eigenvalues J and M of C and K_z. The coherent state $|\zeta, J, j_1, j_2\rangle$, which satisfies (2.25), can be written in terms of this basis as

$$|\zeta, J, j_1, j_2\rangle = N_J \sum_{M=-J}^{\infty} \frac{(\zeta)^{M+J}}{[(M+J)!(M-J-1)!]^{1/2}} |J, M, j_1, j_2\rangle \ . \tag{2.27}$$

The normalization factor N_J is given by

$$N_J = [(|\zeta|)^{2J+1} I_{-2J-1}(2|\zeta|)]^{1/2} \ . \tag{2.28}$$

It is clear from (2.27) that the bi-pair coherent states are complete in the same subspace as the states $|J, M, j_1, j_2\rangle$, i.e., the subspace given by the representation D^J. The resolution of the unit $1_{(J)}$ in *each subspace* characterized by a fixed value of j_1, j_2, and J is

$$\int d^2\zeta \frac{2}{\pi} I_{-2J-1}(2|\zeta|)K_{-2J-1}(2|\zeta|)|\zeta,J,j_1,j_2\rangle\langle\zeta,J,j_1,j_2| = 1_{(J)} . \tag{2.29}$$

The explicit form of this bi-pair coherent state in the product basis $|j_1,m_1\rangle|j_2,m_2\rangle$ using the CG coefficients of SU(1,1) is

$$|\zeta,J,j_1,j_2\rangle = N_J \sum_{M=-J}^{\infty} \frac{(\zeta)^{M+J}}{[(M+J)!(M-J-1)!]^{1/2}} \sum_{m_1} C^{j_1,j_2,J}_{m_1,M-m_1,M}|j_1,m_1\rangle|j_2\ ,M-m_1\rangle , \tag{2.30}$$

where $C^{j_1,j_2,J}_{m_1,M-m_1,M}$ are the CG coefficients in the $|j_i,m_i\rangle$ basis for SU(1,1) whose explicit forms and properties are given in Appendix A [12]. Using the fact that $-J+j_1+j_2=n$ defining $M+J=k$, $M-J-1=k+2n+q_1+q_2$, $n_1+n_2=m_1+m_2+j_1+j_2=M+j_1+j_2=k+n$, and $q=2n+q_1+q_2+1$, and using the relations (2.11), we can write expression (2.30) in the photon number basis as

$$|\zeta,n,q_1,q_2\rangle = N_n \sum_{k=0}^{\infty} \frac{(\zeta)^k}{[(k)!(k+2n+q_1+q_2+1)!]^{1/2}} \sum_{n_1,n_2} C^{q_1,q_2,n}_{n_1,n_2,n+k}\, \delta_{(n_1+n_2,n+k)}|n_1+q_1,n_1\rangle|n_2+q_2,n_2\rangle . \tag{2.31}$$

Substituting the explicit expressions for the CG coefficient given in Appendix A in Eq. (2.30), we get an expression for the BPCS in terms of the CG coefficients in the photon-number basis. In order to understand the properties of the coherent state (2.31) we examine some special cases where the CG coefficient has a simple form.

(a) Vacuum state ($\zeta=0$) of SU(1,1)×SU(1,1). This corresponds to $K^-|0,n,q_1,q_2\rangle=0$. Only the term $k=0$ contributes to (2.31) and $n_1+n_2=n$. Using the recursion relation for the CG coefficient n_1 times we get

$$C^{q_1,q_2,n}_{n_1,n_2,n} = C^{q_1,q_2,n}_{0,n,n}(-1)^{n_1}\left[\frac{(n)!(q_1)!}{(n_1)!(n-n_1)!}\right]^{1/2} \times \left[\frac{(n+q_2)!}{(n_1+q_1)!(n-n_1+q_2)!}\right]^{1/2} , \tag{2.32}$$

where $C^{q_1,q_2,n}_{0,n,n}=\{(n+q_1)!(n+q_1+q_2)!]/[q_1!(2n+q_1+q_2)!]\}^{1/2}$ is a normalization coefficient given in Appendix A. Thus the vacuum state can be written as

$$|0,n,q_1,q_2\rangle = \left[\begin{bmatrix}2n+q_1+q_2\\ n+q_1\end{bmatrix}\right]^{-1/2} \times \sum_{n_1}^{n}(-1)^{n_1}\left[\begin{bmatrix}n\\ n_1\end{bmatrix}\begin{bmatrix}n+q_1+q_2\\ n_1+q_1\end{bmatrix}\right]^{1/2} \times |n_1+q_1,n_1\rangle \times |n-n_1+q_2,n-n_1\rangle . \tag{2.33}$$

It should be noted that expression (2.33) can also be derived directly without going through the CG algebra [8]. As mentioned in the Introduction the vacuum state is degenerate, with the degeneracy parameter given by n.

(b) $q_1=q_2=0$; $q=1$; $\zeta\neq 0$. In this special case, we start with an equal number of photons in the modes a and b and in c and d. We choose $q=1$ so that $n=[q-(q_1+q_2+1)]/2=0$ and $n_1+n_2=k$. The relevant CG coefficient is given by

$$C^{0,0,1}_{n_1,n_2,k} = \frac{1}{(k+1)^{1/2}} = \frac{1}{(n_1+n_2+1)^{1/2}} . \tag{2.34}$$

Therefore (2.31) reduces to

$$|\zeta,1,0,0\rangle = N_1 \sum_k \frac{\zeta^k}{[(k+1)!(k)!]^{1/2}} \times \sum_{n_1,n_2} \frac{1}{(k+1)^{1/2}} \times \delta_{n_1+n_2,k}|n_1,n_1\rangle|n_2,n_2\rangle , \tag{2.35}$$

where

$$N_1 = \frac{(|\zeta|)^{1/2}}{[I_1(2|\zeta|)]^{1/2}} . \tag{2.36}$$

In this case, the joint probability distribution of finding n_1 photons in the mode a and n_2 photons in the mode c is given by

$$P_{n_1,n_2}(|\zeta|) = \frac{N_1^2|\zeta|^{2(n_1+n_2)}}{[(n_1+n_2+1)!]^2} . \tag{2.37}$$

The single-mode probability distribution P_{n_1} and the mean number of photons $\langle n_1\rangle$ are given by

$$P_{n_1}(\zeta) = \sum_{n_2} P_{n_1,n_2}(|\zeta|) = N_1^2|\zeta|^{2n_1}\sum_{n_2}\frac{|\zeta|^{2n_2}}{(n_2+n_1+1)!^2} \tag{2.38}$$

and

$$\langle n_1\rangle = \sum_{n_1,n_2} P_{n_1,n_2}(|\zeta|) = \frac{|\zeta| I_2(2|\zeta|)}{2I_1(2|\zeta|)} . \tag{2.39}$$

A measure of the nonclassical nature of the distribution is given by Mandel's Q parameter, which for the mode a is given by

$$Q = \frac{\langle n_1^2\rangle - (\langle n_1\rangle)^2 - \langle n_1\rangle}{\langle n_1\rangle} \tag{2.40}$$

$$= \frac{2|\zeta| I_3(2|\zeta|)}{3I_2(2|\zeta|)} - \frac{|\zeta| I_2(2|\zeta|)}{2I_1(2|\zeta|)} . \tag{2.41}$$

We have checked, numerically and graphically, the behavior of Q vs $|\zeta|$ and found that for values of $|\zeta|<2$, Q is nearly 0 and slightly negative. This shows that the

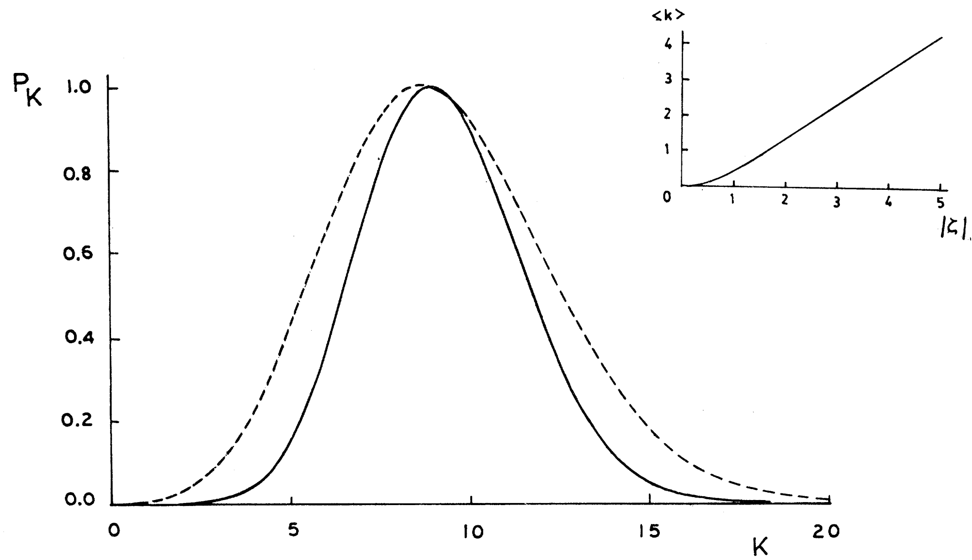

FIG. 1. P_k vs k [expression (2.42)] for $|\zeta|=10$ (solid line). The dashed line is the corresponding Poissonian with mean $\langle k \rangle$. The inset shows the behavior of the mean $\langle k \rangle$ as a function of $|\zeta|$.

distribution P_{n_1} is nearly Poissonian for low values of $|\zeta|$. In fact, upon comparing the distribution P_{n_1} vs n_1 for various values of $|\zeta|$ with the Poissonian of mean $\langle n_1 \rangle$, we find that for $|\zeta|<2$ the two distributions almost coincide, showing the Poissonian nature of the single-mode distributions.

The joint probability distribution $P_{n_1+n_2}$ can be calculated from P_{n_1,n_2} by the relation

$$P_k(|\zeta|)=\sum_{n_1,n_2} \delta_{n_1+n_2,k} P_{n_1,n_2}(|\zeta|)=\frac{N_1^2|\zeta|^{2k}}{k!(k+1)!} . \tag{2.42}$$

The average value $\langle k \rangle$ is given by

$$\sum_k kP_k(|\zeta|)=\frac{|\zeta| I_2(2|\zeta|)}{I_1(|\zeta|)} . \tag{2.43}$$

In Fig. 1 we plot P_k vs k and compare it to the corresponding Poissonian with mean value $\langle k \rangle$, and it is clear that the distribution of the sum n_1+n_2 is sub-Poissonian.

We do not treat in detail the method of production of such states. However, from Eqs. (1.11) and (1.12) and previous works [10,14], the mechanism for the generation of these states is evident.

III. PHASE-SPACE REPRESENTATION OF BPCS IN TERMS OF PAIR COHERENT STATES

In quantum optics, phase-space representations are extensively used to study the quantum statistics of the radiation fields. These representations are useful in extracting precise information on the quantum nature of the field. For a single-mode radiation field, one can expand an arbitrary state $|\psi\rangle$ in the coherent state basis as

$$|\psi\rangle=\frac{1}{\pi}\int \psi(\alpha^*)e^{-|\alpha|^2}|\alpha\rangle\rangle d^2\alpha , \tag{3.1}$$

where

$$\psi(\alpha^*)=\langle\langle \alpha|\psi\rangle\rangle , \tag{3.2}$$

and $|\alpha\rangle\rangle$ is the unnormalized coherent state

$$|\alpha\rangle\rangle=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle . \tag{3.3}$$

The Schrödinger equation can then be transformed into a differential equation for $\psi(\alpha^*)$ as shown by Klauder and Sudarshan [14]. One of the phase-space distributions used in quantum optics is the Q function defined for the single-mode field by $|\psi(\alpha^*)|^2e^{-|\alpha|^2}/\pi$. This can be generalized to other situations such as the SU(1,1) group or equivalently for two-mode fields, as has been done by Barut and Girardello [5]. We will now use a generalization of this method for the case of bi-pair coherent states. Since we consider four-mode radiation fields, we will have to work with an eight-dimensional phase space. However, we could examine a subspace of this eight-dimensional space, keeping in view the occurrence of photons in pairs. Therefore, we introduce an analog of the Q function defined via the projection of the four-mode state $|\psi\rangle$ onto the product of pair coherent states,

$$Q(\zeta_1,\zeta_2,q_1,q_2)=|\langle \zeta_1,q_1,\zeta_2,q_2|\psi\rangle|^2 . \tag{3.4}$$

The relations analogous to (3.1) to (3.3) for this four-mode basis can be obtained by using the unnormalized pair coherent state basis, which we denote by $|\;\rangle\rangle$,

$$\begin{aligned}|\zeta_1,q_1\rangle\rangle&=\sum_{n=0}^{\infty}\frac{\zeta_1^n}{\sqrt{n!(n+q_1)!}}|n+q_1,n\rangle ,\\ |\zeta_2,q_2\rangle\rangle&=\sum_{m=0}^{\infty}\frac{\zeta_2^m}{\sqrt{m!(m+q_2)!}}|m+q_2,m\rangle .\end{aligned} \tag{3.5}$$

The completeness relation for the unnormalized states $|\zeta_i,q_i\rangle\rangle$ is

$$\int d^2\zeta_i \frac{2}{\pi}|\zeta_i|^{q_i}K_{q_i}(2|\zeta_i|)|\zeta_i,q_i\rangle\rangle\langle\langle\zeta_i,q_i| = 1_{q_i} , \quad (3.6)$$

where the unit operator $1_{(q_i)}$ refers to *each subspace* characterized by a fixed value of q_i. Summation over q_i is required to get the unit operator in the whole Hilbert space. The state $|\psi\rangle$ in the subspace of the four-mode Hilbert space specified by the values of q_1 and q_2 is

$$|\psi,q_1,q_2\rangle = \int\int d^2\zeta_1 d^2\zeta_2 \frac{2}{\pi}|\zeta_1|^{q_1}\frac{2}{\pi}|\zeta_2|^{q_2}K_{q_1}(2|\zeta_1|)K_{q_2}(2|\zeta_2|)\psi(\zeta_1^*,\zeta_2^*,q_1,q_2)|\zeta_1,q_1\rangle\rangle|\zeta_2,q_2\rangle\rangle, \quad (3.7)$$

where

$$\psi(\zeta_1^*,\zeta_2^*,q_1,q_2) = \langle\langle\zeta_1,q_1|\langle\langle\zeta_2,q_2|\psi\rangle . \quad (3.8)$$

Summation over q_1 and q_2 would lead to the state $|\psi\rangle$ over the entire four-mode Hilbert space.

The eigenvalue equation for $|\psi\rangle$ can be converted to a differential equation for the function ψ in the basis of pair coherent states by using the differential operator representation for $K_i^{\pm}$ and K_i^z,

$$\begin{aligned} K_i^+|\zeta_i,q_i\rangle\rangle &= \frac{\partial}{\partial\zeta_i}\left[q_i+\zeta_i\frac{\partial}{\partial\zeta_i}\right]|\zeta_i,q_i\rangle\rangle , \\ K_i^z|\zeta_i,q_i\rangle\rangle &= \left[\zeta_i\frac{\partial}{\partial\zeta_i}+q_i\right]|\zeta_i,q_i\rangle\rangle , \\ K_i^-|\zeta_i,q_i\rangle\rangle &= \zeta_i|\zeta_i,q_i\rangle\rangle . \end{aligned} \quad (3.9)$$

The action of the Casimir invariant can be expressed as a differential operator by using Eq. (3.9) and the fact that C_1 and C_2 are conserved quantities,

$$\langle\langle\zeta_i,q_i|C_i = \left[\frac{1-q_i^2}{4}\right]\langle\langle\zeta_i,q_i| . \quad (3.10)$$

We now apply the above prescription for obtaining the projection of the BPCS $|\zeta,q\rangle$ onto the states $|\zeta_1,q_1\rangle|\zeta_2,q_2\rangle$,

$$f(\zeta,\zeta_1^*,\zeta_2^*,q_1,q_2,q) = \langle\zeta_1,q_1,\zeta_2,q_2|\zeta,q\rangle . \quad (3.11)$$

Calling the overlap of the unnormalized $\langle\langle\zeta,q|\zeta_1,\zeta_2,q_1,q_2\rangle\rangle = \tilde{f}^*$, the eigenvalue equations for $|\zeta,q\rangle$ can be converted to equations for $\tilde{f}^*$,

$$\begin{aligned} \langle\langle\zeta,q|K_+|\zeta_1,\zeta_2,q_1,q_2\rangle\rangle &= \zeta^*\langle\langle\zeta,q|\zeta_1,\zeta_2,q_1,q_2\rangle\rangle , \\ \langle\langle\zeta,q|C|\zeta_1,\zeta_2,q_1,q_2\rangle\rangle &= \frac{(1-q^2)}{4}\langle\langle\zeta,q|\zeta_1,\zeta_2,q_1,q_2\rangle\rangle . \end{aligned} \quad (3.12)$$

Using the differential representations (3.9) and (3.10) of $K^+ = K_1^+ + K_2^+$ and C_i we can obtain the following differential equations for the function $\tilde{f}^*$:

$$\left[\frac{\partial}{\partial\zeta_1}\left[q_1+\zeta_1\frac{\partial}{\partial\zeta_1}\right]+\frac{\partial}{\partial\zeta_2}\left[q_2+\zeta_2\frac{\partial}{\partial\zeta_2}\right]\right]\tilde{f}^* = \zeta^*\tilde{f}^* , \quad (3.13)$$

$$\begin{aligned} &\left[\zeta_1\zeta_2\left[\frac{\partial^2}{\partial\zeta_1^2}-2\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}+\frac{\partial^2}{\partial^2\zeta_2}\right]\right]\tilde{f}^* \\ &\quad+\left[(q_1+1)\zeta_2\left[\frac{\partial}{\partial\zeta_1}-\frac{\partial}{\partial\zeta_2}\right] -(q_2+1)\zeta_1\left[\frac{\partial}{\partial\zeta_1}-\frac{\partial}{\partial\zeta_2}\right]\right]\tilde{f}^* \\ &\quad= -\left[\frac{q^2}{4}-\frac{(q_1+q_2+1)^2}{4}\right]\tilde{f}^* . \end{aligned} \quad (3.14)$$

To solve the above set of coupled differential equations we define the complex variables z and θ by

$$\begin{aligned} \zeta_1 &= z^2\cos^2(\theta) , \\ \zeta_2 &= z^2\sin^2(\theta) , \end{aligned} \quad (3.15)$$

Eq. (3.13) for f^* takes the form

$$\frac{d^2\tilde{f}^*}{dz^2}+\frac{1}{z}(2q_1+2q_2+3)\frac{d\tilde{f}^*}{dz}+\frac{L_\theta(\tilde{f}^*)}{z^2} = 4\zeta^*\tilde{f}^* ; \quad (3.16)$$

while (3.14) becomes

$$\begin{aligned} L_\theta(\tilde{f}^*) = \frac{d^2\tilde{f}^*}{d\theta^2} &+ [(2q_2+1)\cot(\theta) \\ &+(2q_1+1)\tan(\theta)]\frac{d\tilde{f}^*}{d\theta} \\ &= -[q^2-(q_1+q_2+1)^2] . \end{aligned} \quad (3.17)$$

We now use the separation of variables

$$\tilde{f}^*(z,\theta) = g(z)h(\theta) , \quad (3.18)$$

to reduce Eq. (3.16) to

$$h(\theta)\left[z^2\frac{d^2g}{dz^2}+z(2q_1+2q_2+3)\frac{dg}{dz}-4\zeta^*z^2g(z)\right] = -g(z)L_\theta[h(\theta)] , \quad (3.19)$$

which after division by $h(\theta)g(z)$ leads to

$$\frac{1}{g(z)}\left[z^2\frac{d^2}{dz^2}+z(2q_1+2q_2+3)\frac{d}{dz}-4\zeta^*z^2g(z)\right] = -\frac{1}{h(\theta)}L_\theta[h(\theta)] . \quad (3.20)$$

Thus the left-hand side is a function of z alone and the right-hand side is a function of θ; hence, we must have

$$\frac{1}{h(\theta)}L_\theta[h(\theta)]=-\Lambda\ , \tag{3.21}$$

$$\frac{1}{g(\theta)}\left[z^2\frac{d^2g}{dz^2}+z(2q_1+2q_2+3)\frac{dg}{dz}-4\zeta^*z^2g(z)\right]=\Lambda\ . \tag{3.22}$$

To determine Λ we use Eq. (3.17) to obtain

$$\frac{1}{h(\theta)}L_\theta[h(\theta)]=-[q^2-(q_1+q_2+1)^2]=-\Lambda\ . \tag{3.23}$$

Thus Eqs. (3.16) and (3.17) reduce to two uncoupled equations,

$$\frac{d^2h}{d\theta^2}+[(2q_2+1)\cot(\theta)+(2q_1+1)\tan(\theta)]\frac{dh}{d\theta}=-[q^2-(q_1+q_2+1)^2]h(\theta)\ , \tag{3.24}$$

$$\left[z^2\frac{d^2g}{dz^2}+z(2q_1+2q_2+3)\frac{dg}{dz}-4\zeta^*z^2g(z)\right]=[q^2-(q_1+q_2+1)^2]g(z)\ . \tag{3.25}$$

The unnormalized regular solution for $g(z)$, which is finite and regular at $z=0$ and consistent with the limit $\zeta^*=0$ is [15]

$$g(z)=D\left[(z)^{-q_1-q_2-1}\frac{I_q(\sqrt{4\zeta^*}z)}{(\sqrt{\zeta^*})^q}\right]\ , \tag{3.26}$$

where D is a normalization constant, which may be a function of ζ and z^*. Now to obtain a solution for h, we use the transformation $x=\cos^2(\theta)-\sin^2(\theta)=(\zeta_1-\zeta_2)/(\zeta_2+\zeta_1)$ to reduce the equation for h (3.24) to the following:

$$(1-x^2)\frac{d^2h}{dx^2}+[(q_1-q_2)-(q_1+q_2+2)x]\frac{dh}{dx}+[-(q_1+q_2+1)^2+q^2]h=0\ . \tag{3.27}$$

By using $n=\frac{1}{2}[q-(q_1+q_2+1)]$, where n is an integer, we recast Eq. (3.24) into

$$(1-x^2)\frac{d^2h}{dx^2}+[q_1-q_2-(q_1+q_2+2)x]\frac{dh}{dx}+n(n+q_1+q_2+1)h=0\ . \tag{3.28}$$

This is the differential equation for the Jacobi polynomial $P_n^{q_2,q_1}(x)=(-1)^nP_n^{q_1,q_2}(-x)$; thus

$$h(\theta)=B(-1)^nP_n^{q_1,q_2}[\sin^2(\theta)-\cos^2(\theta)]=BP_n^{q_2,q_1}[\cos^2(\theta)-\sin^2(\theta)]\ , \tag{3.29}$$

where B is a constant. Thus the expression for $\tilde{f}^*$ is

$$\tilde{f}^*(z,\theta)=A(\sqrt{\zeta^*})^{-q}(z)^{-q_1-q_2-1}I_q(\sqrt{4\zeta^*}z)P_n^{q_2,q_1}(x)\ , \tag{3.30}$$

where A is a constant that can be a function of ζ_i^* and ζ.

Substituting the values of z and θ we have the final expression (apart from normalization factors) for $\tilde{f}$ given by

$$\tilde{f}(\zeta_1^*,\zeta_2^*,\zeta)=A^*(\zeta)^{-q/2}(\zeta_1^*+\zeta_2^*)^{-(q_1+q_2+1)/2}\times I_q[\sqrt{4\zeta(\zeta_1^*+\zeta_2^*)}]P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]\ . \tag{3.31}$$

The normalized overlap function can be obtained from the completeness relation

$$\int\int|\tilde{f}|^2d^2\lambda(\zeta_1)d^2\lambda(\zeta_2)=1\ , \tag{3.32}$$

where

$$d^2\lambda(\zeta_i)=\frac{2}{\pi}K_{q_i}(2|\zeta_i|)|\zeta_i|^{q_i}d^2\zeta_i\ . \tag{3.33}$$

To sum up, using the differential representation of the generators of the SU(1,1) Lie algebra we have derived an expression for the overlap of the unnormalized states: $\langle\langle\zeta_1,q_1,\zeta_2,q_2|\zeta,q\rangle\rangle=\tilde{f}$, given by expression (3.31). The normalized overlap function can be obtained from expression (3.32) to get

$$f=N[\zeta(\zeta_1^*+\zeta_2^*)]^{-q/2}I_q[\sqrt{4\zeta(\zeta_1^++\zeta_2^*)}](\zeta_1^*+\zeta_2^*)^n\times P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]\ . \tag{3.34}$$

Thus the state $|\zeta,q\rangle$ can be obtained from the relation

$$|\zeta,q\rangle=\frac{4N}{\pi^2}\int d^2\zeta_1\int d^2\zeta_2|\zeta_1|^{q_1}K_{q_1}(2|\zeta_1|)|\zeta_2|^{q_2}K_{q_2}(2|\zeta_2|)\times\langle\langle\zeta_1,q_1,\zeta_2,q_2|\zeta,q\rangle\rangle\times|\zeta_1,q_1\rangle\rangle|\zeta_2,q_2\rangle\rangle\ . \tag{3.35}$$

In Appendix B we show by explicit integration that the expression obtained from Eq. (3.35) is identical to that obtained in Eq. (2.31).

We evaluate the quantity $|f|^2=Q_p$ for some special cases of physical interest.

(i) $|\zeta|=0$.

$$\lim_{\zeta\to0}Q_p=|N|^2(|\zeta_1^*+\zeta_2^*|)^{2n}\left|P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]\right|^2\ ,\quad 2n=q-q_1-q_2-1\ , \tag{3.36}$$

where N is the normalization obtained from (3.22). By using the explicit expansion of the Jacobi polynomial [15]

$$P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]=(\zeta_1+\zeta_2)^{-n}\times\sum_m^n\left[\binom{n+q_1}{n-m}\binom{n+q_2}{m}\right]\times(-1)^{(n-m)}(\zeta_1^*)^n(\zeta_2^*)^{n-m}\ , \tag{3.37}$$

and using Eq. (3.35) and the results of Appendix B, after some rearrangement we have

$$\langle \zeta_1,\zeta_2|0,q,q_1,q_2\rangle = N\sum_{n_1,n_2} \frac{C^{q_1,q_2,q}_{n_1,n_2,n}}{(n_1!n_2!(n_1+q_1)!(n_2+q_2)!)^{1/2}} \times (\zeta_1^*)^{n_1}(\zeta_2^*)^{n-n_1} , \quad (3.38)$$

where $C^{q_1,q_2,q}_{n_1,n_2,n}$ is the CG coefficient given in expression (2.31); thus our result is consistent with Eq. (2.33).

(ii) $q=q_1+q_2+1$ and $|\zeta|\neq 0$.

$$Q_P=|N|^2|\zeta(\zeta_1^*+\zeta_2^*)|^{-(q_1+q_2+1)}|I_q[\sqrt{4\zeta(\zeta_1^*+\zeta_2^*}]|^2. \quad (3.39)$$

(iii) $q_1=0$; $q_2=0$. Here we use the fact that $P_n^{0,0}(x)=P_n(x)$, where $P_n(x)$ is the Legendre polynomial and $q=2n+1$. Thus, $|f|^2$ becomes

$$Q_p=|(N)|^2|(\zeta_1^*+\zeta_2^*)|^{-1}\left|\frac{I_{2n+1}(\sqrt{4\zeta(\zeta_1^*+\zeta_2^*)})}{\zeta^{(2n+1)/2}}\right|^2 \times \left|P_n\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]\right|^2 . \quad (3.40)$$

Using (3.35) state $|\zeta,2n+1\rangle$ is

$$|\zeta,2n+1\rangle = N\sum_{k=0}^{\infty}\frac{\zeta^k}{[k!(k+2n+1)!]^{1/2}}\sum_{n_1,n_2}\delta_{n_1+n_2,n+k}(n_1!n_2!n!^2)\left[\frac{k!}{(k+2n+1)!}\right]^{1/2} \times \sum_l (-1)^l\frac{1}{(l!)^2(n-l)!^2(n_2-l)!(n_1-n+l)!}|n_1,n_1\rangle|n_2,n_2\rangle . \quad (3.41)$$

For the case $q=1$ or $n=0$, Q_p becomes

$$Q_p=|N|^2|\zeta(\zeta_1^*+\zeta_2^*)|^{-1}|I_1[\sqrt{4\zeta(\zeta_1^*+\zeta_2^*)}]|^2 , \quad (3.42)$$

and

$$|\zeta,1\rangle = N\sum_{k=0}^{\infty}\frac{\zeta^k}{[k!(k+1)!]^{1/2}}\sum_{n_1,n_2}\delta(n_1+n_2,k)\frac{1}{(k+1)^{1/2}} \times |n_1,n_1\rangle|n_2,n_2\rangle . \quad (3.43)$$

Thus we see that the above the expression for $|\zeta,1\rangle$ is identical to Eq. (2.34) obtained in Sec. II.

IV. CONCLUSION

In conclusion, we have introduced a class of coherent states, the bi-pair coherent states, which can provide a useful tool in tackling problems involving two-photon beams, each polarized in an arbitrary way, or in dealing with problems involving pairs of down-converter crystals. By using the boson realization of SU(1,1) we have shown that the BPCS are related to the group-theoretic-addition properties of two SU(1,1) groups. We can provide a representation of the BPCS in the basis of pair coherent states. In view of the tremendous range of applications of pair coherent states, the utility of the BPCS is not limited to quantum optics, but has applications in other fields as well. An example of such an application in particle physics is in the phenomenon of multipion production in cosmic ray events and particle collisions. Events with unusually large fluctuations in the number of pions known as Centauro events have been recently shown to describe bi-pair coherent states where the charged pion states can be thought to be right and left and circular polarized states [17]. Imposition of isospin conservation leads to pair coherent states, and the BPCS should be useful when pairs of charged pions participate in the dynamical process. These and other related problems are the subject of future research.

APPENDIX A

In this appendix we list some properties and expressions for the Clebsch-Gordon coefficients for the coupling of two positive discrete unitary representations of SU(1,1), which are required in Sec. II [12]. The CG coefficients are defined by the relation

$$|J,M\rangle = \sum_{m_1} C^{j_1,j_2,J}_{m_1,M-m_1,M}|j_1,m_1\rangle|j_2,M-m_1\rangle . \quad (A1)$$

Applying the lowering operator $K^-=K_1^-+K_2^+$ on both sides, the following recursion relation can be derived:

$$[(m_1+j_1)(m_1-j_1-1)]^{1/2}C^{j_1,j_2,J}_{m_1,m_2,M} = [(M+J)(M-J-1)]^{1/2}C^{j_1,j_2,J}_{m_1-1,m_2,M-1} - [(m_2-j_2)(m_2+j_2+1)]^{1/2}C^{j_1,j_2,J}_{m_1-1,m_2+1,M} . \quad (A2)$$

After some algebra and repeated applications of the recursion relation we have

$$C^{j_1,j_2,J}_{m_1,m_2,M}=(-1)^{j_1+j_2-J}C^{j_1,j_2,J}_{-j_1,j_1-J,-J}\left[\frac{(M+J)!(m_2+j_2)!(m_1+j_1)!(-2j_1-1)!(-2J-1)!(j_1+j_2-J)!}{(m_1-j_1-1)!(m_2-j_2-1)!(M-J-1)!(j_1-j_2-J-1)!(j_2-j_1-J-1)!^2}\right]^{1/2}$$
$$\times\sum_l(-1)^l\frac{(m_1-j_1-1+l)!(m_2+j_1-J-1-l)!}{(l)!(m_1+J-j_2+l)!(m_2+j_2-l)!(j_2-J+j_1-l)!}\ , \tag{A3}$$

with

$$C^{j_1,j_2,j}_{-j_1,j_1-J,-J}=(-1)^{j_1+j_2-J}[-2J-1]^{1/2}\left[\frac{(j_2-j_1-J-1)!(-j_1-j_2-J-2)!}{(-2j_1-1)!(-2J-1)!}\right]^{1/2}.$$

Thus,

$$C^{j_1,j_2,J}_{m_1,m_2,M}=\sqrt{-2J-1}\left[\frac{(M+J)!(m_2+j_2)!(m_1+j_1)!(-j_1-j_2-j-2)!(j_1+j_2-J)!}{(m_1-j_1-1)!\ (m_2-j_2-1)!(M-J-1)!(j_1-j_2-J-1)!(j_2-j_1-J-1)!}\right]^{1/2}$$
$$\times\sum_l(-1)^l\frac{(m_1-j_1-1+l)!(m_2+j_1-J-1-l)!}{(l)!(m_1+J-j_2+l)!(m_2+j_2-l)!(j_2-J+j_1-l)!}\ . \tag{A4}$$

Using the fact that $-J+j_1+j_2=n$ and defining

$$M+J=k\ ,\quad M-J-1=k+q\ ,\quad q=2n+q_1+q_2+1\ ,\quad n_1+n_2=m_1+m_2+j_1+j_2=M+j_1+j_2=k+n\ , \tag{A5}$$

the expressions (A2) and (A4) for the CG coefficients in the photon-number basis become

$$[(n_1)(n_1+q_1)]^{1/2}C^{q_1,q_2,q}_{n_1,n_2n_1+n_2}=[(k)(k+q)]^{1/2}C^{q_1,q_2,q}_{n_1-1,n_2n_1+n_2-1}-[(n_2+q_2+1)(n_2+1)]^{1/2}C^{q_1,q_2,q}_{n_1-1,n_2+1,n_1+n_2} \tag{A6}$$

and

$$C^{q_1,q_2,n}_{n_1,n_2,n+k}=(2n+q_1+q_2+1)^{1/2}\left[\frac{(k)!(n_1)!(n_2)!(n)!(n+q_1+q_2)!}{(n_1+q_1)!(n_2+q_2)!(k+2n+q_1+q_2+1)!(n+q_2)!(n+q_1)!}\right]^{1/2}$$
$$\times\sum_l(-1)^l\frac{(n+n_2+q_2+l)!(n_1+q_1+l)!}{l!(n_1-n+l)!(n_2-l)!(n-l)!}\ , \tag{A7}$$

with

$$C^{q_1,q_2,q}_{0,n,n}=\left[\frac{(n+q_1)!(n+q_1+q_2)!}{(q_1)!(2n+q_1+q_2)!}\right]^{1/2}\ . \tag{A8}$$

The expression due to Biedenharn and Holman can be converted into a more convenient form for the purposes of comparing (2.31) and (3.35) using the method of Racah [18] and the identity

$$\frac{a!}{b!c!}=\sum_s\frac{(a-b)!(a-c)!}{(a-b-s)!(a-c-s)!(b+c-a+s)!s!}\ , \tag{A9}$$

with $a=n+n_2=q_2-l$, $b=n_2-l$, and $c=n-l$ we get

$$C^{q_1,q_2,n}_{n_1,n_2,n+k}=(2n+q_1+q_2+1)^{1/2}\left[\frac{k!(n_1)!(n_2)!(n)!(n+q_1+q_2)!(n_2+q_2)!(n+q_2)!}{(n_1+q_1)!(k+2n+q_1+q_2+1)!(n+q_1)!}\right]^{1/2}$$
$$\times\sum_{l,s}(-1)^l\frac{(n_1+q_1+l)!}{l!(n_1-n+l)!(s)!(-q_2+s-l)!(n+q_2-s)!(n_2+q_2-s)!}\ . \tag{A10}$$

Then summing over 1 using the identity

$$\sum_l\frac{(-1)^l(u-l)!}{l!(x-l)!(z-l)!}=(-1)^z\frac{(u-z)!(x+z-u-1)!}{x!z!(x-u-1)!}\ , \tag{A11}$$

with $u=n-n_1-1$, $x=-n_1-q_1-1$, and $z=-q_2+s$, and defining $m=-q_2+s$, we have

$$C^{q_1,q_2,n}_{n_1,n_2,n+k}=(2n+q_1+q_2+1)^{1/2}\left[\frac{n_1!n_2!(n_1+q_1)!(n_2+q_2)!k!}{(k+2n+q_1+q_2+1)!}\right]^{1/2}[(n+q_1)!(n+q_2)!]^{1/2}$$
$$\times\sum_m(-1)^m\frac{1}{(q_2+m)!(n-m)!(n_2-m)!(n_1-n+m)!(n+q_1-m)!}\ . \tag{A12}$$

As will be seen from Appendix B this form is convenient when it comes to comparing expressions (2.31) and (3.35).

APPENDIX B

In this appendix we will show that the two expressions for $|\zeta,q\rangle$, Eqs. (2.31) and (3.35), are equivalent by the explicit evaluation of the integral given by

$$|\zeta,q\rangle=\frac{4N}{\pi^2}\int\int d^2\zeta_1 d^2\zeta_2|\zeta_1|^{q_1}|\zeta_2|^{q_2}K_{q_1}(2|\zeta_1|)K_{q_2}(2|\zeta_2|)\bar{f}(\zeta_1^*\zeta_2^*,\zeta)|\zeta_1,q_1\rangle\rangle|\zeta_2,q_2\rangle\rangle\ . \tag{B1}$$

Using Eq. (3.30) we have

$$\bar{f}(\zeta_1^*,\zeta_2^*,\zeta)=\langle\langle\zeta_1,q_2,\zeta_2,q_2|\zeta,q\rangle\rangle=[\zeta(\zeta_1^*+\zeta_2^*)]^{-q/2}I_q[\sqrt{4\zeta(\zeta_1^*+\zeta_2^*)}](\zeta_1^*+\zeta_2^*)^n P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]\ . \tag{B2}$$

Thus

$$|\zeta,q\rangle=\frac{4N}{\pi^2}\int d^2\zeta_1\int d^2\zeta_2|\zeta_1|^{q_1}|\zeta_2|^{q_2}K_{q_1}(2|\zeta_1|)K_{q_2}(2|\zeta_2|)[\zeta(\zeta_1^*+\zeta_2^*)]^{-q/2}$$
$$\times I_q(\sqrt{4\zeta(\zeta_1^*+\zeta_2^*)})(\zeta_1^*+\zeta_2^*)^n P_n^{q_2,q_1}\left[\frac{\zeta_1^*-\zeta_2^*}{\zeta_1^*+\zeta_2^*}\right]|\zeta_1,q_1\rangle\rangle|\zeta_2,q_2\rangle\rangle\ . \tag{B3}$$

Substituting the values of $|\zeta_1,q_1\rangle\rangle$ and $|\zeta_2,q_2\rangle\rangle$, substituting $\zeta_1=|\zeta_1|e^{i\theta_1}$ and $\zeta_2=|\zeta_2|e^{i\theta_2}$, and using the expansion for the Jacobi polynomial (3.37) as well as the expansion of the Bessel function I_q [15]

$$I_q(z)=(\tfrac{1}{2}z)^q\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!(k+q)!}\ , \tag{B4}$$

we get

$$|\zeta,q\rangle=\frac{4N}{\pi^2}\sum_{n_1,n_2}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{n}\int d\theta_1 e^{i\theta_1(n_1-m-1)}\int d\theta_2 e^{i\theta_2(n_2-n+m+l-k)}$$
$$\times\int|\zeta_1|^{n_1+q_1+l+m+1}K_{q_1}(2|\zeta_1|)d|\zeta_1|\int|\zeta_2|^{n_2+q_2+n-m+k-l+1}K_{q_2}(2|\zeta_2|)d|\zeta_2|$$
$$\times\frac{1}{(n_1!n_2!(n_1+q_1)!(n_2+q_2)!)^{1/2}}\frac{\zeta^k}{k!(k+q)!}\begin{bmatrix}k\\l\end{bmatrix}\begin{bmatrix}n+q_1\\n-m\end{bmatrix}\begin{bmatrix}n+q_2\\m\end{bmatrix}(-1)^{n-m}|n_1+q_1,n_1\rangle|n_2+q_2,n_2\rangle\ . \tag{B5}$$

The integrations over θ_1 and θ_2 give us two Kronecker deltas $\delta[n_1-(m+l)]$ and $\delta[n_2-(n-m+k-l)]$, and the integral over ζ_1 and ζ_2 can be carried out using the identity

$$4\int|\zeta_1|^{2n_1+q_1+1}K_{q_1}(2|\zeta_1|)d|\zeta_1|=(n_1)!(n_1+q_1)!\ . \tag{B6}$$

By carrying out these integrations using the δ functions and rearranging we have

$$|\zeta,q\rangle=N'\sum_{k=0}^{\infty}\frac{\zeta^k}{(k!(k+q)!)^{1/2}}\sum_{n_1,n_2}\delta_{(n_1+n_2,n+k)}\left[\frac{n_1!n_2!(n_1+q_1)!(n_2+q_2)!k!}{(k+q)!}\right]^{1/2}(n+q_1)!(n+q_2)!$$
$$\times\sum_l(-1)^l\frac{1}{l!(q_2+l)!(n-l)!(n_2-l)!(n_1-n+l)!(n+q_1-l)!}|n_1+q_1,n_1\rangle|n_2+q_2,n_2\rangle\ . \tag{B7}$$

This is the same as expression (2.31) (apart from normalization) derived in Sec. II using the CG coefficient in the form given in Appendix A. In fact, this method can actually be used to derive the CG coefficient of SU(1,1).

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