Time-dependent solution of the nonlinear Schrödinger equation for Bose-condensed trapped neutral atoms

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We present numerical results from solving the time-dependent nonlinear Schrödinger equation (NLSE) that describes an inhomogeneous, weakly interacting Bose-Einstein condensate in a small harmonic trap potential at zero temperature. With this method we are able to find solutions for the NLSE for ground-state condensate wave functions in one dimension or in three dimensions with spherical symmetry. These solutions corroborate previous ground-state results obtained from the solution of the time-independent NLSE. Furthermore, we can examine the time evolution of the macroscopic wave function even when the trap potential is changed on a time scale comparable to that of the condensate dynamics, a situation that can be easily achieved in magneto-optical traps. We show that there are stable solutions for atomic species with both positive and negative s-wave scattering lengths in one-dimensional (1D) and 3D systems for a fixed number of atoms. In both the 1D and 3D cases, these negative scattering length solutions have solitonlike properties. In 3D, however, these solutions are only stable for a modest range of nonlinearities. We analyze the prospects for diagnosing Bose-Einstein condensation in a trap using several experiments that exploit the time-dependent behavior of the condensate.

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I. INTRODUCTION

Recent experimental advances toward achieving Bose-Einstein condensation (BEC) in trapped atomic vapors [1], [2] provide continuing impetus for theoretical descriptions of the properties of such a condensate. In particular, diagnosis of BEC in a trap will depend on accurate knowledge of condensate behavior in the presence of various probes.

While the condensate ground-state wave function can be determined by solving the time-independent nonlinear Schrödinger equation (NLSE) [3], some behavior is best investigated using the time-dependent version. For example, a time-dependent description is particularly useful in the examination of condensate stability. By solving the NLSE, it is also straightforward to model the condensate response, even in the nonlinear regime, to an external perturbation of arbitrary strength and temporal profile. This response may include the nonlinear coupling between different modes of excitation of the condensate. The time-dependent condensate characteristics are often significantly different from those of an uncondensed atomic sample, and may thus provide a useful diagnostic of the presence of BEC. In this paper, we describe a method for solving the time-dependent NLSE for a Bose-Einstein condensate at zero temperature, and use it to explore several of the cases described above. This technique can be extended to finite temperatures, as will be described elsewhere.

In the past, most detailed studies have concentrated on homogeneous or quasihomogeneous condensates with effectively infinite spatial extents. However, critical differences arise in the response of an inhomogeneous condensate, for which the detailed effects of its finite size must be taken into consideration, and for which the effects of the external trap potential may have important consequences. One notable set of work into strongly inhomogeneous systems has involved liquid He adsorbed in Vycor glass [4]. The particular aspect of these studies that is most closely linked to our present work describes how individual pores in the Vycor are filled with He atoms. In this process, one has to determine the properties of the inhomogeneous assembly of atoms in a pore [5]. Of particular concern in this case is the evolution of the density of states available to the incoming atoms, especially near the superfluid phase transition. In this paper, rather than considering the spectrum of elementary excitations, we look directly at the behavior of the condensate in the trap in space and time. These issues are, however, closely linked.

A situation of considerable interest arises when the trapped atomic species has a negative s-wave scattering length a. In this case, which holds for a number of the alkali atoms [6], the nonlinear interaction between atoms is

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attractive. For 1D systems or spatially homogeneous 3D systems, the theory of the NLSE with $a < 0$ is well developed. In 1D, the attractive interaction can compensate exactly for the dispersion of a wave packet, leading to an integrable, and highly stable, soliton solution [7]. In 3D, for the homogeneous case, all solutions are predicted to be unstable [8]. For an inhomogeneous condensate, however, previous work into the $a < 0$ case has been limited. We find that in 3D the spatial localization provided by an external trap potential can stabilize the condensate against collapse, provided that the nonlinearity is relatively weak. This result supports previous suggestions by Hulet [9] and Clark [10] regarding the stabilizing effects of a positive external potential on a small trapped condensate. For the homogeneous case, Stoof [11] has shown how a negative scattering length actually changes the nature of the condensate order parameter. The extension of that analysis to an inhomogeneous gas has yet to be done. We do not, therefore, know the precise relevance of the 3D solutions we find for $a < 0$ to a phase transition at nonzero temperature. We shall, however, make some general comments on their potential importance in future experiments.

We also model several prototypical experiments that demonstrate a number of aspects of condensate behavior in space and time; these might be used to determine whether a Bose condensate has been produced. Specifically, in Secs. V–VII, we describe the evolution of a condensate that has been released from a trap, the behavior of the wave function when the trap potential is subject to sharp perturbations, and the response of a ground-state condensate when the scattering length is tuned using an external magnetic field.

II. NUMERICAL METHODS

At zero temperature, the condensate wave function $\psi(\mathbf{r},t)$ may be described by a self-consistent, mean field NLSE known as the Gross-Pitaevskii equation [8]. If a harmonic trap potential is included, this equation becomes

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + \tfrac{1}{2}m\omega_t^2 r^2\psi(\mathbf{r},t) + NU_0|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t), \quad (1)$$

where m is the atomic mass, ω_t is the (isotropic) angular trap frequency, and N is the number of atoms in the condensate. U_0 describes the interaction between atoms in the condensate and for very low temperatures has the form

$$U_0 = \frac{4\pi\hbar^2 a}{m}. \quad (2)$$

First, consider a one-dimensional condensate. In order to facilitate the numerical solution of the one-dimensional version of the NLSE, we scale the 1D version of Eq. (1) in terms of harmonic oscillator units,

$$\xi = \left(\frac{\hbar}{2m\omega_t}\right)^{-\frac{1}{2}} x, \quad \tau = \omega_t t. \quad (3)$$

The NLSE thus becomes

$$i\frac{\partial\psi(\xi,\tau)}{\partial\tau} = -\frac{\partial^2\psi(\xi,\tau)}{\partial\xi^2} + \tfrac{1}{4}\xi^2\psi(\xi,\tau) + N\frac{4\pi\hbar a}{\omega_t m}|\psi(\xi,\tau)|^2\psi(\xi,\tau). \quad (4)$$

Because N appears as part of the nonlinear potential, it is necessary to ensure that the wave function is properly normalized. Specifically, we require

$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1. \quad (5)$$

Clearly, this 1D model does not correspond in an obvious way to a specific situation in BEC, and so the normalization must be defined somewhat arbitrarily. However, we still include a brief discussion of some of the properties of Eq. (4) because it yields general insight into the important effect that dimensionality has on nonlinear dynamics and because of its close link with nonlinear optics [7], [12].

In 3D, for an isotropic trap potential, the ground-state solution must be spherically symmetric [13], so we need to consider only the radial part of the wave function, which we write in the form

$$\psi(r) = A\frac{\phi(r)}{r}, \quad (6)$$

where A is a constant used to ensure proper normalization. With this substitution and the transformation to harmonic oscillator units

$$\rho = \left(\frac{\hbar}{2m\omega_t}\right)^{-\frac{1}{2}} r, \quad \tau = \omega_t t, \quad (7)$$

we find that the 3D NLSE becomes an effectively 1D equation,

$$i\frac{\partial\phi(\rho,\tau)}{\partial\tau} = -\frac{\partial^2\phi(\rho,\tau)}{\partial\rho^2} + \tfrac{1}{4}\rho^2\phi(\rho,\tau) + 8A^2N\pi a\frac{|\phi(\rho,\tau)|^2}{\rho^2}\phi(\rho,\tau). \quad (8)$$

As before, we require the norm of $\psi(\mathbf{r})$ to be one, which means that

$$4\pi\left(\frac{\hbar}{2m\omega_t}\right)^{\frac{1}{2}} A^2\int_0^{\infty}|\phi(\rho,\tau)|^2 d\rho = 1. \quad (9)$$

We can take advantage of the time-dependent nature of Eqs. (4) and (8) to determine numerically their time-independent ground-state solutions. We begin with the analytic, normalized, ground-state solutions of Eqs. (4) and (8) in the absence of the nonlinear terms. We then apply the full NLSE to these solutions and propagate them through time using the semi-implicit Crank-Nicolson numerical method for diffusive, initial value, partial differential equations [14]. At each time step we can increase or decrease the value of the nonlinear constant ($C_{\rm nl}^{\rm 1D} = 4\pi N\hbar a/\omega_t m$, $C_{\rm nl}^{\rm 3D} = 8A^2N\pi a$) adiabati-

cally until reaching a desired value of $C_{\rm nl}$. The resulting solution is thus the ground state of the NLSE corresponding to that $C_{\rm nl}$. Physically, this technique corresponds to increasing or decreasing either the number of particles in the condensate or the scattering length. During our 3D calculations, we maintain a normalization of $\int_0^\infty |\phi(\rho)|^2 = 1$. This means that the values of the nonlinear constant that we quote are for $C_{\rm nl}^{\rm 3D} = 2Na(2m\omega_t/\hbar)^{\frac{1}{2}}$ in order to satisfy Eq. (9).

The presence of the nonlinear potential creates a limit to the numerical stability of this method. The Crank-Nicolson algorithm that we use is stable and unitary for any size of time step; that is, numerical errors will not grow with time [14]. However, as a result of the nonlinear coupling between different vibrational modes of the condensate (as described in the next section), the tiny perturbations in the wave function caused by random numerical noise, even at the double precision level, can mimic a real excitation of the condensate and thus the input of kinetic energy into the condensate. This effect limits our ability to find ground-state solutions at high values of $C_{\rm nl}$.

The time-dependent NLSE can also be used to calculate the energy per particle for the ground states found using the method above. While the probabilities $|\psi|^2$ for these ground states are stationary in time, the wave function amplitudes contain a time-dependent phase factor $\exp(-i\mu\tau)$, where μ is the desired energy eigenvalue in units of $\hbar\omega_t$. Now, note that at a given spatial point, the wave function variation over a time step $\delta\tau$ can be characterized by

$$\frac{\psi(\tau+\delta\tau)}{\psi(\tau)} = e^{-i\mu\,\delta\tau}. \tag{10}$$

Thus,

$$\begin{aligned}\mu &= -\frac{1}{\delta\tau}\arg\left(\frac{\psi(\tau+\delta\tau)}{\psi(\tau)}\right)\\ &= -\frac{1}{\delta\tau}\tan^{-1}\left(\frac{\mathrm{Im}\,\{\psi(\tau+\delta\tau)/\psi(\tau)\}}{\mathrm{Re}\,\{\psi(\tau+\delta\tau)/\psi(\tau)\}}\right),\end{aligned} \tag{11}$$

and so the energy is straightforward to determine from consecutive time steps of a numerical simulation. We usually calculate μ from an average over several hundred pairs of consecutive time steps to reduce the effect of any random numerical noise in the individual phases.

Once a ground-state wave function is known, it is possible to monitor its response to a radially symmetric, time-dependent perturbation in the external potential by solving the NLSE while changing the potential at each time step. We describe the results of such calculations in later sections.

Before continuing, however, we must address the criteria for the lifetime of the condensate in the face of high frequency excited modes.

III. CONDENSATE LIFETIME AND DAMPING

There are several issues to consider regarding the stability of the overall motion of the condensate. The first is the possibility of the coupling of the condensate motion to any uncondensed atoms that may be present. The second is the set of conditions for which excitation into very high energy modes leads to depletion of the condensate. A related, but nonphysical, issue concerns the stability of the numerical algorithm used. Because our time-dependent models allow us to fully monitor the condensate motion, we can ensure that our simulations satisfy the stability criteria set out in the remainder of this section.

First, let us briefly examine how coupling to atoms that are thermally excited, that is, not part of the condensate, might affect its motion. This effect will not be present here because we assume that the system is at zero temperature. It will, however, affect real experiments, for which the temperature may be only slightly below the critical condensation temperature. In that case, it is expected that thermal atoms will damp the motion of the condensate via Landau and collisional damping. In the homogeneous case, Landau damping is the dominant mechanism. In the case of a trapped gas, however, the relative importance of collisional damping will increase due to the lack of a continuum of excited modes, but should generally occur at a lower rate than for Landau damping [15]. The assumption that these damping rates are small compared to the frequency of the excited modes is equivalent to being outside the region of critical fluctuations, as shown by Eckern [16].

A more important consideration in the present case is the effect of the direct coupling of the condensate motion into high frequency modes due to the nonlinear interaction. Any perturbation in the potential causes a corresponding, time-dependent, change in the wave function. This change in the wave function in turn causes further variation in the nonlinear potential, and so on. As a result, the nonlinearity acts to couple the excited vibrational modes of the condensate to one another, which can eventually lead to the population of modes with a variety of momenta. (Note that in a finite sized, inhomogeneous condensate, there may exist components of different momenta. In this case, the important criterion for the existence of a condensate is a spatial phase coherence, *not* necessarily a macroscopic population in a single momentum state.) Because we evolve the solutions to the NLSE in time, we can observe directly the growth of such excitations on the condensate following a change in the external potential. The time evolution of the condensate motion for modes of modest frequency, and including the nonlinear coupling, is thus fully accounted for in our model.

It should be noted that vibrational modes with very high energy become essentially independent of the condensate, and for these the mean field approximation inherent in the NLSE becomes inappropriate. We identify three related criteria that an excitation must satisfy for condensate coherence to be maintained, and thus for the mean field approximation to remain valid. First, for an excited mode to be considered a part of the condensate, its kinetic energy, must not be significantly greater than the magnitude of the condensate self-energy $|NU_0|$. Furthermore, the wavelength of the excitation should not

be small relative to the coherence length of the mean field, $\hbar(mNU_0/V)^{-\frac{1}{2}}$, where V is a unit volume. Finally, the density of particles in a given region of the condensate must remain sufficiently large that a notion of phase coherence among the assembly of particles is sensible. Clearly these criteria are difficult to define in a strict, quantitative way. They are also closely related to the onset of coherence around the condensation point [17].

Another important but poorly understood subject is the precise physical mechanism of the nonlinear coupling between vibrational modes and the time scales associated with this process. We believe that this coupling will depend on the shapes of the excited modes and their relative energies as well as on the size of the nonlinearity, and will thus be very system dependent. A more detailed numerical study of the properties of the excited modes in the $a > 0$ case will be the subject of a future paper.

We believe that the considerations outlined in this section should not be of great concern in the work presented here. Because we can observe the development of excited modes, we can ensure that our simulations are such that they unambiguously satisfy the coherence criteria above. Furthermore, we find that the time required for numerical errors to manifest themselves as visible excitations of the condensate ground state is quite large. In fact, those times are on the same order as the expected lifetimes of real atomic condensates, which will likely decay as a result of three-body collisions [18]. Indeed, the only inherently unstable behavior we observe in this work is for solutions with negative scattering lengths and sufficiently large nonlinearities.

IV. GROUND-STATE SOLUTIONS

As a first demonstration of the method of Sec. II, we have determined ground-state wave functions in both 1D and 3D for a range of values of the nonlinear constant. These appear in Figs. 1(a) and 1(b). To check the accuracy of the 3D solutions, we have compared them with the results of a previous, time-independent, study of Eq. (8) [3]. The wave functions in Fig. 1(b), for $a > 0$, agree very well with those previous results. We have normalized all of the wave functions plotted in this paper to unity over the radial coordinate for ease of comparison.

As a test of the stability of these solutions, we have evolved them through time for $\tau > 70$, over which they showed no visible change. However, for $a > 0$ the nonlinear coupling between vibrational modes, as mentioned in the preceding section, limits our ability to propagate a ground-state wave function indefinitely through time without excitation. In fact, this effect prevents us from even finding pure ground-state solutions for higher values of the nonlinear constants than those shown in Fig. 1.

To place these results in the context of current experimental parameters, we must convert out of harmonic oscillator units. For Cs, $m = 2.21 \times 10^{-25}$ kg and $a \approx 3.45$ nm (for a collision of $F = 3, m_F = -3$ on $F = 3, m_F = -3$ [19], although a may be tuned using a magnetic field in this case [20]). A usual magnetic trap frequency is $\omega_t \approx 20\pi$ rad s^{-1} [1]. For these parameters, the harmonic oscillator unit of length is equivalent to 1.95 μm and a $C_{\rm nl}^{\rm 3D}$ of 100 corresponds to a condensate population of $N \approx 28\,000$.

Figures 2(a) and 2(b) show the variation of the ground-state energy per particle with $C_{\rm nl}$ for the 1D and 3D cases, as determined by the method outlined above. Again, we find good agreement with the results from a time-independent calculation.

We would expect the 1D condensate to be stable for all values of the $C_{\rm nl}$. We do, indeed, find stable solutions for both positive and negative values, with a smooth variation of μ over the range shown in Fig. 2(a). In 3D with $a > 0$, stable ground states are also to be expected, and Fig. 2(b) shows the variation of μ in that region. For $a < 0$, in contrast, disturbances with wave vectors k, such that

$$\frac{\hbar^2 k^2}{4m} < |NU_0|\,, \tag{12}$$

are all unstable [8]. In a homogeneous condensate, long wave length disturbances will grow and destroy the con-

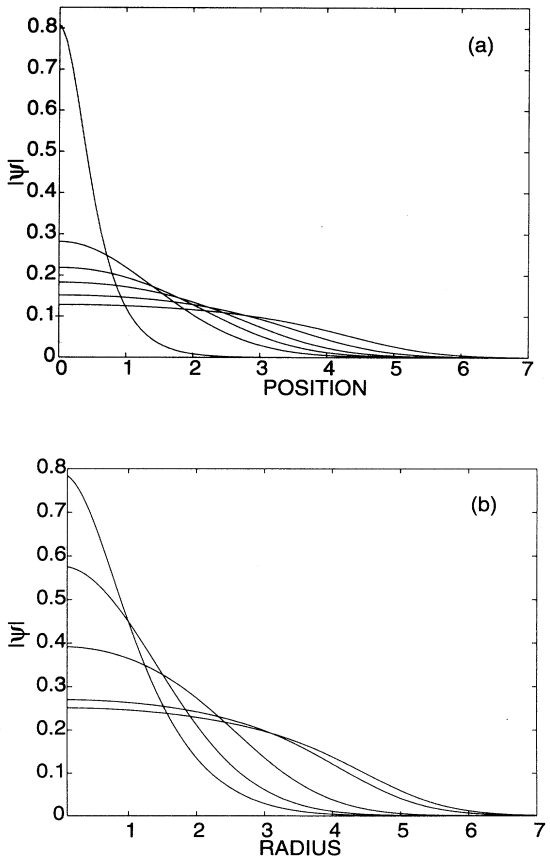

FIG. 1. (a) 1D condensate wave functions with $C_{\rm nl}^{\rm 1D}$ of -10, 0, 4.5, 10, 20, and 35 (in order of increasing width). (b) 3D condensate wave functions with $C_{\rm nl}^{\rm 3D}$ of -1.5, 0, 10, 70, and 100 (in order of increasing width).

densate. However, for a condensate that is localized in space, e.g., by a trap potential, low momentum components that satisfy Eq. (12) are not present, provided that the nonlinearity NU_0 is not too great. Indeed, Fig. 2(b) shows that values of μ can be found for relatively small nonlinearities, but that this energy has an asymptotic cutoff at $C_{\rm nl}^{\rm 3D} \approx -1.62$. Beyond this point, the trap potential can no longer compensate for the negative nonlinear potential and thus a stable condensate cannot occur. Nevertheless, the fact that stable ground-state solutions can exist at all for atoms with negative scattering lengths should be of interest, in light of the number of alkali species with states for which $a < 0$. Note that a small $C_{\rm nl}$ does not necessarily mean that the condensate contains only a small number of particles. Recent results have suggested that a can be tuned using an external magnetic field [20]; by tuning a to an arbitrarily small negative value, it would be possible to have a correspondingly large value for N while staying within the region of stability.

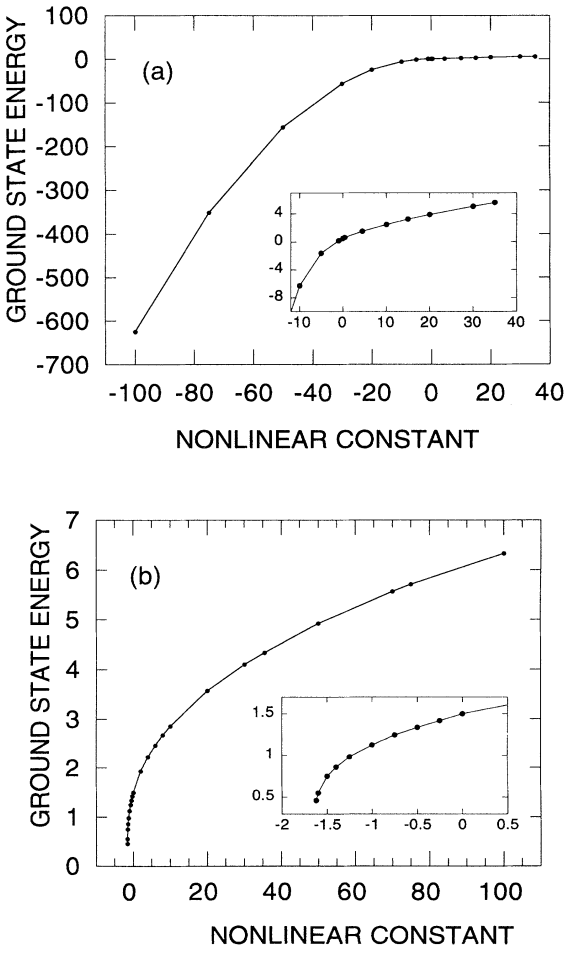

FIG. 2. Ground-state condensate energy per particle as a function of nonlinear constant for 1D (a) and 3D (b).

We emphasize, however, that it may be difficult to reach these $a < 0$ solutions experimentally via a normal condensation process [11]. Whether these ground states can be "grown" using other laser cooling processes is another issue that needs to be addressed. Such a growth process would, of course, need to stabilize the total number of atoms in the condensate. When $T \neq 0$, thermal atoms will be present and available for scattering into the condensate. Because the energy of the condensate decreases as atoms are added, N will diverge in this case, and a stable condensate cannot exist. It is possible that a stable situation might occur if the increase in particle number due to condensation were balanced by the enhanced collisional loss that is predicted to occur in trapped condensates [18].

V. EVOLUTION IN FREE SPACE

In this and the next two sections, we consider three relatively simple computational "experiments" to demonstrate several important aspects of condensate dynamics.

We first consider an atomic sample in a trap. If the trap potential is turned off, the sample will fall freely through space and may be imaged at a later time using any of a variety of standard techniques [21].

We begin with 3D ground-state solutions as determined in the preceding section for three cases: $a > 0$, $a = 0$ (uncondensed), and $a < 0$. We then turn off the trap potential and follow the subsequent condensate free evolution. This wave packet development appears in Figs. 3(a)–(c).

These plots display several instructive features. First, the expansion of the wave function with time is quite uniform between the three cases. This effect shows that straightforward wave packet dispersion is the dominant influence on the spatial evolution of the wave function. The repulsive or attractive interaction in the nonlinear potential has a much smaller effect. Since the ratio of the radii of the $a > 0$ and uncondensed wave functions decreases toward unity quickly with time, it seems that such simple time-of-flight measurements do not provide a particularly good diagnostic of BEC.

These results also help to clarify a somewhat counterintuitive aspect of the $a < 0$ solutions, namely that a confining trap potential can stabilize the condensate against a collapse brought on by attractive atomic interactions. The fact that the $a < 0$ wave function expands so quickly confirms that confinement in the trap had simply excluded unstable long wave length components, as required by Eq. (12). As the trap disappears, the kinetic energy $\hbar^2 k^2/2m$ due to confinement is sufficient to keep the wave packet expanding with little impedance from the attractive interaction NU_0.

It is interesting to compare the above $a < 0$ result with the behavior of the 1D case for the same scenario. Equation (4) can have stable solutions even in the absence of a trap potential; these solutions may be solitons, for which the attractive interaction exactly compensates for the wave packet dispersion. As a result, the spatial form of the wave function does not change in time. Figure 4

shows the time evolution of a solution with $C_{\rm nl}^{\rm 3D} = -10.0$ over $\tau = 28.0$. Between $\tau = 0$ and $\tau = 4$, the trap potential is smoothly turned off. The wave function does not change much during this time because due to its very narrow form, it is not strongly influenced by the trap. After $\tau = 4$, the solution is allowed to evolve in free space, during which time it does not change shape; such behavior is one criterion for the existence of a soliton.

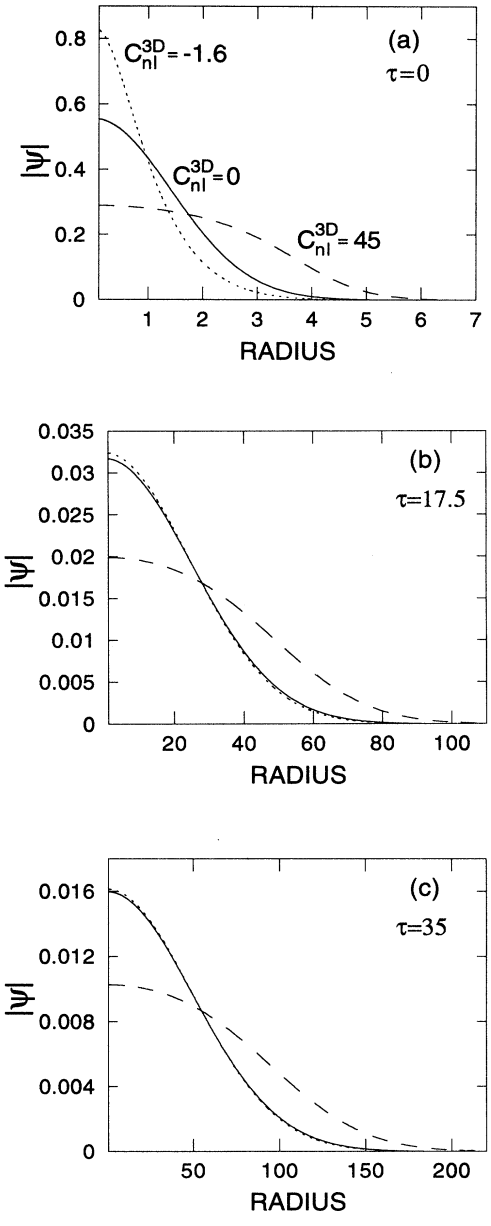

FIG. 3. Evolution of 3D wave functions in free space. Wave functions with nonlinear constants of -1.6, 0, and 45 are shown with dotted, solid, and dashed lines, respectively. These wave functions at $\tau = 0$, $\tau = 17.5$, and $\tau = 35$ appear in (a), (b), and (c).

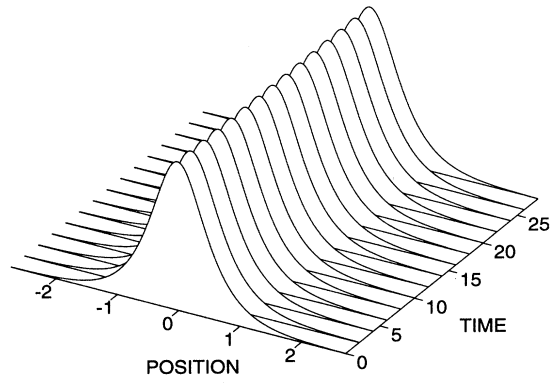

FIG. 4. Evolution through time of a 1D soliton solution as the trap potential is turned off over $0 \leq \tau \leq 4$ and subsequently in free space. Note that the wave function exhibits no variation in form during the free space evolution.

VI. RESPONSE TO EXTERNAL PERTURBATIONS

We now consider a situation in which the condensate is subjected to radial perturbations while still confined by the trap potential. Most analytical methods rely on linear response, which limits investigation to weak per-

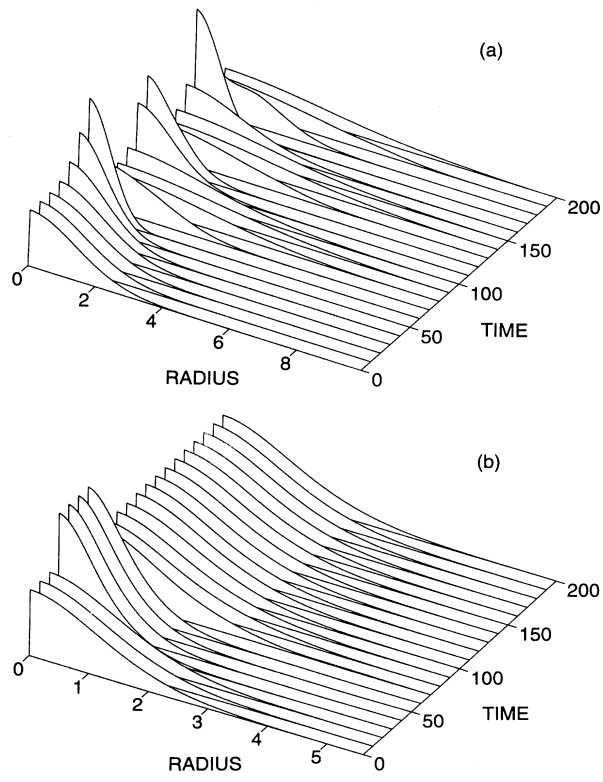

FIG. 5. Response of 3D wave functions to perturbations in the trap potential. Between $\tau = 20$ and $\tau = 60$, the trap potential term is multiplied by 5; at other times it has the constant default value. (a) $C_{\rm nl}^{\rm 3D} = 0$. (b) $C_{\rm nl}^{\rm 3D} = -1.0$, showing solitonlike behavior.

turbations. By solving the time-dependent NLSE numerically, however, we can model the condensate response to perturbations of virtually any strength or time dependence. We should recall that the treatment here is strictly limited to $T = 0$. The effect of uncondensed thermal atoms on the condensate response should be to produce a modest damping of the collective motion [15].

As an example, we follow a 3D condensate response as the shape of the trap potential changes abruptly. Figures 5(a) and (b) show the time evolution of an uncondensed wave function ($a = 0$) and that of a condensate for which $a = -1.0$. At $\tau = 20$, the trap potential term is multiplied by 5, and at $\tau = 60$ it is returned to its original value. (The time taken for the potential to change is $\delta\tau \approx 0.07$.) As expected, this perturbation excites a few vibrational modes in the uncondensed sample, causing it to ripple back and forth within the trap.

However, any similar excitation is damped out very quickly in the $a < 0$ condensate; it settles almost immediately into steady-state solutions after both trap changes. [The short-lived rippling that does occur is not visible on the time scale of Fig. 5(b).] This striking result is analogous to the solitonlike behavior of the 1D wave function in the previous section. However, in the 3D case, the trap potential must always be present to ensure that a solution can exist. We believe that the high degree of stability of condensates with negative scattering lengths may make them attractive candidates for study in systems for which N can be stabilized against divergence, such as the atom laser.

VII. VARYING SCATTERING LENGTH

As a final example, we simulate the condensate response when the scattering length is tuned adiabatically over some range, as is possible by applying an external magnetic field. In Fig. 6, we show the shape of the ground-state 3D wave function over a range of values of the nonlinear constant, which is equivalent to a scan of a. The resulting change in the size of the condensate can be substantial, especially for small and negative values of a; compare also Fig. 1(b). Such a variation in the condensate radius could be an effective and straightforward diagnostic of the presence of BEC, since no change would be expected in the absence of a condensate. Furthermore, by tuning a to a sufficiently negative value, it should be possible to force the condensate into the region of instability, thus inducing the macroscopic wave function to disappear suddenly and entirely.

VIII. CONCLUSION

In this paper, we have outlined a method for solving the time-dependent NLSE that describes Bose-Einstein condensed neutral atoms in a small trap. With it, we have found solutions corresponding to ground-state condensates in 1D and 3D for atoms with both positive and negative scattering lengths. Furthermore, we have shown that the condensate time-dependent response to a variety of perturbations can be modeled, and we have used several computational experiments that take advantage of condensate behavior in time to demonstrate some of its interesting properties.

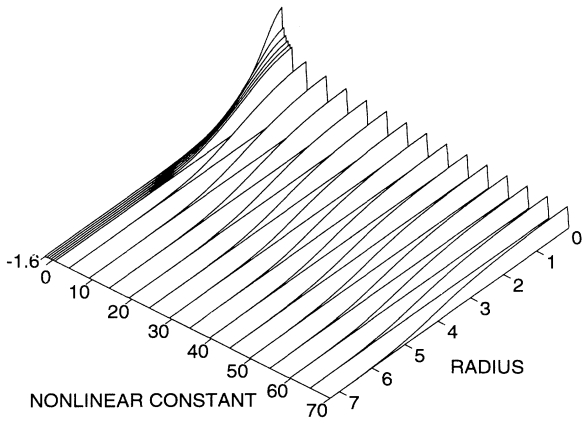

FIG. 6. Variation in the shape of a 3D wave function as the nonlinear constant is scanned over a wide range, which is equivalent to the scattering length being tuned with an external magnetic field.

A number of further areas can be explored by solving the time-dependent NLSE as well. The spectrum of vibrational modes within an inhomogeneous condensate is straightforward to determine, and will be the subject of a forthcoming paper. This technique could be extended to include the effects of the uncondensed atoms that would be present at nonzero temperatures, and to study various other trap geometries, possibly with nonsymmetric perturbations. One particularly interesting configuration would be that of an atom laser, in which a condensate contained in a suitable cavity could undergo stimulated amplification of its particle number, while outputting a coherent atomic beam. In this case, if the atomic species used had a negative scattering length, it might even be possible to produce an "atomic soliton laser."

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