

## Master equation for sympathetic cooling of trapped particles

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A model for cooling a system of bosons in a harmonic trap via their interactions with a thermal bath of other particles is studied. The master equation describing the evolution of the system is derived for an arbitrary number of spatial dimensions. This equation is characterized by transition rates between trap levels. We present an analytic approximation for these rates and compare it with exact formulas, derived for the case of an even number of spatial dimensions. Analytic expressions show very good agreement with the exact ones for a wide range of parameters. We also discuss the cooling dynamics in terms of the approximated rates.

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### I. INTRODUCTION

Much of the work of atomic physicists has recently been devoted to observing effects related to the quantum statistical properties of weakly interacting gases of atoms [1]. Apart from the fundamental interest in such phenomena, quantum statistical effects might lead to intriguing applications such as the development of coherent sources of atoms. It is clear that to observe quantum statistical effects it is necessary to reach very low temperatures and high atomic densities. Thus part of both theoretical and experimental research in atomic physics is now focused on cooling mechanisms for atoms confined in traps at high densities [2].

One of the well-established cooling procedures is laser cooling [3–5]. In laser cooling, atoms absorb energy from a laser beam and then emit spontaneously in such a way that the energy balance in each absorption-emission process is negative. With some of the laser cooling mechanisms proposed so far, such as velocity selective population trapping [6] or Raman cooling [7], atoms can be cooled down to temperatures (or energies) below the photon recoil  $E_R [= (\hbar k)^2/(2M)]$ , where  $k$  is the laser wave vector and  $M$  the atomic mass. However, such schemes have been designed for atoms in the free space (and not in a trap) and are limited to low atomic densities [8], where atom-atom interactions are unimportant. An alternative procedure that is not restricted by those limitations is evaporative cooling [9,10]. In evaporative cooling, the most energetic atoms are removed from the trap. With this technique, although the number of atoms in the trap decreases, the temperature is drastically reduced so that the phase space density increases. Here, as opposed to laser cooling, atom-atom collisions are necessary ingredients for thermalization.

Another route to cool samples of particles is sympathetic cooling [11–14]. With this technique, a gas of particles ( $A$ ) is cooled via its interactions with another gas

( $B$ ), which is already at low temperature. Typically, one can assume that either the number of particles in  $B$  is very large and/or that the particles are kept cold by another mechanism (such as laser cooling or evaporation cooling). Then  $B$  can be regarded as a thermal bath and therefore the final temperature of  $A$  will be very close to the original one of  $B$ . Here, as in the case of evaporative cooling, the required thermalization occurs due to particle-particle collisions.

Sympathetic cooling was proposed in the late 1970s and initial applications of this method concerned cooling of charged particles, such as different isotopes of magnesium ions [11,12], or cooling of  $\text{Hg}^+$  ions via interactions with laser cooled  $\text{Be}^+$  ions. These pioneering experiments have had a direct impact on experiments involving storage and cooling of antiparticles [15]. In particular, cooling of antiprotons in an ion trap via interactions with electrons have been proposed [16] and demonstrated by Gabrielse and collaborators [17]. The same group recently proposed positron cooling of ions [18]. Sympathetic cooling of molecules interacting with a buffer gas of  $^3\text{He}$  in a cell has also been proposed [19]. In that case the  $^3\text{He}$  atoms were in thermal equilibrium with the walls of the cell. The molecules were cooled through collisions with the  $^3\text{He}$  atoms and loaded into a trap.

To our knowledge, the idea of sympathetic cooling of neutral particles, and in particular atoms, has not been exploited in the literature. In this paper we discuss such possibility concentrating on the following physical situation: the gas of alkali atoms  $B$  is confined in a large and rather loose trap, such as a magneto-optical trap (MOT) (cf. [5]). Typically, for alkali atoms such traps have frequencies of the order of 10–100 Hz and sizes of few micrometers [20]. The gas  $B$  is cooled by some mechanism (e.g., laser cooling or evaporation cooling) to a temperature  $T_B$ . The temperature  $T_B$  might still be relatively high for the  $B$  atoms, which are additionally assumed to be relatively heavy. The gas  $A$  is composed of other

alkali atoms, that are assumed to be stored in a tight trap such as a far-off resonance dipole trap (FORT) [21]. Tight traps may have frequencies in the range of few kilohertz and sizes of  $\simeq 0.1 \mu\text{m}$  [22].  $A$  atoms have smaller mass, but not necessarily much smaller than that of  $B$  atoms. Of course, in principle, the same physical forces can be used to store both atomic species. In the latter case the difference in trap frequencies would also result from the differences of atomic masses, dipole moments, etc. We think, however, that the idea of using a FORT inside a MOT is interesting itself, since it leads to the extreme frequency and size differences and moreover can be also used in the context of evaporation cooling of a single atomic species.

Although the temperatures reached with sympathetic cooling cannot be too low (i.e., lower than the temperature of  $B$  atoms), the main advantage of this method consists in its passive character. For this reason it may be employed to cool  $A$  atoms even when all other procedures fail. Second, even though the temperature of the gas  $B$  might be relatively high, it might at the same time be sufficiently low for the gas  $A$ , so that the  $A$  atoms at this temperature might already exhibit quantum statistical effects. The reason for that is that in general  $A$  particles are characterized by a mass different (and charge, in the case of ionic systems) from that of the  $B$  particles. This appealing property of the sympathetic cooling can be illustrated when one considers the conditions for the appearance of Bose-Einstein condensation (BEC). As is well known, BEC in three dimensions occurs when  $n\Lambda^3 \gtrsim 2.612$ , where  $n$  is the atomic density and  $\Lambda = (2\pi\hbar^2/M\kappa T)^{1/2}$  is the de Broglie wavelength [23]. In the physical situation that we consider, for a fixed density (temperature), the lighter  $A$  atoms reach the critical point at a higher temperature (at lower density) than the heavier  $B$  atoms.

The quantum dynamics of many-body cooling is a very complex problem in general. A master equation describing the dynamics of a small sample of the laser cooled atoms in a microtrap has been proposed and analyzed recently [24]. The quantum statistical nature of the atoms is reflected in the dynamics of the cooling process. For evaporative cooling, the quantum kinetic equations describe the evolution at least in the first stages of the cooling process before the Bose-Einstein condensation takes place [25,26]. A master equation for evaporation cooling can also be derived under certain circumstances [27]. This master equation has the advantage that it permits one to study atom number fluctuations in each of the trap levels and thus provides a more complete description of the cooling process. In particular, it can, in principle, be used to describe the dynamics of condensate formation.

In this paper we present and analyze a simple model describing the quantum dynamics of sympathetic cooling. We consider a gas of particles  $A$  trapped in a harmonic potential and interacting with other particles  $B$  that can be regarded as a bath at a given temperature. The interaction between the particles are due to atom-atom collisions, which we model using the standard shape-independent potential approximation [28,29]. We use methods borrowed from quantum optics to de-

rive a master equation for the reduced density operator of the system  $A$ . The master equation describes cooling through transitions between different trap levels. The rates at which these transitions occur depend on the specific properties of the atomic collisions, as well as on the characteristics of the trap and the temperature of the atoms of the bath. In principle, they contain all the information concerning the cooling process. It is the main goal of this paper to analyze these rates and to present accurate analytic formulas for them. The results and the techniques developed can be generalized to study other problems, such as the above mentioned problem of evaporation cooling of atoms in a loose MOT with a tight FORT in the center.

This paper is organized as follows. In Sec. II we introduce the model, describing separately the atomic bath, the system, and the system-bath interactions. In Sec. III we derive a master equation for sympathetic cooling, under Born and Markov approximations. Exact formulas for the transition rates between different levels are given in Sec. IV and are compared to analytic approximations using the saddle point method. In Sec. V we discuss qualitatively the process of sympathetic cooling in terms of the master equation derived in Sec. III. In Sec. VI we summarize the results.

## II. MODEL

We consider a system of particles  $A$  that are confined in a trap and interact with other particles  $B$  in a finite region of space. We assume that the particles  $B$  are practically unaffected by their interactions with the system  $A$ , so that  $B$  can be regarded as a reservoir for  $A$ . We assume that the reservoir is in thermal equilibrium at some given temperature. The thermalization in the bath  $B$  occurs due to some external cooling mechanism (e.g., laser cooling or evaporation cooling) and as a result of collisions between the  $B$  particles. In this section we introduce the Hamiltonian for the bath  $B$ , the system  $A$ , and their mutual interactions. The formalism will be developed for the case of  $d$  spatial dimensions.

### A. Bath

We consider that the system  $B$  contains a practically infinite number of bosons of mass  $M_B$  embedded in a practically infinite volume, with finite density  $n_B$ . The free Hamiltonian for the bath  $B$  of particles in a second quantized form is

$$H_B = \int d\mathbf{k} \epsilon(\mathbf{k}) b(\mathbf{k})^\dagger b(\mathbf{k}). \quad (1)$$

Here  $\mathbf{k}$  is a wave vector in a  $d$ -dimensional space,

$$\epsilon(\mathbf{k}) = \frac{(\hbar\mathbf{k})^2}{2M} \quad (2)$$

is the corresponding energy, and  $b(\mathbf{k})^\dagger$  and  $b(\mathbf{k})$  are cre-

ation and annihilation operators of bath particles in the plane wave states with momentum  $\mathbf{k}$ . They fulfill the usual commutation relations

$$\begin{aligned} [b(\mathbf{k}), b(\mathbf{k}')] &= [b(\mathbf{k})^\dagger, b(\mathbf{k}')^\dagger] = 0, \\ [b(\mathbf{k}), b(\mathbf{k}')^\dagger] &= \delta^{(d)}(\mathbf{k} - \mathbf{k}'). \end{aligned}$$

One could easily extend this model to the case of a bath of fermions by simply introducing anticommutation relations between creation and annihilation operators. However, we will be interested in a situation in which the bath is far from the range of temperatures and densities at which quantum statistical properties are important. For this reason the results of the paper will not depend on the bosonic or fermionic character of the bath. Note that, formally, Eq. (1) does not take into account collisions between the bath particles. At high temperatures and low densities, however, the major role of those collisions is to thermalize, i.e., to keep the temperature of the bath constant. It is therefore legitimate to use the Hamiltonian (1) and assume appropriate thermal distribution of the  $B$  particles. Alternatively, one could view Eq. (1) as a Hamiltonian for quasiparticles that describes effective excitations of the system  $B$  with the collisions between the bare particles accounted for. Quasiparticle creation and annihilation operators can be constructed using a self-consistent Bogoliubov-Hartree method [30]. In such a case, the dispersion relation (2) would, in general, be modified, but again the modifications are negligible in the high-temperature low-density limit considered here.

In thermal equilibrium, the density operator describing the state of the bath  $\rho_B$  corresponds to the usual Bose-Einstein distribution (BED). In this situation, we have

$$\langle b(\mathbf{k})b(\mathbf{k}') \rangle = \langle b(\mathbf{k})^\dagger b(\mathbf{k}')^\dagger \rangle = 0, \quad (3a)$$

$$\langle b(\mathbf{k})^\dagger b(\mathbf{k}') \rangle = n(\mathbf{k})\delta^{(d)}(\mathbf{k} - \mathbf{k}'), \quad (3b)$$

where  $n(\mathbf{k})$  defines the number of particles with wave vector  $\mathbf{k}$  and is given by

$$n(\mathbf{k}) = \frac{ze^{-\beta_B \epsilon(\mathbf{k})}}{1 - ze^{-\beta_B \epsilon(\mathbf{k})}}. \quad (4)$$

In the above expression,  $\beta_B = 1/(\kappa T_B)$  is the inverse temperature and  $z$  is the fugacity. Note that both  $n(\mathbf{k})$  and  $\epsilon(\mathbf{k})$  depend only on  $k \equiv |\mathbf{k}|$ . Particle and energy densities are connected to these quantities by the relations

$$n_B = \frac{1}{(2\pi)^d} \int d\mathbf{k} n(\mathbf{k}), \quad (5a)$$

$$\epsilon_B = \frac{1}{(2\pi)^d} \int d\mathbf{k} n(\mathbf{k}) \epsilon(\mathbf{k}), \quad (5b)$$

respectively.

Here we will be interested in the case in which the bath is at relatively high temperature so that its statistical properties do not play an important role. In that case, the BED can be replaced by a Boltzmann distribution, for which

$$n(\mathbf{k}) = n_B \Lambda_B^d e^{-\beta \epsilon(\mathbf{k})}, \quad (6)$$

where  $\Lambda_B = (2\pi\hbar^2\beta_B/M_B)^{1/2}$  is the thermal de Broglie wavelength for the bath particles. Note that the temperature  $T_B$ , although high for the  $B$  atoms, might still be quite low for the  $A$  particles.

## B. System

The system  $A$  consists of  $N$  particles (bosons) of mass  $M_A$  confined by a harmonic potential in  $d$  dimensions. In a second quantized form, the Hamiltonian describing the system can be written as the sum of two contributions. The first one is the free Hamiltonian, which in the Fock representation takes the form

$$H_A = \sum_{\mathbf{n}} \hbar\nu(n_x + n_y + \dots) a_{\mathbf{n}}^\dagger a_{\mathbf{n}}, \quad (7)$$

where  $\nu$  is the trap frequency,  $\mathbf{n} = (n_x, n_y, \dots)$  with  $n_x, n_y, \dots = 0, 1, 2, \dots$ , and  $a_{\mathbf{n}}^\dagger$  and  $a_{\mathbf{n}}$  are creation and annihilation operators of particles in the  $\mathbf{n}$ th level of the harmonic potential, respectively. For the sake of clarity, we have written Hamiltonian  $H_A$  (7) for the simplest case of an isotropic harmonic trap. With slight modifications, the theory presented here can be easily extended to non-isotropic and nonharmonic traps (see Sec. VB below). The second contribution to the total Hamiltonian  $H_{A-A}$  is due to the atom-atom interactions and its role will be specified later.

## C. Interactions

The interactions between the particles belonging to the system and those of the bath are due to atom-atom collisions. For sufficiently low temperatures, we can use the *shape-independent approximation* [28,29] to write down the corresponding Hamiltonian. Physically, this approximation means that the wave functions of both kinds of particles do not change significantly over distances characterizing the interparticle potential for low-energy collisions. The interparticle potential can be then safely replaced by a zero-range Dirac  $\delta$  potential. In the Fock representation the interaction Hamiltonian is thus given by

$$H_{A-B} = \sum_{\mathbf{n}, \mathbf{n}'} \int d\mathbf{k} d\mathbf{k}' \gamma_{\mathbf{n}, \mathbf{n}'}(\mathbf{k}, \mathbf{k}') a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} b^\dagger(\mathbf{k}) b(\mathbf{k}'), \quad (8)$$

where

$$\gamma_{\mathbf{n}, \mathbf{n}'}(\mathbf{k}, \mathbf{k}') = \frac{C}{(2\pi)^d} \int d\mathbf{x} \psi_{\mathbf{n}}(\mathbf{x})^* \psi_{\mathbf{n}'}(\mathbf{x}) e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}, \quad (9)$$

$\psi_{\mathbf{n}}(\mathbf{x})$  is the wave function corresponding to the  $\mathbf{n}$ th level of the harmonic oscillator, and  $C$  is a constant depending on the  $d$ -dimensional scattering length  $a_{sc}$ . For example, in three dimensions

$$C = \frac{4\pi\hbar^2 a_{sc}}{2\mu}, \quad (10)$$

with  $\mu$  the reduced mass.

Without loss of generality, we can exclude from the integration over  $\mathbf{k}$  and  $\mathbf{k}'$  in (8) the values for which  $\mathbf{k} = \mathbf{k}'$ . This is clear since

$$\sum_{\mathbf{n}, \mathbf{n}'} \int d\mathbf{k} \gamma_{\mathbf{n}, \mathbf{n}'}(\mathbf{k}, \mathbf{k}) a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} b^\dagger(\mathbf{k}) b(\mathbf{k})$$

$$\propto \left( \sum_{\mathbf{n}} a_{\mathbf{n}}^\dagger a_{\mathbf{n}} \right) \left( \int d\mathbf{k} b^\dagger(\mathbf{k}) b(\mathbf{k}) \right) \quad (11)$$

is a constant operator, proportional to the product of the number of particles in the system  $A$  and the bath  $B$ ; therefore we can always remove it from the Hamiltonian. Hence, in the integrals including  $\mathbf{k}$  and  $\mathbf{k}'$  it will be implicitly assumed that  $\mathbf{k} \neq \mathbf{k}'$ . On the other hand, we anticipate that it will be useful to make a rotating-wave approximation (RWA) in the master equation derived from the Hamiltonian (8). To this aim, it is convenient to rewrite Eq. (8) in the form

$$H_{A-B} = H_0 + \sum_{\alpha=1}^{\infty} (H_\alpha + H_\alpha^\dagger). \quad (12)$$

Here  $H_0$  contains the part of  $H_{A-B}$  given in (8) in which the sum is extended over values with  $\sum_{l=x,y,\dots} (n_l - n'_l) = 0$ .  $H_\alpha$  contains the part of  $H_{A-B}$  with the sum extended over the values for which  $\sum_{l=x,y,\dots} (n_l - n'_l) = \alpha$  and the integral is extended to  $|\mathbf{k}'| > |\mathbf{k}|$ . We wish to point out that in the master equation derived in the next section, an additional RWA will be made at the end of derivation. Due to this RWA the Hamiltonian part of the master equation (that describes shifts of the energy levels) will not be correct in general [31,32]. However, in this paper we will be interested in only the decaying part of the master equation, which is correctly described under the mentioned RWA, provided the trap frequency  $\nu$  is larger than the cooling rates. The latter assumption will be made throughout the present paper.

### III. MASTER EQUATION

The Schrödinger equation describing the evolution of  $A$  and  $B$  does not give much information about the cooling process. However, when  $B$  can be regarded as a bath, one can derive a master equation for the system  $A$  only. This gives clear physical insight into the problem and simplifies it enormously. In this section we present a derivation of such a master equation, following well-established procedures in the field of quantum optics [31,32].

We first move to an interaction picture defined by the unitary operator  $\exp[-i(H_A + H_B)t]$ . In this picture, the density operator  $\tilde{\rho}$  describing system-plus-bath degrees of freedom fulfills

$$\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar} [\tilde{H}_{A-B}(t), \tilde{\rho}(t)], \quad (13)$$

where the tilde indicates that the operator is expressed in

the interaction picture. Integrating formally this equation and substituting it back into (13), we obtain

$$\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar} [H_{A-B}(0), \rho(0)]$$

$$- \frac{1}{\hbar^2} \int_0^t d\tau [\tilde{H}_{A-B}(t), [\tilde{H}_{A-B}(t-\tau), \tilde{\rho}(t-\tau)]]]. \quad (14)$$

Now we define the reduced density operator for system  $A$  only,  $\rho_A = \text{Tr}_B(\rho)$ , where  $\text{Tr}_B$  stands for the trace over the bath states. It can be easily shown that in the interaction picture it satisfies

$$\frac{d\tilde{\rho}_A(t)}{dt} = -\frac{1}{\hbar^2} \int_0^t d\tau \text{Tr}_B \{ [\tilde{H}_{A-B}(t),$$

$$\times [\tilde{H}_{A-B}(t-\tau), \tilde{\rho}(t-\tau)]] \}. \quad (15)$$

In the above expression we have made use of the fact that  $\text{Tr}_B \{ [H_{A-B}(0), \rho(0)] \} = 0$ , since the density operator for the bath  $\rho_B(0)$  is diagonal in the Fock basis (with respect to  $H_B$ ), whereas  $H_{A-B}$  does not contain any diagonal matrix elements [the reader should recall that we have extracted the terms with  $\mathbf{k} = \mathbf{k}'$  in  $H_{A-B}$ ; see Eq. (11)].

Next we perform Born and Markov approximations. The latter is related to the fact that the bath atoms are practically unaffected by their interactions with the system; this allows us to write  $\tilde{\rho}(t-\tau) = \tilde{\rho}_A(t-\tau) \otimes \rho_B(0)$ . For the Born approximation we have to assume that the correlation time  $\tau_c$  of the system-bath interaction is much shorter than the typical time over which  $\tilde{\rho}_A(t)$  changes, i.e., the cooling time [32]. From the technical point of view, the correlation time  $\tau_c$  can be defined as a time for which the integrand of (15) practically vanishes. It can be shown that

$$\nu \tau_c \simeq \frac{M_B}{M_A} \min \left[ 1, \sqrt{\beta_B \hbar \nu M_A / M_B} \right]. \quad (16)$$

The cooling time, on the other hand, depends on the collision processes (scattering lengths, densities, etc.) between atoms and it is assumed to be the longest time scale of the problem. Since for two alkali-atom species the mass ratios cannot be too high, Eq. (16) indicates that the sufficient condition for the Born approximation to hold is that the trap frequency should be much larger than the cooling rate [33]. In this case, we can safely substitute  $\tilde{\rho}_A(t-\tau)$  by  $\tilde{\rho}_A(t)$  in the integral (15) and extend the upper limit of the integral to infinity.

Let us now make use of Eq. (12) to obtain

$$\tilde{V}_{A-B}(t) = \int_0^\infty d\tau \tilde{H}_{A-B}(t-\tau)$$

$$\equiv \tilde{V}_0(t) + \sum_{\alpha=1}^{\infty} [\tilde{V}_\alpha(t) + \tilde{V}_\alpha(t)^\dagger]. \quad (17)$$

Substituting this expression into Eq. (15) and using Eq. (12) again, we obtain

$$\begin{aligned}
\frac{d\tilde{\rho}_A(t)}{dt} = & -\frac{1}{\hbar^2} \text{Tr}_B \{ \tilde{H}_0(t) \tilde{V}_0(t) \tilde{\rho}_A(t) \rho_B(0) - \tilde{V}_0(t) \tilde{\rho}_A(t) \rho_B(0) \tilde{H}_0(t) + \text{H.c.} \} \\
& -\frac{1}{\hbar^2} \sum_{\alpha=1}^{\infty} \text{Tr}_B \{ \tilde{H}_\alpha(t) \tilde{V}_\alpha(t)^\dagger \tilde{\rho}_A(t) \rho_B(0) - \tilde{V}_\alpha(t)^\dagger \tilde{\rho}_A(t) \rho_B(0) \tilde{H}_\alpha(t) + \text{H.c.} \} \\
& -\frac{1}{\hbar^2} \sum_{\alpha=1}^{\infty} \text{Tr}_B \{ \tilde{H}_\alpha(t)^\dagger \tilde{V}_\alpha(t) \tilde{\rho}_A(t) \rho_B(0) - \tilde{V}_\alpha(t) \tilde{\rho}_A(t) \rho_B(0) \tilde{H}_\alpha(t)^\dagger + \text{H.c.} \}. \tag{18}
\end{aligned}$$

To get this expression, we perform an additional RWA, i.e., neglect terms rotating at multiples of the trap frequency. Again, this approximation is based on large trap frequencies compared to the cooling rates.

Note that in the derivation of Eq. (18) we have excluded from the very beginning the interactions between system particles. However, when the trap frequency is the largest frequency in the problem one can make the *independent rates approximation* [32] and simply introduce directly the Hamiltonian  $H_{A-A}$  in the final master equation.

Finally, performing the trace in Eq. (18), taking into account the bath properties (3), and returning to the Schrödinger picture we obtain the master equation

$$\frac{d\rho_A}{dt} = -\frac{i}{\hbar} [H_A + H'_{A-A}, \rho_A] + \mathcal{L}\rho_A, \tag{19}$$

where  $H'_{A-A}$  is the original Hamiltonian  $H_{A-A}$  plus the terms corresponding to shifts produced by the elimination of the bath in the master equation. In the following we omit those terms, but they can be easily derived from Eq. (18). Physically, they account for the energy level shifts due to the effective interaction between system particles via their collisions with bath particles. The Liouvillian  $\mathcal{L}$  describes the cooling process and is given by

$$\mathcal{L} = \sum_{\alpha=-\infty}^{\infty} \mathcal{L}_\alpha, \tag{20}$$

where

$$\begin{aligned}
\mathcal{L}_\alpha \rho_A = & \sum_{\mathbf{n}, \mathbf{n}', \mathbf{m}, \mathbf{m}'} \Gamma_{\mathbf{n}, \mathbf{n}'}^{\mathbf{m}, \mathbf{m}'} (2a_{\mathbf{m}}^\dagger a_{\mathbf{m}'} \rho_A a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} \\
& - a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} a_{\mathbf{m}}^\dagger a_{\mathbf{m}'} \rho_A - \rho_A a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} a_{\mathbf{m}}^\dagger a_{\mathbf{m}'}). \tag{21}
\end{aligned}$$

The sum in this expression is extended to  $\mathbf{n}, \mathbf{n}', \mathbf{m}, \mathbf{m}'$ , fulfilling

$$\sum_{l=x, y, \dots} (n_l - n'_l) = \alpha, \quad \sum_{l=x, y, \dots} (m_l - m'_l) = -\alpha. \tag{22}$$

The Liouvillian (21) accounts for transitions of particles from one level of the harmonic oscillator to another, experiencing a change in the energy of  $\alpha\hbar\nu$ . Thus the term with  $\alpha = 0$  conserves the energy, whereas the terms with  $\alpha > 0$  ( $\alpha < 0$ ) describe processes increasing (decreasing) the energy. These transitions are characterized by

$$\begin{aligned}
\Gamma_{\mathbf{n}, \mathbf{n}'}^{\mathbf{m}, \mathbf{m}'} = & \frac{\pi}{\hbar} \int d\mathbf{k} d\mathbf{k}' \gamma_{\mathbf{n}, \mathbf{n}'}(\mathbf{k}, \mathbf{k}') \gamma_{\mathbf{m}, \mathbf{m}'}(\mathbf{k}', \mathbf{k}) \\
& \times n(\mathbf{k}) [n(\mathbf{k}') + 1] \delta[\epsilon(\mathbf{k}) - \epsilon(\mathbf{k}') + \alpha\hbar\nu], \tag{23}
\end{aligned}$$

where  $\alpha$  is defined through (22).

Note that, in general, the master equation (19) is not written in the Lindblad form [34] and therefore its interpretation in terms of transitions between trap levels may be misleading [31]. However, one can see by simple inspection of this equation that the  $\Gamma$ 's with  $\mathbf{m} = \mathbf{n}'$  and  $\mathbf{m}' = \mathbf{n}$  do correspond to transition rates between level  $|\mathbf{n}\rangle$  and  $|\mathbf{n}'\rangle$ . These are precisely the rates that enter into the evolution equation for the energy and thus the ones responsible from the cooling process (see Sec. V).

One should also stress that the master equation has three important properties: it conserves the trace of the density matrix, it describes physical *decay* processes, and it conserves the positivity of the density matrix. The first property follows readily from Eq. (21) and the second and the third from the fact that, using expressions (23) and (9), one can show that the matrix  $\Gamma_{\mathbf{n}, \mathbf{n}'}^{\mathbf{m}, \mathbf{m}'}$  is positively defined. Diagonalizing this matrix one can reduce each of the Liouvillians (21) to a sum of Lindblad-like terms of a form proportional to  $C\rho_A C^\dagger - \frac{1}{2}\rho_A C^\dagger C - \frac{1}{2}C^\dagger C\rho_A$ , where the operators  $C$  and  $C^\dagger$  are appropriate bilinear combinations of  $a_{\mathbf{m}}^\dagger$ 's and  $a_{\mathbf{m}'}$ 's. In such a representation the physical soundness of the master equation is evident. Unfortunately, such a representation is generally difficult to handle, unless the matrix (23) has some very special properties.

#### IV. RATES

In the preceding section we derived the master equation describing the sympathetic cooling of atoms by their interaction with a bath composed of other atoms. The cooling is described in terms of rates as given in Eq. (23). In this section we derive analytic expression for them. In particular, for an even number of dimensions  $d$ , we derive an *exact* expression in terms of the parameters entering in the problem. We compare it with an analytic approximation based on the saddle point method, valid for both even and odd dimensions.

We start by writing Eq. (23) in a slightly different form

$$\begin{aligned}
\Gamma_{\mathbf{n}, \mathbf{n}'}^{\mathbf{m}, \mathbf{m}'} = & \frac{1}{2\hbar^2} \int_{-\infty}^{\infty} d\tau \int d\mathbf{k} d\mathbf{k}' \gamma_{\mathbf{n}, \mathbf{n}'}(\mathbf{k}, \mathbf{k}') \gamma_{\mathbf{m}, \mathbf{m}'}(\mathbf{k}', \mathbf{k}) \\
& \times n(\mathbf{k}) [n(\mathbf{k}') + 1] e^{i[\epsilon(\mathbf{k}) - \epsilon(\mathbf{k}') + \alpha\hbar\nu]\tau/\hbar}. \tag{24}
\end{aligned}$$

Now we make the following change of variables:  $\hbar\nu\tau M_A/M_B = t$ ,  $\mathbf{ka} = \boldsymbol{\eta}$ , and  $\mathbf{k}'a = \boldsymbol{\eta}'$ , where  $a = (\hbar/2M_A\nu)^{1/2}$  is the size of the ground state of the trap. In the limit where the bath follows a Boltzmann distribution,  $n(\mathbf{k}') + 1 \simeq 1$ ; using (2) and (6), we obtain

$$\Gamma_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'} = \frac{M_B}{M_A} \frac{n_B \Lambda_B^d C^2}{2\hbar^2\nu(2\pi a)^{2d}} I_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'} . \quad (25)$$

The dimensionless quantities  $I_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'}$  are defined by

$$I_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'} = \int_{-\infty}^{\infty} dt \int d\boldsymbol{\eta} d\boldsymbol{\eta}' e^{-\delta\eta^2} e^{i(\boldsymbol{\eta}^2 - \boldsymbol{\eta}'^2 + \alpha')t} \times f_{\mathbf{n},\mathbf{n}'}(\boldsymbol{\eta} - \boldsymbol{\eta}') f_{\mathbf{m},\mathbf{m}'}(\boldsymbol{\eta}' - \boldsymbol{\eta}), \quad (26)$$

where  $\delta = \beta_B \hbar\nu M_A/M_B$ ,  $\alpha' = \alpha M_B/M_A$ , and

$$f_{\mathbf{n},\mathbf{n}'}(\boldsymbol{\eta}) = \int d\mathbf{x} \psi_{\mathbf{n}}(\mathbf{x})^* \psi_{\mathbf{n}'}(\mathbf{x}) e^{-i\boldsymbol{\eta}\cdot\mathbf{x}/a}. \quad (27)$$

To calculate this quantity, we first perform the change of variables  $\boldsymbol{\eta}'' = \boldsymbol{\eta}' - \boldsymbol{\eta}$  in the integral. The quantities  $f_{\mathbf{n},\mathbf{n}'}$  can be easily calculated, taking into account that the eigenstates of the harmonic oscillator in the  $l$ th dimension  $|n_l\rangle$  fulfill

$$\langle n_l | e^{i\eta_l''(a_l + a_l^\dagger)} | n_l' \rangle = e^{-\eta_l''^2/2} \sum_{r_l=0}^{\min(n_l, n_l')} \frac{(i\eta_l'')^{n_l + n_l' - 2r_l} \sqrt{n_l!} \sqrt{n_l'!}}{r_l!(n_l - r_l)!(n_l' - r_l)!}, \quad (28)$$

where  $a_l$  ( $a_l^\dagger$ ) is the annihilation (creation) operator for the harmonic oscillator (in the first quantization picture). We obtain

$$I_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'} = \int_{-\infty}^{\infty} dt e^{i\alpha't} \left[ \prod_{l=x,y,\dots} \sum_{r_l=0}^{\min(n_l, n_l')} \sum_{s_l=0}^{\min(m_l, m_l')} C_{n_l, n_l', r_l} \times C_{m_l, m_l', s_l} \right] U(n_l + n_l' + m_l + m_l' - r_l - s_l), \quad (29)$$

where

$$C_{n, n', r} = \frac{(-1)^{n+r} \sqrt{n!} \sqrt{n'!}}{r!(n-r)!(n'-r)!} \quad (30)$$

and

$$U(n) = \int d\boldsymbol{\eta} d\boldsymbol{\eta}' (\boldsymbol{\eta}')^n e^{-\eta'^2} e^{-\delta\eta^2} e^{(2\boldsymbol{\eta}\boldsymbol{\eta}' - \eta'^2)t}. \quad (31)$$

In this derivation we have used the fact that  $\sum_{l=x,y,\dots} (n_l + n_l' - m_l - m_l') = 2 \sum_{l=x,y,\dots} (n_l - m_l)$  [see (22)]. The Gaussian integrals in the definition of  $U(n)$  can be performed easily, giving

$$U(n) = \frac{\pi}{\sqrt{\delta}} \frac{(n+1)!!}{2^{n/2} A(t)^{(n+1)/2}} \quad (32)$$

for  $n$  even and  $U(n) = 0$  for  $n$  odd. Here

$$A(t) = 1 + it + t^2/\delta. \quad (33)$$

Substituting these results in (26), we obtain

$$I_{\mathbf{n},\mathbf{n}'}^{\mathbf{m},\mathbf{m}'} = \left[ \frac{\pi^2}{\delta} \right]^{d/2} \left[ \prod_{l=x,y,\dots} \sum_{r_l=0}^{\min(n_l, n_l')} \sum_{s_l=0}^{\min(m_l, m_l')} C_{n_l, n_l', r_l} \times C_{m_l, m_l', s_l} \frac{(2q_l + 1)!!}{2^{q_l}} \right] V_q \quad (34)$$

if  $n_l + n_l' + m_l + m_l'$  is even for all  $l$  and zero otherwise. In (34)  $q_l = (n_l + n_l' + m_l + m_l' - 2r_l - 2s_l)/2$  and  $q = d/2 + \sum_{l=x,y,\dots} q_l = d/2 + \sum_{l=x,y,\dots} (n_l + m_l - r_l - s_l)$  [see Eq. (22)]. The quantity  $V_q$  is defined by

$$V_q = \int_{-\infty}^{\infty} dt \frac{e^{i\alpha't}}{A(t)^q}, \quad (35)$$

where  $q$  is an integer number plus  $d/2$ . The calculation of the rates  $\Gamma$  has thus been reduced to the evaluation of the above integral. Note that the integrand in Eq. (35) tends to zero as  $t$  increases. The typical width of  $1/|A(t)|$  is of the order of  $\min[1, \sqrt{\delta}]$ , which gives for  $\tau_c$  the result given in (16).

The integration in (35) can be performed exactly when the number of dimensions  $d$  is *even*. In this case, using the residues technique one obtains (see the Appendix)

$$V_q^{\text{res}} = 2\pi \sum_{l=0}^{q-1} \binom{q-1+l}{q-1} \times \frac{(\alpha')^{q-1-l} e^{-\alpha'[\sqrt{\delta(\delta+4)}-\delta]/2}}{(q-1-l)! [\delta(\delta+4)]^{(q+l)/2}}. \quad (36)$$

On the other hand, one can obtain a very good approximation for  $V_q$  using the saddle point method, for both even and odd  $d$ . As shown in the Appendix, we obtain

$$V_q^{\text{SP}} \simeq \sqrt{\frac{2\pi}{|t^s - t_+^0|^{-2} + |t^s - t_-^0|^{-2}} \frac{e^{i\alpha't^s}}{|(t^s - t_+^0)(t^s - t_-^0)|^q}}, \quad (37)$$

where  $t_{\pm}^0$  are the zeros of  $A(t)$

$$t_{\pm}^0 = \frac{i}{2} [\pm \sqrt{\delta(\delta+4)} - \delta] \quad (38)$$

and  $t^s$  the only saddle point that contributes to the integral

$$t^s = \frac{i}{2} [\sqrt{\delta(\delta+4)} + 4q^2/(\alpha')^2 - \delta - 2q/\alpha']. \quad (39)$$

In Fig. 1 we have plotted the results given by formulas (36) and (37). More specifically, we have plotted the error (in percent) given by the saddle point approximation, that is,  $|(V_q^{\text{SP}} - V_q^{\text{res}})/V_q^{\text{res}}|$ , as a function of  $q$  for several values of the  $\delta$  and  $\alpha'$ . Apart from the case  $\alpha' = 0$ , which

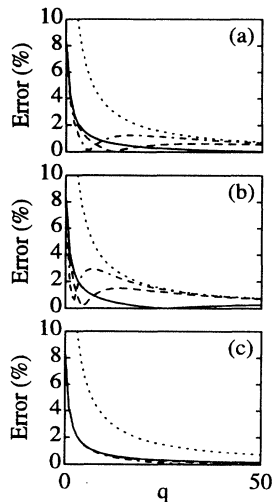


FIG. 1. Error given by the saddle point approximation (in percent) for the value of  $V_q$  as a function of  $q$ . (a)  $\delta = 1$ ; (b)  $\delta = 0.2$ ; (c)  $\delta = 10$ . Solid, dashed, dash-dotted, and dotted lines correspond to  $\alpha' = 100, 20, 10$ , and  $0$ , respectively.

gives a maximum error of about 50%, the saddle point approximation gives the correct result with an accuracy of less than 8% for any  $\delta$ , as the figure shows. The error decreases as  $q$  increases, tending to a very small value as  $q \rightarrow \infty$ .

This is a very important observation since it strongly supports the validity of the saddle point approximation in calculating  $V_q$ . We conjecture that Eq. (37) can be used with a great confidence to evaluate the rates  $\Gamma$ , also in the case of odd dimension  $d$ . The analysis presented in this section is thus one of the main results of this paper. The techniques that we have developed here can be used in other problems involving the derivation of the master equation for cooled atomic systems.

## V. DISCUSSION

In general, the master equation (19) is rather difficult to solve. The main problem is that it couples different Fock states with the same (bare) energy and therefore it cannot be reduced to a set of rate equations, i.e., to a Lindblad form [31]. In other words, due to the degeneracies of the Fock states there is no clear distinction between the rates of decay of coherences and diagonal elements of the reduced density matrix  $\rho_A$ . There are, however, at least two situations in which such a reduction of the master equation to the Lindblad form is possible: the case of an anisotropic and anharmonic trap and the case of rapid thermalization of the system  $A$  due to collisions between  $A$  atoms. We discuss both of these cases below. Moreover, in spite of the presence of degeneracies, we can obtain the steady state solution of Eq. (19) in the limit when the  $A$  system is not too dense. In this limit the collisions between the  $A$  atoms can be treated perturbatively, so that the quantum kinetic equations are valid [26].

## A. Steady state

As could be expected from thermodynamic considerations, the steady state solution of the master equation (19) is the BED

$$\rho_A = \mathcal{Z} e^{-\beta_A H_A}, \quad (40)$$

with  $\beta_A = \beta_B$ . This statement can be proved quite generally from the following three observations. First, by direct inspection one can show that  $\mathcal{L}\rho_A = 0$ , i.e., that  $\rho_A$  is a steady state solution. This is a consequence of the fact that the Liouvillians  $\mathcal{L}_\alpha$  always contain products of operators that increase and then decrease the energy by  $\pm \hbar\alpha\nu$ . Second, following the same procedure as the one used in Sec. II to derive the master equation, one can easily see that the part of the Hamiltonian  $H'_{A-A} - H_{A-A}$  that describes shifts induced by interactions with the bath particles commutes with  $\rho_A$ . Third, the collisions between the  $A$  atoms described by  $H_{A-A}$  can be treated using the standard theory of the quantum Boltzmann equations [26]. In effect they lead to quantum kinetic equations for which (40) is a steady state solution (for any  $\beta_A$ ). Since the steady state solutions of the quantum kinetic equations must be of the form (40), this justifies the uniqueness of the solution (40) for  $\beta_A = \beta_B$ . This last point is well established in quantum statistical mechanics and is based on ergodicity assumptions [26].

Knowing the steady state solution we can analyze under what circumstances quantum statistical effects can be observed in the system  $A$ . Roughly speaking, they occur when the number of particles in the ground level is of the order of 1 or greater. We can estimate when this happens, using, for the system  $A$ , the Boltzmann distribution for the harmonic trap in  $d$  dimensions. Requiring that  $N_0 \gtrsim 1$ , one can easily show that this occurs when

$$N(\beta_B \hbar\nu)^d \gtrsim 1. \quad (41)$$

We shall use the condition (41) even though it is slightly exaggerated, since in the case  $d = 3$  it is essentially stronger than the condition for Bose-Einstein condensation. On the other hand, our theory is valid in the regime when the bath does not display quantum statistical effects, i.e., when the number of particles in the ground level is much smaller than one. According to (6), we then have

$$n_B(2\pi\hbar^2\beta_B/M_B)^{d/2} \ll 1. \quad (42)$$

These two conditions can be satisfied when

$$N \gg (4\pi)^d n_B^2 a^{2d} (M_A/M_B)^d. \quad (43)$$

Thus one of the following conditions must hold: (a) For similar densities of atoms,  $M_B \gg M_A$ , or (b) for similar atomic masses, the density of  $A$  atoms must be larger than that of  $B$  atoms. Let us analyze the condition (43) more closely for  $d = 3$ . For alkali atoms the masses are 7 (lithium), 23 (sodium), 85 (rubidium), and 133 (cesium).

The mass ratio can change from 0.05 (lithium:cesium) to 0.64 (rubidium:cesium). Assuming that the bath consists of  $10^7$  cesium atoms in a 10- $\mu\text{m}$  trap and taking  $a = 0.1 \mu\text{m}$ , we obtain from (43) that the number of  $A$  atoms  $N$  must be larger than approximately 25 for the case of lithium and larger than  $5 \times 10^2$  for the case of rubidium. Amazingly, these numbers are within the reach of present day experiments.

### B. Anisotropic and anharmonic traps

The main difficulty in analyzing the master equation (19) lies in the fact that this equation does not have a Lindblad form and cannot, in general, be reduced to equations for diagonal density matrix elements. The reason that such reduction is not possible is that the system  $A$  is highly degenerated. There are two kind of degeneracies: (a) the degeneracy of energy levels due to isotropy, that is, for the states for which  $n_x + n_y + \dots = n'_x + n'_y + \dots$ , and (b) the *dynamical* degeneracy, which occurs even in the case of  $d = 1$ , for instance, for the states  $\mathbf{n} = (1, 0, 1, 0, \dots)$  and  $\mathbf{n}' = (0, 2, 0, \dots)$ . Both kinds of degeneracies are lifted up if one considers an anisotropic trap with anharmonic energy levels. If one then assumes that the resulting energy level shifts are larger than cooling rates, one can evoke standard secular arguments to reduce the master equation to the diagonal form. Moreover, if one additionally assumes that the resulting level shifts are small (so that they can be neglected in the evaluation of the transition rates), the resulting master equation will take the form

$$\frac{d\rho_A}{dt} = \sum_{\alpha=-\infty}^{\infty} \mathcal{L}_{\alpha} \rho_A, \quad (44)$$

with

$$\begin{aligned} \mathcal{L}_{\alpha} \rho_A = & \sum_{\mathbf{n}, \mathbf{m}} \Gamma_{\mathbf{n}, \mathbf{m}}^{\mathbf{m}, \mathbf{n}} (2a_{\mathbf{m}}^{\dagger} a_{\mathbf{n}} \rho_A a_{\mathbf{n}}^{\dagger} a_{\mathbf{m}} \\ & - a_{\mathbf{n}}^{\dagger} a_{\mathbf{m}} a_{\mathbf{m}}^{\dagger} a_{\mathbf{n}} \rho_A - \rho_A a_{\mathbf{n}}^{\dagger} a_{\mathbf{m}} a_{\mathbf{m}}^{\dagger} a_{\mathbf{n}}), \end{aligned} \quad (45)$$

$n - m = \alpha$ , and the rates given by the same expressions as discussed in Sec. IV.

The above master equation can be then easily simulated using the standard Monte Carlo procedures (see Ref. [24]). It describes Poisson jump process in which the  $A$  atoms may jump from one energy level to another. Each term in Eq. (44) corresponds to an energy change by  $\hbar\nu\alpha$ . We expect that, qualitatively, the dynamics described by Eq. (44) will be very similar to the one discussed in Ref. [24] in which we studied the dynamics of the laser cooled quantum gas in a tight trap, i.e., in the so-called Lamb-Dicke limit. In that case the Liouvillian has a form analogous to Eq. (44), but contains only the terms corresponding to  $\alpha = \pm 1$ , so that only the jumps corresponding to the energy change  $\pm\hbar\nu$  are possible.

For the laser cooled gas of bosons the quantum dynamics exhibits interesting collective behavior. If one starts, for instance, with all atoms in the  $n$ th level, they will, in

the case of low temperature, jump collectively to subsequent levels  $n - 1$ ,  $n - 2$ , and so on. At a given instant, most of the population will be concentrated in a single level of the trap. More strikingly, similar effects occur when one starts the dynamics from the state in which all bosons are in different trap levels. After some time, because of their bosonic character, the atoms will “bunch” and tend to occupy only a few trap levels. After that they will again perform subsequent collective jumps to the lower states. In the case of the laser cooled gas the dynamics slows down slightly as the lower energy levels are reached. Obviously similar effects will occur for the more general dynamics described by Eq. (44), except that now collective jumps might take place between the states that differ in energy by  $\pm\hbar\nu\alpha$ . The analysis of the behavior of the rates  $\Gamma$  indicates that for high-energy levels there are no preferable values of  $\mathbf{n}, \mathbf{m}$ , and  $\alpha$  and all the rates are more or less of the same order. That means that in the dynamics described by Eq. (19) one should not expect significant differences in the cooling rates as the system cools down, whereas one should expect that the jumps corresponding to all values of  $\alpha$  are, more or less, equally probable. Only when the system cools down close to the ground state do the jumps corresponding to small values of  $\alpha$  become dominant.

The behavior of the rates  $\Gamma$  with the different parameters can be deduced from Eqs. (25) and (37). They are proportional to the density of bath atoms  $n_B$  and the square of the scattering length through the factor  $C^2$ . Their dependence on the bath temperature is displayed in Figs. 2 and 3. In Fig. 2 we plot  $V_q^{\text{SP}}$  as a function of  $q$  for the same parameters as in Fig. 1(a). Figure 2(a) corresponds to integer values of  $q$  (i.e., even  $d$ ), whereas Fig. 2(b) corresponds to semi-integer values of  $q$  (i.e., odd  $d$ ). The differences between these figures are nearly negligible. For  $\alpha' = 0$  (transitions that do not change the energy)  $V_q$  decreases monotonically with  $q$ . For  $\alpha' \neq 0$  it has a maximum at a given  $q$ . The value of  $q$  for which  $V_q$  is maximum increases with  $\alpha'$ . This means that for  $\alpha'$  small the most important rates are those between the lowest levels of the harmonic oscillator. However, as  $\alpha'$  increases, the important rates are those corresponding to transitions between higher and higher levels.

In Fig. 3 we analyze the behavior of  $V_q$  as a function of  $\delta$  for several values of  $q$  and  $\alpha' = 25$ . Note that the dependence on  $\delta$  of the prefactor of the  $\Gamma$ 's and  $I$  in (25) and (34), respectively, cancels out. As the figure shows,

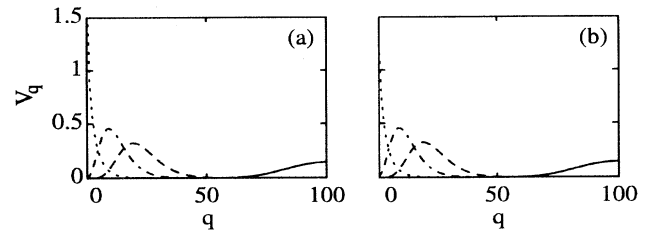


FIG. 2.  $V_q$  as a function of  $q$  for  $\delta = 1$  for (a) integer values of  $q$  and (b) semi-integer values of  $q$ . Solid, dashed, dash-dotted, and dotted lines correspond to  $\alpha' = 100, 20, 10$ , and 0, respectively.



$V_q$  decreases as  $\delta$  increases (i.e., as the temperature of the bath decreases) and therefore the cooling rate becomes faster as the bath temperature decreases (note the logarithm scale for  $V_q$ ). In this low- (bath) temperature regime ( $\delta \rightarrow 0$ ), the function  $V_q$  behaves as

$$V_q \simeq \sqrt{\pi} \frac{e^{\delta(1-\alpha'/q)/2}}{\delta^{q-1/2}}. \quad (46)$$

This expression clearly shows the above statement. It also displays that the rates are larger for larger  $q$ . Then, in this limit, the terms with  $r_j = s_j = 0$  give the maximum contribution to  $I$  in Eq. (34). The rates are then larger for larger values of  $\mathbf{n}, \mathbf{m}$  and, therefore, as the cooling process takes place, it becomes slower and slower. Note also that formally, in the limit  $\delta \rightarrow 0$ , the rates are not bounded and therefore the Born approximation breaks down. Even for  $\delta < 1$  and for sufficiently large values of  $\mathbf{n}, \mathbf{m}$ , the rates calculated with the expressions given here become larger than the trap frequency and therefore the assumptions made in Sec. III do not hold.

### C. Rapid thermalization of A

Another situation in which the analysis of the dynamics governed by the master equation (19) is particularly simple occurs when collisions between atoms in  $A$  are more frequent than those between different atoms. This will be the case when the density of  $A$  is larger than that of  $B$ . In this case, the system  $A$  thermalizes rapidly and its state at each moment is given by a BED, with some time-dependent inverse temperature  $\beta_A(t)$ . The density operator is thus always diagonal in the Fock basis. The mean occupation of the  $\mathbf{m}$ th levels evolves as

$$\begin{aligned} \frac{dN_{\mathbf{m}}}{dt} &= \frac{d}{dt} \langle a_{\mathbf{m}}^\dagger a_{\mathbf{m}} \rangle \\ &= 2 \sum_{\alpha \neq 0} \left[ \sum_n^+ \Gamma_{\mathbf{n}, \mathbf{m}}^{\mathbf{m}, \mathbf{n}} N_{\mathbf{n}} (N_{\mathbf{m}} + 1) \right. \\ &\quad \left. - \sum_n^- \Gamma_{\mathbf{m}, \mathbf{n}}^{\mathbf{n}, \mathbf{m}} N_{\mathbf{m}} (N_{\mathbf{n}} + 1) \right] + \frac{dN_{\mathbf{m}}}{dt} \Big|_{A-A}, \end{aligned} \quad (47)$$

where the last term refers to the contribution due to atom-atom interactions between system particles. This contribution has the standard form of quantum kinetic equations [26] and ensures that within a fast thermalization time the system will reach a state of instantaneous equilibrium described by BED with the inverse temperature  $\beta_A(t)$ . After the thermalization time this contribution simply vanishes and the  $N_{\mathbf{m}}$ 's become uniquely determined functions of  $\beta_A(t)$ . The  $\pm$  in the sums indicate that they extended to values of  $\mathbf{n}$  for which  $\sum_{l=x,y,\dots} (n_l - m_l) = \pm \alpha$ . To derive (47), we have utilized the master equation (19) and Wick's theorem. Now, using the fact that at instantaneous equilibrium

$$\frac{N_{\mathbf{n}}}{N_{\mathbf{n}} + 1} = \frac{N_{\mathbf{m}}}{N_{\mathbf{m}} + 1} e^{-\beta_A(t) \hbar \nu (n_x + n_y + \dots - m_x - m_y - \dots)}, \quad (48)$$

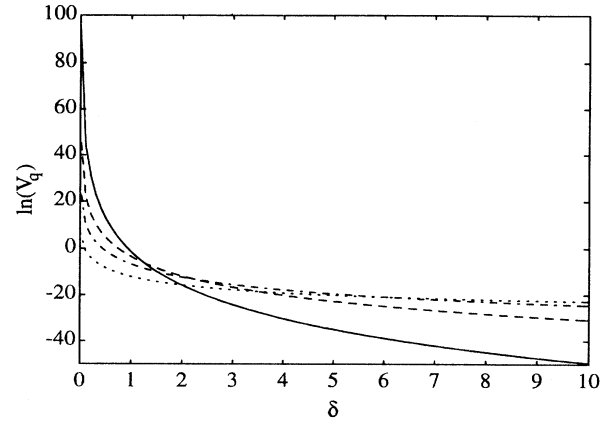


FIG. 3. Logarithm of ( $V_q$ ) as a function of  $\delta$  for  $\alpha' = 25$ . Solid, dashed, dash-dotted, and dotted lines correspond to  $q = 1, 5, 10$ , and  $20$ , respectively.

we can derive, for the total energy of the system  $E = \sum_{\mathbf{m}} \hbar \nu (m_x + m_y + \dots) N_{\mathbf{m}}$ , the evolution equation

$$\begin{aligned} \frac{dE}{dt} &= -\hbar \nu \sum_{\alpha=1}^{\infty} \alpha [1 - e^{\alpha \hbar \nu [\beta_B - \beta_A(t)]}] \\ &\quad \times \sum_{\mathbf{n}, \mathbf{m}}^+ \Gamma_{\mathbf{n}, \mathbf{m}}^{\mathbf{m}, \mathbf{n}} N_{\mathbf{n}} (N_{\mathbf{m}} + 1). \end{aligned} \quad (49)$$

Note that at instantaneous equilibrium  $E$  is a unique function of  $\beta_A(t)$ , so that Eq. (49) can be viewed as an equation determining slow changes of  $\beta_A(t)$  due to atom-bath interactions. Equation (49) clearly shows that the steady state is reached when  $\beta_A = \beta_B$ , given that  $\Gamma_{\mathbf{n}, \mathbf{m}}^{\mathbf{m}, \mathbf{n}}$  and  $N_{\mathbf{n}}$  are positive numbers. The rate at which the energy changes is characterized by the rates  $\Gamma$  and therefore they determine the cooling. This is the main and very useful result of this subsection, since it allows us to compute directly the cooling rate using the techniques developed in Sec. IV.

Finally, we wish to emphasize that the present formulation of the problem of sympathetic cooling, as is the case of other cooling mechanism based on collisions, is expressed in terms of physical quantities such as the scattering length  $a_{sc}$ . These quantities are not well known for many specific atom-atom collisions (as is the case, e.g., for Li-Cs). Thus it is difficult to estimate the cooling rates from our formulation until these quantities can be measured in experiments or calculated numerically. Note that the shape-independent approximation is valid under the assumption  $ka_{sc} \ll 1$ , where  $k \sim 2\pi/\lambda$  is the thermal wave vector of the atoms. Assuming that the atoms are initially at a temperature equivalent to 100 recoils (with respect to an internal optical transition), as obtained with optical molasses, this approximation is valid for  $a_{sc} \ll 10^{-2} \lambda$ , where  $\lambda$  is the optical wavelength. Although the specific value of  $a_{sc}$  is not well known, typical results for other atom-atom interactions [35] indicate that it fulfills this condition, i.e., the shape-independent approximation is well satisfied.

## VI. CONCLUSIONS

In this paper we have studied sympathetic cooling of one alkali-metal species  $A$  by another one  $B$ . We have considered a situation in which the lighter atoms  $A$  are trapped in a tight harmonic trap (such as a FORT) and interact with heavier atoms  $B$  trapped in a loose trap (such as a MOT). The atoms  $B$  are cooled by some other mechanism. We have derived a master equation describing sympathetic cooling of a system  $A$ , treating the  $B$  particles as a bath at a fixed temperature. The validity conditions for this master equation are that the cooling rates appearing in it must be much smaller than the trap frequency of the harmonic potential. We have derived an exact analytic expression for these cooling rates in the case of an even number of dimensions. We have developed powerful mathematical techniques to calculate these rates in any dimension using a saddle point method. The techniques can be generalized to other problems involving a master equation for cooled atom systems.

We have shown that the steady state solution of the master equation is described by the BED with the inverse temperature  $\beta_A$  equal to that of the bath  $\beta_B$ . We have identified regions of parameters in which the system  $A$  exhibits quantum statistical effects and argued that those regions lie within the reach of present day experiments. We have also identified two cases when the master equation derived here can be solved numerically via Monte Carlo method: the case of anharmonic and anisotropic traps and the case of rapid thermalization in the system  $A$ . We have discussed the dynamics in those cases in qualitative terms and argued that it would present interesting collective behaviors, similar to those found in the case of many-atom laser cooling [24].

## ACKNOWLEDGMENTS

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## APPENDIX: CALCULATION OF $V$

In this appendix we evaluate (35) by two methods.

### 1. Residues

Let us take the case with  $d$  even. The poles of the integrand in the definition of  $V_q$  are at  $t_{\pm}^0$ . Closing the integration path in the upper half of the complex plane, we obtain

$$V_q = 2\pi i \text{Res} \left[ \frac{e^{i\alpha't}}{[(t-t_+^0)(t-t_-^0)]^q}, t_+^0 \right]. \quad (\text{A1})$$

This residue can be easily calculated, giving (36).

### 2. Saddle point

Now we evaluate (35) by the saddle point approximation technique for any  $d$ . To do this, we first express  $V_q$  as

$$V_q = \int_{-\infty}^{\infty} dt e^{F(t)}, \quad (\text{A2})$$

where

$$F(t) = i\alpha't - q \ln[A(t)]. \quad (\text{A3})$$

Solving  $F(t)' = 0$ , we find two saddle points

$$t_{\pm}^s = \frac{i}{2} [\pm \sqrt{\delta(\delta+4) + 4q^2/(\alpha')^2} - \delta - 2q/\alpha']. \quad (\text{A4})$$

On the other hand,

$$F(t)'' = q \left[ \frac{1}{(t-t_+^0)^2} + \frac{1}{(t-t_+^0)^2} \right], \quad (\text{A5})$$

which is negative at the saddle points. Note that all  $t_{\pm}^0$  and  $t_{\pm}^s$  are located at in the imaginary axis. It is easy to see that  $t_+^0/i > t_+^s/i > t_-^0/i > t_-^s/i$ . Thus the integration path from  $-\infty$  to  $\infty$  can only be deformed (without changing the result of the integral) to cross  $t_+^s$ , i.e., this is the only saddle point that contributes to the integral. Using the formula

$$V_q = \sqrt{\frac{2\pi}{|F(t_+^s)''|}} e^{F(t_+^s)}, \quad (\text{A6})$$

we find (37).

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