Quantum inverse problem for an unstable nonlinear Schrödinger equation: A functional Bethe ansatz

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We have formulated the quantum inverse problem for the unstable nonlinear Schrödinger equation that forms the basis of nonlinear optics, following the modified approach of Sklyanin. Due to the nonexistence of the vacuum for the corresponding Lax L operator, the algebraic Bethe ansatz cannot be formulated. This difficulty is removed by recourse to the functional Bethe ansatz. The whole analysis is performed by considering different boundary conditions at the two ends of the interval on the x axis. Both signs of the nonlinearity are treated, corresponding to "bright" and "dark" solitons.

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INTRODUCTION

Quantization of an integrable system forms one of the most interesting problems in the study of two-dimensional nonlinear systems [1]. Of late, various equations have been treated on the basis of the formalism developed by Thacker [2] and Faddeev [3]. An interesting aspect of the formulation is its close relation with the Yang-Baxter equation [4], and this is perhaps the reason for which the second approach has been widely applied. Of late, two very important problems have been analyzed while applying the quantum inverse scattering method (QISM). One is the question of the finiteness of the interval on which the quantization is performed, for relaxing the condition of periodicity [5], and the other is the difficulty associated with the nonexistence of a vacuum state due to the algebraic structure of the Lax operator (L) [6]. Here in this paper we report on the QISM study of a nonlinear Schrödinger equation (NLS) obtained by interchanging the (x and t) variables. It can be written as

$$iq_x+q_{tt}\pm 2|q|^2q=0 \ .$$

Originally such an equation was deduced by Yasima and Wadati [7] from the equations describing the propagation of waves inside a plasma. More importantly this equation also forms the basis of contemporary nonlinear optics, where t is the retarded time measured in a frame of reference moving with the group velocity [8]. Fuchasteiner and Oevel studied the integrability through the use of master symmetry [9].

FORMULATION

The nonlinear Schrödinger equation is written as

$$iq_x+q_{tt}\pm 2|q|^2q=0 \tag{1}$$

and is second order in time, whereas the original NLS was first order in time. The Lax pair associated with Eq. (1) can be written as

$$\Psi_x=L\Psi \ , \quad \Psi_t=M\Psi \ , \tag{2}$$

where

$$L=\begin{bmatrix} \pm i|q|^2-2i\lambda^2 & iq_t+2\lambda q \\ iq_t^*-2\lambda q^* & \mp i|q|^2+2i\lambda^2 \end{bmatrix} , \tag{3}$$

$$M=\begin{bmatrix} -i\lambda & q \\ -q^* & +i\lambda \end{bmatrix} . \tag{4}$$

It is now easily observed that Eq. (1) can be generated from the following Hamiltonian:

$$H=\int \{ i(q_xq^*-q_x^*q)+2q_tq_t^*\pm 2|q|^2|q|^2\} dx \ , \tag{5}$$

with the following canonical Poisson bracket:

$$\begin{aligned} \{q,q_t^*\} &= \delta(x-y) \ , \\ \{q^*,q_t\} &= \delta(x-y) \ . \end{aligned} \tag{6}$$

For the quantization, we start with the definition of the scattering matrix $T(x,y,\lambda)$ (via the discretization procedure adopted by Faddeev [3]),

$$T(\lambda)=\prod Ln \ , \tag{7}$$

where we have set

$$\begin{aligned} T(x,y,\lambda) &= \exp\left[\int_x^y L\, dz\right]=\left[1+\Delta\int_x^y L\, dz\right] \\ &=(1+\Delta\hat{L}n)=Ln \ . \end{aligned} \tag{8}$$

It is to be remembered that the interval (x,y) has been divided into n equal intervals, each of length Δ, and since our theory is ultralocal, it is sufficient to keep only terms of the order Δ. We now set

$$q_n=\frac{1}{\Delta}\int_x^y q\, dz \ , \ q_{nt}=\frac{1}{\Delta}\int_x^y q_t dz \ , \tag{9}$$

whence we get

$$L_n=\begin{bmatrix} 1+\Delta\{\pm i|q_n|^2-2i\lambda^2\} & \Delta(iq_{nt}+2\lambda q_n) \\ \Delta(iq_{nt}-2\lambda q_n^*) & 1+\Delta\{\mp i|q_n|^2+2i\lambda^2\} \end{bmatrix} , \tag{10}$$

and from the Poisson brackets (6) we can at once write the commutation rules:

$$[q_n, q^*_{mt}] = \frac{\hbar}{\Delta}\delta_{nm} ,$$
$$[q^*_n, q_{mt}] = \frac{\hbar}{\Delta}\delta_{nm} . \tag{11}$$

Using the L_n given in Eq. (10) in

$$R(\lambda,\mu)L^1_m(\lambda)L^2_n(\mu) = L^2_n(\mu)L^1_m(\lambda)R(\lambda,\mu) , \tag{12}$$

we can solve for the $R(\lambda,\mu)$,

$$R(\lambda,\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2(\lambda-\mu)}{2(\lambda-\mu)\pm\gamma} & \frac{\pm\gamma}{2(\lambda-\mu)\pm\gamma} & 0 \\ 0 & \frac{\pm\gamma}{2(\lambda-\mu)\pm\gamma} & \frac{2(\lambda-\mu)}{2(\lambda-\mu)\pm\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{13}$$

where we have used the standard notation $L^1_n(\lambda) = L_n(\lambda)\otimes 1 L^2_n(\lambda) = 1 \otimes L_n(\lambda)$. Finally it is possible to write the following equivalent form of the quantum R matrix, which is more convenient:

$$R(\theta) = \begin{bmatrix} \theta\pm\gamma & 0 & 0 & 0 \\ 0 & \theta & \pm\gamma & 0 \\ 0 & \pm\gamma & \theta & 0 \\ 0 & 0 & 0 & \theta\pm\gamma \end{bmatrix} , \tag{14}$$

with $\theta = 2(\lambda-\mu)$. It may be noted that though the form of the L operator is quite distinct from that of the usual NLS equation, the R matrix has the same structure.

COMMUTATION RULES FOR THE SCATTERING DATA AND BOUNDARY CONDITION

We now want to formulate the QISM with different boundary conditions at the two ends of the interval. It has already been demonstrated by Sklyanin [10] that such boundary conditions could be easily considered with the help of the matrices K_+ and K_-, obtained as solutions of the following equations:

$$R(\lambda_{12})K'_-(\lambda_1)R(\bar\lambda_{12}\mp\gamma)K^2_-(\lambda_2) = K^2_-(\lambda_2)R(\bar\lambda_{12}\mp\gamma)K'_-(\lambda_1)R(\lambda_{12}) , \tag{15}$$

$$R(-\lambda_{12})K^{t_1}_+(\lambda_1)R(-\lambda_{12}\mp\gamma)K^{t_2}_+(\lambda_2) = K^{t_2}_+(\lambda_2)R(-\bar\lambda_{12}\mp\gamma)K^{t_1}_+(\lambda_1)R(-\lambda_2) , \tag{16}$$

where $K^{t_1}_+$ denote the transpose. In the present situation we get

$$K_+ = \begin{bmatrix} \xi_+ + \lambda\pm\gamma & 0 \\ 0 & \xi_- - \lambda\mp\gamma \end{bmatrix} , \tag{17}$$

where we have used the notation

$$\lambda_{12} = \lambda_1 - \lambda_2 \bar\lambda_{12} = \lambda_1 + \lambda_2 .$$

The commutation rules for the scattering data are given by

$$R_{12}(u-v)\mathcal{T}'(u)R_{12}(u+v)\mp\gamma\mathcal{T}^2(v) = \mathcal{T}^2(v)R_{12}(u+v\mp\gamma)\mathcal{T}'(u)R_{12}(u-v) , \tag{18}$$

where $\mathcal{T}(u)$ denotes the scattering data, modified due to the presence of the boundary condition. It is defined as

$$\mathcal{T}(u) = T(u)K_-(u)T^{-1}(-u\pm\gamma) = \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} , \tag{18a}$$

and we shall also write

$$T(u) = \begin{bmatrix} A'(u) & B'(u) \\ C'(u) & D'(u) \end{bmatrix} . \tag{18b}$$

We quote here only a few, which can be obtained from (18), needed for the setting up of the Bethe ansatz equation:

$$(u-v\pm\gamma)(u+v\mp\gamma)B(u)A(v) = (u-v)(u+v)A(v)B(u) \pm(u-v)\gamma[B(v)D(u)\pm(u+v)\mp\gamma]B(v)A(u) , \tag{19}$$

$$B(v)B(u) = B(u)B(v) , \tag{20}$$

$$(u-v)[D(u),A(v)] = (u+v\pm\gamma)[A(u),D(v)] , \tag{21}$$

$$(u-v)[A(u),D(v)] = (u+v\pm\gamma)[D(u),D(v)] , \tag{22}$$

$$(u+v)[A(u),A(v)] = \pm\gamma\{B(v)C(u) - B(u)C(v)\} . \tag{23}$$

The next important step is to define a pseudovacuum $|0\rangle$, which would serve as the basis for the construction of the excited states; but due to the form of the commutation rules, if we interpret q^*_t, q_t as annihilation operators, then (q,q^*) will be the operators for creating the states. Unfortunately, then the operator L_n on $|0\rangle$ will not be triangular, and so we cannot interpret either $B(u)$ or $C(u)$ as a creation or annihilation operator, and hence the usual route for the construction of the algebraic Bethe ansatz cannot be taken. Of late, two methods have been proposed to circumvent such a difficulty. One method is due to Baxter (for the case of the eight-vertex model) and the other due to Sklyanin [11]. The latter one was called the functional Bethe ansatz. Here we adopt this latter approach to set up the equation determining the momenta of the Bethe states.

Let us refer back to Eqs. (10) and (7) and observe that, due to the form of the operator L_n, we can at once write

$$A'(\lambda) = K'_2\lambda^{2m} + p'\lambda^{2m-2} + \cdots ,$$
$$B'(\lambda) = Q'_{+1}\lambda^{2m-1} + Q'_{+2}\lambda^{2m-2} + Q'_{+3}\lambda^{2m-3} + \cdots , \tag{24}$$
$$C'(\lambda) = Q'_{-1}\lambda^{2m-1} + Q'_{-2}\lambda^{2m-2} + Q'_{-2}\lambda^{2m-3} + \cdots ,$$
$$D'(\lambda) = K'_2\lambda^{2m} + R'\lambda^{2m-2} + \cdots ,$$

where the coefficients occurring in (24) are dependent on various nonlinear fields. Now, from the definition of $\mathcal{T}(u)$, as given in (18a), we get

$$A(\lambda)=K_2\lambda^N+P\lambda^{N-2}+\cdots\ ,$$
$$B(\lambda)=Q_1\lambda^{N-1}+Q_2\lambda^{N-2}+Q_3\lambda^{N-3}+\cdots\ , \tag{25}$$
$$D(\lambda)=K_2\lambda^N+R\lambda^{N-2}+\cdots\ ,$$

where the coefficient sets used in (25) are some combinations of those in (24), and their actual form will not be necessary in the ensuing discussion.

In the functional approach of Sklyanin it is very necessary to construct some general functional form of A,B,C, etc., from the information given in (25) and the zeros of the function $B(\lambda)$. Let us suppose that these are at $\lambda=\lambda_1$, whence by Lagrange interpolation formulas we can at once write

$$B(\lambda)=\lambda Q_1\prod_{j=1}^{N=2}(\lambda-\lambda_j)\ ,$$
$$-Q_1\sum_j\lambda_j=Q_2\ , \tag{26}$$
$$Q_1\sum\lambda_i\lambda_j=Q_3\ ,$$
$$A(\lambda)=\sum_{j=1}^{N-2}\prod_{K=1}^{N-2}\frac{\lambda-\lambda_K}{\lambda_j-\lambda_K}\alpha_j+S(\lambda)\prod_{j=1}^{N-2}(\lambda-\lambda_j)\ ,$$
$$S(\lambda)=\lambda^2K_2+K_2\lambda\sum_j\lambda_j+P_+K_2\sum_j\lambda_j^2\ , \tag{27}$$
$$D(\lambda)=\sum_j\prod_K\frac{\lambda-\lambda_K}{\lambda_j-\lambda_K}\beta_j+X(\lambda)\prod_{j=1}^{N-2}(\lambda-\lambda_j)\ ,$$

where

$$X(\lambda)=\lambda^2K_2+R+K_2\lambda\sum_j\lambda_j+K_2\sum_j\lambda_j^2\ . \tag{28}$$

Here we interpret $\lambda_j,Q_1,\alpha_j,P,R,\beta_j$ as operators. Our chief motivation will be to determine the communication rules of these from those of A, B, C, etc. Using these forms of A, B, and D in Eqs. (19) to (23), we obtain

$$\alpha_j\sum_{i=K}^{n}\lambda_K^2=\left\{\sum_K\hat{\lambda}_K^2+(\lambda_j\pm\gamma)^2\right\}\alpha_j\ , \tag{29}$$

where λ_K^2 means that the jth term is dropped from the summation. The other important relations are

$$Q_1P=(P_- -\gamma^2K_2)Q_1\ , \tag{30}$$
$$Q_1R=(R+\gamma^2K_2)Q_1\ , \tag{31}$$
$$\beta_j\sum\lambda_K^2=\{\hat{\lambda}_K^2+(\lambda_{j+}-\gamma)^2\}\beta_j\ . \tag{32}$$

All other commutation relations are zero.

FUNCTIONAL BETHE ANSATZ

Equations (29)–(32) form the basis for setting up to the functional Bethe ansatz equation. Actually we can consider any symmetric function of the variables λ_i^2, and there equations will then imply that

$$\begin{aligned}\alpha_if(\lambda_1^2\cdots\lambda_n^2)&=f(\lambda_1^2\cdots(\lambda_j\pm\gamma)^2\cdots\lambda_n^2)\alpha_j\ ,\\ \beta_jf(\lambda_1^2\cdots\lambda_n^2)&=f(\lambda_1^2\cdots(\lambda_j\mp\gamma)^2\cdots\lambda_n^2)\alpha_j\ .\end{aligned} \tag{33}$$

Here we observe an important effect due to the change in sign of γ. So, it actually interchanges the role of the creation and annihilation operator, due to a change in sign of γ in nonlinear terms, which creates the important distinction between the bright and dark solitons [12].

The Hamiltonian of the system with finite boundary condition can be written as

$$\begin{aligned}t_n(\lambda)&=\mathrm{tr}K_+(\lambda)\mathcal{T}(\lambda)\\ &=(\lambda\pm\gamma+\xi_+)A(\lambda)+(\xi_+-\lambda\mp\gamma)D(\lambda)\ ,\end{aligned} \tag{34}$$

whence let us consider the action of such an operator on the symmetric function $O(\lambda_1^2\cdots\lambda_{n-1}^2)$:

$$\begin{aligned}t_n(\lambda)\phi(\lambda_1^2\cdots\lambda_{n-1}^2)&\\ =(\lambda\pm\gamma+\xi_+)&\left\{\left[\lambda^2K_2+K_2\lambda\sum_j\lambda_j+PK_2\sum_j\lambda_j^2\right]\prod_j^{2n-2}(\lambda-\lambda_j)+\sum_{j=1}^{2n-2}\frac{\lambda-\lambda_K}{\lambda_j-\lambda_K}\alpha_j\right\}\phi(\lambda_1^2\cdots\lambda_{n-1}^2)\\ &+(\xi_+\mp\gamma-\lambda)\left\{\left[\lambda^2K_2+K_2\sum_j\lambda_j^2+K_2\lambda\sum_j\lambda_j+R\right]\prod_{j=1}^{2n-2}(\lambda-\lambda_j)+\sum_{\substack{k=1\\ j\neq K}}^{2n-2}\frac{\lambda-\lambda_K}{\lambda_j-\lambda_K}\beta_j\right\}\phi(\lambda_1^2\cdots\lambda_{n-1}^2)\end{aligned} \tag{35}$$

Setting $\lambda\to\lambda_j$, we get

$$t_n(\lambda_j)\phi(\lambda_1^2\cdots\lambda_{n-1}^2)=(\lambda_j\pm\gamma+\xi_+)\phi(\lambda_1^2\cdots(\lambda_j\pm\gamma)^2\cdots\lambda_{n-1}^2)+(\xi_+-\lambda_j\mp\gamma)\phi(\lambda_1^2\cdots(\lambda_{j+}-\gamma)^2\cdots\lambda_{n-1}^2)\ . \tag{36}$$

We now seek ϕ as a separable function of the form

$$\phi=\prod_j \Psi(\lambda_j^2) . \tag{37}$$

Substituting in (36) we get

$$t(\lambda_j)\Psi(\lambda_j^2)=(\lambda_j\pm\gamma+\xi_+)\Psi(\lambda_j\pm\gamma^2)+(\xi_+-\lambda_j\mp\gamma)\Psi(\lambda_j-h) , \tag{38}$$

and we set

$$\Psi(u)^2=\prod_m (u-u_m)(u+u_m) ,$$

so we get

$$\prod_m \frac{(u_n-u_m\pm\gamma)(u_n+u_m\pm\gamma)}{(u_n-u_m\mp\gamma)(u_m+u_n\mp\gamma)}=\frac{(u_n\mp\gamma+\xi_+)}{(\xi_+-u_{n+}-\gamma)} , \tag{39}$$

which is the familiar type of coupled equations for the momenta U_m, which can be solved in the usual way after converting it into an integral equation.

DISCUSSION

In the above analysis we have considered the problem of quantizing the (x,t) interchanged nonlinear Schrödinger equation. Apart from the different form of L operator, the difficulty is the nonexistence of a pseudovacuum. Such a difficulty can be overcome by recourse to the functional Bethe ansatz approach. It may be noted that both cases of bright and dark solitons can be treated on the same footing, though the Bethe-type equation for the quasimomenta becomes different due to the interchanged roles of creation and annihilation operators. Lastly the equation giving the eigenmomenta is given explicitly.

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