Heisenberg-picture approach to the exact quantum motion of a time-dependent harmonic oscillator

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The generalized invariant and the exact quantum motions are found in the Heisenberg picture for a harmonic oscillator with time-dependent mass and frequency in terms of classical solutions. It is shown that the Heisenberg picture gives a relatively simpler picture than the Schrödinger picture and also manifestly exhibits the time independency of the invariant. We apply this method to the system with a linear sweep of frequency and Paul trap and study the squeezing properties.

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The explicitly time-dependent harmonic oscillators have been of great interest in various branches of physics with the coherent and squeezed-state formalism. There have been many methods for finding the exact quantum states, such as the generalized invariant method, path integral method, direct method to find quantum states in the Gaussian form, and Hermite polynomials, etc., and the solutions are found in many interesting cases. As one of the most typical and powerful methods, the generalized invariant method first introduced by Lewis [1], and Lewis and Riesenfeld (LR) [2] finds a generalized invariant, a quantum mechanical invariant in whose eigenstates the exact quantum states are found. Even though the generalized-invariant method has an advantage in providing the Fock space of number states easily compared with other methods, the generalized invariant does not rely on the Heisenberg picture, which is to play an important role in time-dependent quantum systems; in the original articles the generalized invariant and the Fock space are constructed in the Schrödinger picture. Thus the generalized invariant is still lacking in the Heisenberg picture.

In this paper, we shall establish the generalized invariant method in the Heisenberg picture to find the exact quantum motion in the general cases, where the mass and frequency change in time explicitly. We shall also find the exact quantum evolution of the position operator $q(t)$ and momentum operator $p(t)$ explicitly by solving Heisenberg equations of motion, where we use the generalized invariant in the Heisenberg picture. The coherent and squeezed-state formalism from the generalized invariant point of view [3] will be developed in the Heisenberg picture. Finally, we shall find the time evolution operator that converts the Heisenberg picture into Schrödinger picture. In this paper, all operators are Heisenberg-picture operators unless mentioned otherwise (using subscript S to denote the Schrödinger-picture operators).

Consider a time-dependent harmonic oscillator of the form

$$H(t) = \frac{1}{2m(t)}p^2(t) + \frac{1}{2}m(t)\omega^2(t)q^2(t) \tag{1}$$

whose mass and angular frequency depend on time explicitly. By introducing the Hermitian basis,

$$\begin{aligned} L_-(t) &= \frac{p^2(t)}{2}, \\ L_0\ (t) &= \frac{p(t)q(t)+q(t)p(t)}{2}, \\ L_+(t) &= \frac{q^2(t)}{2}. \end{aligned} \tag{2}$$

This basis forms the Lie algebra su(2) with the following group structure:

$$\left[\frac{i}{2}L_0, L_\pm\right] = \pm L_\pm,\ [L_+, L_-] = 2\left(\frac{i}{2}L_0\right). \tag{3}$$

We look for the generalized invariant of the form

$$I(t) = \sum_{k=0,\pm} g_k(t)L_k(t). \tag{4}$$

(Note that in this paper we are working in the Heisenberg picture, but in the original LR invariant method [2] in the Schrödinger picture.) The generalized invariant is truly a quantum mechanical conserved quantity in the Heisenberg picture and satisfies the Heisenberg equation of motion (in units of $\hbar = 1$)

$$\frac{d}{dt}I(t) = \frac{\partial}{\partial t}I(t) - i[I(t), H(t)] = 0. \tag{5}$$

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From the Lie algebra (3), Eq. (5) leads to a linear system of the first-order differential equations

$$\dot g_-(t) = -\frac{2}{m(t)} g_0(t),$$
$$\dot g_0(t) = m(t)\omega^2(t) g_-(t) - \frac{1}{m(t)} g_+(t), \qquad (6)$$
$$\dot g_+(t) = 2m(t)\omega^2(t) g_0(t).$$

Recently, we obtained the most general form of $g_k(t)$ [3],

$$\begin{aligned} g_-(t) &= c_1 f_1^2(t) + c_2 f_1(t) f_2(t) + c_3 f_2^2(t),\\ g_0(t) &= -m(t)\Big\{c_1 f_1(t)\dot f_1(t) + \frac{c_2}{2}\Big[\dot f_1(t) f_2(t) + f_1(t)\dot f_2(t)\Big] + c_3 f_2(t)\dot f_2(t)\Big\}, \qquad (7)\\ g_+(t) &= m^2(t)\Big[c_1 \dot f_1^2(t) + c_2 \dot f_1(t)\dot f_2(t) + c_3 \dot f_2^2(t)\Big], \end{aligned}$$

where $f_1(t)$ and $f_2(t)$ are two independent solutions of the classical equation of motion

$$\frac{d}{dt}\left(m(t)\frac{d}{dt}f(t)\right) + m(t)\omega^2(t) f(t) = 0. \qquad (8)$$

Since the generalized invariant thus obtained has still the same algebra as a harmonic oscillator, it can be expressed in terms of the creation and annihilation operators. By introducing the time-dependent creation and annihilation operators of the form

$$\begin{aligned} A^\dagger(t) &= \left(\sqrt{\frac{\omega_I}{2g_-(t)}} - i\sqrt{\frac{1}{2\omega_I g_-(t)}} g_0(t)\right) q(t) - i\sqrt{\frac{g_-(t)}{2\omega_I}} p(t),\\ A(t) &= \left(\sqrt{\frac{\omega_I}{2g_-(t)}} + i\sqrt{\frac{1}{2\omega_I g_-(t)}} g_0(t)\right) q(t) + i\sqrt{\frac{g_-(t)}{2\omega_I}} p(t), \qquad (9) \end{aligned}$$

where $\omega_I = \sqrt{g_+(t)g_-(t) - g_0^2(t)}$ is a constant of motion, we can rewrite the generalized invariant in the following form:

$$I(t) = \omega_I\left(A^\dagger(t)A(t) + \frac{1}{2}\right). \qquad (10)$$

The eigenstates are the number states

$$|n,t\rangle_I = \frac{A^{\dagger n}(t)}{\sqrt{n!}}|0,t\rangle_I, \qquad (11)$$

where the ground state is defined, as usual, by

$$A(t)|0,t\rangle_I = 0. \qquad (12)$$

On the other hand, the Hamiltonian can be re-expressed as

$$H(t) = \sum_{j=0,\pm} h_j(t) K_j(t) \qquad (13)$$

in terms of the spectrum generating algebra su(1,1) of the harmonic oscillator

$$K_-(t) = \frac{A^2(t)}{2},$$
$$K_0(t) = \frac{A^\dagger(t)A(t) + A(t)A^\dagger(t)}{4}, \qquad (14)$$
$$K_+(t) = \frac{A^{\dagger 2}(t)}{2},$$

where

$$h_0(t) = \frac{g_0^2(t) + m^2(t)\omega^2(t)g_-^2(t) + \omega_I^2}{g_-(t)m(t)\omega_I} \qquad (15)$$

and

$$h_\pm(t) = \frac{g_0^2(t) + m^2(t)\omega^2(t)g_-^2(t) - \omega_I^2 \mp 2ig_0(t)\omega_I}{2g_-(t)m(t)\omega_I}. \qquad (16)$$

A direct calculation using Eq. (6) and $[q(t),p(t)] = i$, results in the Heisenberg equations of motion of the creation and annihilation operators:

$$\frac{dA^\dagger(t)}{dt} = \frac{i\omega_I}{m(t)g_-(t)}A^\dagger(t),$$
$$\frac{dA(t)}{dt} = \frac{-i\omega_I}{m(t)g_-(t)}A(t),$$

whose solutions are given by

$$A^\dagger(t) = e^{i\Omega(t)}A^\dagger,\ A(t) = e^{-i\Omega(t)}A, \qquad (17)$$

where A and $A^\dagger$ denote the creation and annihilation operators at some initial time t_0 and the integral of generalized frequency is defined by

$$\Omega(t) = \int \frac{\omega_I}{m(t)g_-(t)}dt. \qquad (18)$$

Then we can write the generalized invariant in the form

$$I(t) = \omega_I\left(A^\dagger A + \frac{1}{2}\right), \qquad (19)$$

which clearly exhibits the time independency. This is a remarkable feature of the invariant in the Heisenberg picture but not in the Schrödinger picture. By applying Eq. (17) repeatedly and after fixing the phase of the $|0\rangle_I$ state properly, the eigenstates of the generalized invariant are given by

$$|n,t\rangle_I = e^{i\Omega(t)(n+\frac{1}{2})}|n\rangle_I, \qquad (20)$$

where $|n\rangle_I$ is an eigenstate at the fixed time t_0.

Furthermore, equating the Hermitian and anti-Hermitian parts of both sides of (17) separately, we deduce the quantum evolution of the Heisenberg operators $q(t)$ and $p(t)$:

$$q(t) = q(t_0)\sqrt{\frac{g_-(t)}{g_-(t_0)}}\left(\cos\Omega(t) + \frac{g_0(t_0)}{\omega_I}\sin\Omega(t)\right) + p(t_0)\frac{\sqrt{g_-(t)g_-(t_0)}}{\omega_I}\sin\Omega(t),$$

$$p(t) = q(t_0)\frac{1}{\sqrt{g_-(t_0)g_-(t)}}\left[[g_0(t_0) - g_0(t)]\cos\Omega(t) - \left(\omega_I + \frac{g_0(t_0)g_0(t)}{\omega_I}\right)\sin\Omega(t)\right] + p(t_0)\sqrt{\frac{g_-(t_0)}{g_-(t)}}\left(\cos\Omega(t) - \frac{g_0(t)}{\omega_I}\sin\Omega(t)\right). \quad (21)$$

When the oscillator has an asymptotic region, where the mass and frequency are constant in time, the generalized invariant can be set equal to the Hamiltonian itself and the invariant eigenstates become the energy eigenstates in this region. In this case, if the oscillator is initially in the ground state at the asymptotic region, then using Eq. (9) and Eq. (17) we can get the following result:

$$\left\langle [\Delta q(t)]^2 \right\rangle = \frac{g_-(t)}{2\omega_I},$$
$$\left\langle [\Delta p(t)]^2 \right\rangle = \frac{\omega_I}{2g_-(t)}\left(1 + \frac{g_0^2(t)}{\omega_I^2}\right). \quad (22)$$

In addition, if the oscillator is initially in the state

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle_I, \quad (23)$$

Eq. (22) remains valid. However, the expectation value $\langle q(t)\rangle$ oscillates with the generalized frequency

$$\langle\alpha|q(t)|\alpha\rangle = \left(\frac{2|\alpha|^2 g_-(t)}{\omega_I}\right)^{1/2}\cos[\Omega(t) + \delta], \quad (24)$$

the shape of the wave packet being changed by the squeezing. The physical properties of the state (23) will become apparent in the following.

Now we apply our method to several interesting harmonic oscillators. First, in the case of the harmonic oscillator with a time-independent mass m and frequency ω, by adjusting three parameters – or, equivalently, by adjusting two linearly independent solutions – we can set $g_-(t) = 1/m$ and $\omega_I = \omega$; then the quantum motions become

$$q(t) = q(t_0)\cos\omega(t - t_0) + \frac{p(t_0)}{m\omega}\sin\omega(t - t_0)$$
$$p(t) = -m\omega q(t_0)\sin\omega(t - t_0) + p(t_0)\cos\omega(t - t_0), \quad (25)$$

which are the same as the classical motions, and Eq. (23) becomes a coherent state. In addition, by adjusting those parameters we can also get the quantum motion of the squeezed state. For example, by setting $c_1 = 1/m$, $c_2 = 0$ and $c_3 = s^4/m$, and selecting two independent solutions to be $f_1(t) = \cos\omega t$ and $f_2(t) = \sin\omega t$, we get

$$g_-(t) = (1/m)(\cos^2\omega t + s^4\sin^2\omega t),$$
$$g_0\ (t) = (s^4 - 1)\omega\cos\omega t\sin\omega t, \quad (26)$$
$$g_+(t) = m\omega^2(\sin^2\omega t + s^4\cos^2\omega t).$$

The second case is the harmonic oscillator considered in Ref. [4] with a constant unit mass and the frequency changing linearly for some finite interval, i.e.,

$$\omega^2(t) = \begin{cases} \omega_0^2 \text{ for } -\infty < t < 0, \\ \omega_0^2(1 + \beta_0 t/T) \text{ for } 0 \le t \le T, \\ \omega_1^2 = \omega_0^2(1 + \beta_0) \text{ for } T < t < \infty. \end{cases} \quad (27)$$

If the oscillator is initially($t < 0$) in the ground state, we set two independent solutions to be $\cos\omega_0 t$ and $\sin\omega_0 t$ and adjust parameters $c_1 = c_3 = 1$ and $c_2 = 0$, so that $g_-(t) = 1$ and $\omega_I = \omega_0$ and, therefore, $|0\rangle_I$ is the ground state. However, the oscillator passes through the intermediate sweeping-frequency region, where the classical solutions are described by the Bessel functions of order 1/3, $\sqrt{\tau}J_{1/3}(z)$ and $\sqrt{\tau}Y_{1/3}(z)$, where $\tau = (\omega_0 t + \omega_0 T/\beta_0)^{1/2}$ and $z = 2/3(\beta_0\tau^3/\omega_0 T)^{1/2}$. Finally, the two classical solutions evolve into the form $C\cos\omega_1 t + D\sin\omega_1 t$, where the constants C and D are determined by the boundary condition at $t = 0$ and $t = T$ and the sweeping parameters β_0 and T. In the special case where two independent classical solutions evolve into $\cos\omega_1 t$ and $\sin\omega_1 t$ separately (this case will seldom or never occur), $g_-(t)$ may settle down to a constant and $g_0(t)$ may vanish, and we find from Eq. (22) that $\Delta q(t)$ and $\Delta p(t)$ are constant in time; however, since $g_-(t)$ suffers a certain amount of change, we find a significant amount of squeezing. In the general case, neither C nor D vanishes; as a result, $g_-(t)$ is the linear combination of $\cos^2\omega_1 t$, $\sin^2\omega_1 t$ and $\cos\omega_1 t\sin\omega_1 t$. Thus, the variance of the quadrature $q(t)$ presented numerically in Ref. [4] can be found analytically here. In the particular case of sudden jump ($T = 0$), we obtain explicitly

$$f_1(t) = \begin{cases} \cos\omega_0 t & \text{for } t < 0, \\ \cos\omega_1 t & \text{for } t > 0, \end{cases}$$
$$f_2(t) = \begin{cases} \sin\omega_0 t & \text{for } t < 0, \\ (\omega_0/\omega_1)\sin\omega_1 t & \text{for } t > 0, \end{cases}$$

and therefore

$$g_-(t) = \begin{cases} 1 & \text{for } t < 0 \\ (\omega_0/\omega_1)^2 + [1 - (\omega_0/\omega_1)^2]\cos^2\omega_1 t & \text{for } t > 0. \end{cases}$$

It is remarkable that although the states evolve into cosine and sine functions separately, $g_-(t)$ is time-dependent and thus the squeezing shape changes in time. Even in the case of adiabatic change, it is not evident that $g_-(t) = 1$ ($t > T$) and it is expected that the squeezing is produced, but slowly.

Our method also applies to the Paul trap, which is described by an oscillator with a constant mass m and periodic frequency $\omega^2(t) = a + b\cos\omega_0 t$. It is straightforward to get the quantum motion of the Paul trap, because its classical solutions are known to be Mathieu functions. In Ref. [5], its quantum state has been expressed as the eigenstates of the transformed Hamiltonian

$$\tilde{H} = \frac{W}{|f(t)|^2}\left(A^\dagger(t)A(t) + \frac{1}{2}\right), \quad (28)$$

where the definition of $A(t)$ and $A^\dagger(t)$ are equivalent to Eq. (9) provided that $f(t)$ and its complex conjugate $f^*(t)$ are chosen for two independent solutions and $c_1 = c_3 = 0$ and $c_2 = 1$. Then the Wronskian W is found to be $W = \omega_I/m$. In fact, the expression (28) is the generalized invariant (11) divided by $mg_-(t)$. This result arose from the wrong transformation $H_{\rm new} = H_{\rm old}(p_{\rm old}, q_{\rm old}) + U_c^{-1} i \frac{\partial U_c}{\partial t}$ (where U_c is a unitary operator), while the correct transformation should take the form

$$H_{\rm new} = U_c^{-1} H_{\rm old} U_c + U_c^{-1} i \frac{\partial U_c}{\partial t} = H_{\rm old}(p_{\rm new}, q_{\rm new}) + U_c^{-1} i \frac{\partial U_c}{\partial t}. \quad (29)$$

Therefore, the eigenstates (called quasienergy eigenstates in Ref. [5]) of (28) are not those of Hamiltonian $H(t)$ itself, but correspond to $|n,t\rangle_I$, which are the eigenstates of the generalized invariant. So one should pay attention to the fact that $:H(t): |0,t\rangle_I \neq 0$, where $:\;:$ denotes the normal ordering for the omission of the zero-point energy, and this is easily understandable from the form of Eq. (13).

Finally, we wish to find the time-evolution operator $U(t,t_0)$. In the Heisenberg picture, the position and momentum operators transform according to $q(t) = U^\dagger(t,t_0)q(t_0)U(t,t_0)$ and $p(t) = U^\dagger(t,t_0)p(t_0)U(t,t_0)$. By expressing these in terms of the creation and annihilation operators at t_0 and using the Eq. (9) and Eq. (17), we obtain

$$U^\dagger(t,t_0)A^\dagger(t_0)U(t,t_0) = u_1^* e^{i\Omega(t)}A^\dagger(t_0) - u_2 e^{-i\Omega(t)}A(t_0),$$

$$U^\dagger(t,t_0)A(t_0)U(t,t_0) = u_2^* e^{-i\Omega(t)}A^\dagger(t_0) - u_1 e^{i\Omega(t)}A(t_0),$$

where

$$u_1(t,t_0) = \frac{1}{2}\left(\sqrt{\frac{g_-(t)}{g_-(t_0)}} + \sqrt{\frac{g_-(t_0)}{g_-(t)}}\right) + \frac{i}{2}\left(\frac{g_0(t_0)}{\omega_I}\sqrt{\frac{g_-(t)}{g_-(t_0)}} - \frac{g_0(t)}{\omega_I}\sqrt{\frac{g_-(t_0)}{g_-(t)}}\right),$$

$$u_2(t,t_0) = \frac{1}{2}\left(-\sqrt{\frac{g_-(t)}{g_-(t_0)}} + \sqrt{\frac{g_-(t_0)}{g_-(t)}}\right) + \frac{i}{2}\left(\frac{g_0(t_0)}{\omega_I}\sqrt{\frac{g_-(t)}{g_-(t_0)}} - \frac{g_0(t)}{\omega_I}\sqrt{\frac{g_-(t_0)}{g_-(t)}}\right). \quad (30)$$

Using disentangling theorems for the su(1,1) [4], we can find

$$U(t,t_0) = e^{i\phi_1 A^\dagger(t_0)A(t_0)} \exp\left(\frac{\nu}{2}e^{-i(\phi_1-\phi_2)}A^{\dagger 2}(t_0) - \frac{\nu}{2}e^{i(\phi_1-\phi_2)}A^2(t_0)\right), \quad (31)$$

where ϕ_1, ϕ_2, and ν are related to $g_i(t_0)$ and $g_i(t)$ by

$$u_1^*(t,t_0)e^{i\Omega(t,t_0)} = (\cosh\nu)e^{-i\phi_1},$$
$$-u_2(t,t_0)e^{-i\Omega(t,t_0)} = (\sinh\nu)e^{-i\phi_2}. \quad (32)$$

And this time-evolution operator is the same form as Ref. [6], and is related to the squeeze operator of Ref. [3] as

$$U(t,t_0) = S^\dagger(t,t_0)e^{-i\Omega A^\dagger(t_0)A(t_0)}, \quad (33)$$

where the squeeze operator $S(t,t_0)$ is defined by

$$A_S^\dagger(t) = S^\dagger(t,t_0)A_S^\dagger(t_0)S(t,t_0) = u_1(t,t_0)A_S^\dagger(t_0) + u_2(t,t_0)A_S(t_0). \quad (34)$$

The state vector in Ref. [7] can also be found easily from $|\Psi(t)\rangle = U(t,t_0)\,|\Psi(t_0)\rangle$ with the global phase fixed properly.

In summary, we have found exactly the quantum motions of a time-dependent harmonic oscillator in the Heisenberg picture in terms of its classical solutions. The descriptions of this problem are given in Ref. [6]. In addition, our method enables one to find quantum motions in analytically closed forms, and to establish a connection between classical and quantum motions.

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