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Reconstruction of the quantum state of multimode light

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Phase-controlled homodyne $2(N+R)$-port detection with R local oscillators is analyzed with the aim of reconstructing the quantum state of a correlated N-mode signal field. It is shown that both the $(N+R)$-fold joint count distributions and the N-fold joint difference-count distributions contain all knowable information on the state of the field. In any case, the minimum number of local oscillators is given by the number of different signal-field frequencies. Two Fourier integrals per mode are found to be required to reconstruct the density matrix of the signal field from the joint difference-count distributions measured in balanced homodyning. To illustrate the theory, it is applied to balanced homodyne $4N$-port detection ($R=N$). In this scheme, the joint difference-count distributions directly yield the joint field-strength distributions of a correlated N-mode signal field.

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I. INTRODUCTION

Photoelectric detection plays an important role in gaining insight into the statistical properties of light. It is well known that in direct detection the photon statistics can be measured [1–3]. However, this knowledge does not comprise the phase information required for the determination of the quantum state of light.

Information on phase-sensitive light properties can be obtained by means of homodyne detection, where in the simplest case a signal-field mode whose properties are desired to be observed and a strong local-oscillator mode are combined by a beam splitter. Detecting the superimposed light, field strengths of the signal-field mode can be measured [4,5].

Optical homodyne detection has been studied intensively and applied successfully. The effect of detection efficiencies has been analyzed [6] and various multiport detection schemes have been considered [7]. A detailed analysis of balanced homodyne four-port detection has been given with special emphasis on the quantum properties of the local oscillator, the detection efficiencies, and the relation between the measured difference-count distributions and the field-strength distributions (also called rotated-quadrature distributions) of the signal-field mode [8,9]. Moreover, homodyne detection has been applied to the measurement of phase-sensitive properties of light, such as squeezing [10,11] and the phase statistics of quantized light modes [12–14].

Balanced homodyne four-port detection with input-phase control can be used to obtain the quantum state of the signal-field mode. It is well known that knowledge of the field-strength distributions of a radiation-field mode for all phases within a π interval is equivalent to knowledge of the s-parametrized pseudodistributions [15,16], which contain all information on the quantum state of the mode. Since the field-strength distributions are given by the difference-count distributions measured in (phase-controlled) balanced homodyne four-port detection (cf. [8,9,17]), the quantum state of the signal-field mode can be reconstructed in terms of pseudodistributions. It has further been shown that the measured field-strength distributions can also be used to directly reconstruct the quantum state of the signal-field mode in terms of the density matrix in a field-strength representation [18], without introduction of pseudodistributions. Alternatively, more complex homodyne detection schemes have been proposed to measure s-parametrized pseudodistributions (with $s\leq -1$) [19–22] or the positive P distribution [23,24].

Recently, experimental reconstruction of the quantum state of a radiation-field mode has been performed by using optical homodyne tomography [25]. The Wigner function of the signal mode has been reconstructed from the difference-count distributions measured in balanced homodyne four-port detection using inverse radon transformation [26], and the density matrix of the field has been obtained as the Fourier transform of the Wigner function. Although the results clearly demonstrate the experimental feasibility of the reconstruction of the quantum state of a radiation-field mode, there have been some open questions, such as the effect of the additional noise introduced by nonperfect detection and the influence of the multimode structure of the light pulses used in the experiments.

Apart from pulses, correlated multimode fields play an important role in a number of nonlinear optical processes, such as parametric down conversion and four-wave mixing. When the modes of a field are correlated to each other, single-mode measurements are not sufficient to obtain the full information on the quantum statistics of the field. In the present paper the problem of reconstruction of the quantum state of a correlated N-mode radiation field in phase-controlled homodyne $2(N+R)$-port detection (with a R-mode local-oscillator field) is considered. It is shown that the measured $(N+R)$-fold joint count distribution as a function of the phase parameters contains all knowable information on the quantum state of the N-mode signal field and can, therefore, be used to reconstruct the state of the field. In particular, the density matrix of the signal field can be obtained from the count distributions by Fourier-transformation techniques, which, in principle, enables one to also include the effects of non-perfect detection. It is further shown that the desired information on the quantum state of the signal field can also be obtained from the N-fold joint difference-count distributions measured in balanced homodyning. This may be more suitable for practical applications, due to the elimination of classical excess noise of the local oscillators. In this case the reconstruction of the signal-field density matrix can be accomplished with two Fourier integrals per mode.

The paper is organized as follows. In Sec. II the multimode-field density matrix is related to the expectation value of the (coherent) multimode-field displacement operator, which is shown to be a characteristic function of the density matrix. The relation between this characteristic function of the density matrix and the joint count distributions measured in multiport homodyne detection is studied in Sec. III. In Sec. IV the theory is extended to balanced multiport homodyning and illustrated by an example. A summary and some concluding remarks are given in Sec. V.

II. MULTIMODE-FIELD DENSITY MATRIX IN FIELD-STRENGTH REPRESENTATION

Let us consider a quantized N-mode radiation field, with the creation and destruction operators $\hat{a}_i^\dagger$ and $\hat{a}_i$, respectively,

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \quad i,j=1,2,\ldots,N. \tag{1}$$

Using standard quantum mechanics, the state of the field is described in terms of the density operator $\hat{\varrho}$. As we will see later, the density matrix of an N-mode signal field and the joint (difference-) count distributions that can be measured in (balanced) $2(N+R)$-port homodyning (R being the number of local-oscillator modes) are directly related to each other through their characteristic functions expressed in terms of the expectation value of the (coherent) multimode-field displacement operator. We first show that the expectation value of the (coherent) multimode-field displacement operator is a characteristic function of the density matrix of the field.

For this purpose we represent the density operator $\hat{\varrho}$ in the basis of the field-strength operator

$$\hat{F} = \sum_{j=1}^{N} \hat{F}_j(\varphi_{F_j}), \tag{2}$$

where

$$\hat{F}_j(\varphi_{F_j}) = F_j\,\hat{a}_j + F_j^*\,\hat{a}_j^\dagger, \quad F_j = |F_j|\,e^{i\varphi_{F_j}}. \tag{3}$$

The eigenkets of $\hat{F}$ may be written as

$$\left|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right\rangle = \prod_{j=1}^{N} \left|\mathcal{F}_j,\varphi_{F_j}\right\rangle, \tag{4}$$

where the $|\mathcal{F}_j,\varphi_{F_j}\rangle$ are the eigenkets of the single-mode operators $\hat{F}_j(\varphi_{F_j})$,

$$\hat{F}_j(\varphi_{F_j})\,\left|\mathcal{F}_j,\varphi_{F_j}\right\rangle = \mathcal{F}_j\,\left|\mathcal{F}_j,\varphi_{F_j}\right\rangle. \tag{5}$$

To calculate the density matrix elements in the $\hat{F}$ basis,

$$\begin{aligned}&\left\langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right|\hat{\varrho}\left|\{\mathcal{F}'_j\},\{\varphi_{F_j}\}\right\rangle \\ &= \mathrm{Tr}\left\{\hat{\varrho}\,\left|\{\mathcal{F}'_j\},\{\varphi_{F_j}\}\right\rangle\left\langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right|\right\},\end{aligned} \tag{6}$$

we note that $\hat{F}_j(\varphi_{F_j})$ and $\hat{F}_j(\varphi_{F_j}+\pi/2)$ satisfy the same type of commutation rule as position and momentum, so that standard concepts of quantum mechanics apply. Using in Eq. (6) the relations

$$\left|\{\mathcal{F}'_j\},\{\varphi_{F_j}\}\right\rangle = \exp\left[i\sum_{j=1}^{N}\frac{\mathcal{F}'_j-\mathcal{F}_j}{2|F_j|^2}\,\hat{F}_j\left(\varphi_{F_j}+\tfrac{1}{2}\pi\right)\right]\left|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right\rangle \tag{7}$$

and

$$\begin{aligned}\left|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right\rangle\left\langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}\right| &= \hat{\delta}\left(\{\mathcal{F}_j-\hat{F}_j\},\{\varphi_{F_j}\}\right)\\ &= \left(\frac{1}{2\pi}\right)^N\int dy_1\ldots\int dy_N\,\exp\left(-i\sum_{j=1}^{N}y_j\mathcal{F}_j\right)\exp\left(i\sum_{j=1}^{N}y_j\hat{F}_j(\varphi_{F_j})\right),\end{aligned} \tag{8}$$

straightforward calculation yields

$$\begin{aligned}\left\langle\{\mathcal{F}_j^{(1)}+\mathcal{F}_j^{(2)}\},\{\varphi_{F_j}\}\right|\hat{\varrho}\left|\{\mathcal{F}_j^{(1)}-\mathcal{F}_j^{(2)}\},\{\varphi_{F_j}\}\right\rangle &= \frac{1}{(2\pi)^N}\int dy_1\ldots\int dy_N\,\exp\left(-i\sum_{j=1}^{N}y_j\mathcal{F}_j^{(1)}\right)\\ &\quad\times\left\langle\hat{D}\left(\left\{-\frac{\mathcal{F}_j^{(2)}F_j^*}{|F_j|^2}+iy_jF_j^*\right\}\right)\right\rangle,\end{aligned} \tag{9}$$

where

$$\hat{D}(\{\beta_j\}) = \prod_{j=1}^{N} \hat{D}_j(\beta_j) = \exp\left[\sum_{j=1}^{N} (\beta_j\, \hat{a}_j^\dagger - \beta_j^*\, \hat{a}_j)\right] \quad (10)$$

is the familiar (coherent) multimode-field displacement operator, and

$$\mathcal{F}_j = \mathcal{F}_j^{(1)} + \mathcal{F}_j^{(2)}, \quad \mathcal{F}_j' = \mathcal{F}_j^{(1)} - \mathcal{F}_j^{(2)}. \quad (11)$$

Equation (9) indeed reveals that (for appropriately chosen arguments) the expectation value of the (coherent) displacement operator of a multimode radiation field is a characteristic function of the density matrix of the field. Knowing this characteristic function, the density matrix in a field-strength basis can simply be obtained by Fourier transformation, where one Fourier integral per mode must be performed. In this way we find that any measurement scheme that enables one to determine $\langle\hat{D}(\{\beta_j\})\rangle$ for all complex values of β_j $(j = 1,\ldots,N)$ yields all knowable information on the quantum state of a N-mode radiation field. Note that if we let $\mathcal{F}_j^{(2)} = 0$ $(j \neq k)$ in Eq. (9) and integrate over all $\mathcal{F}_j^{(1)}$ $(j \neq k)$, we obtain the single-mode density matrix of the kth mode $(k = 1,\ldots,N)$ [18], which of course does not comprise information on the quantum correlations of the modes.

The state of the field in terms of pseudoprobabilities can also be obtained from $\langle\hat{D}(\{\beta_j\})\rangle$, since this quantity is uniquely related to the characteristic functions of s-parametrized pseudodistributions [16]. However, one more Fourier integral per mode must be performed, because of the complex integration variables.

Finally, it should be noted that when $\mathcal{F}_j^{(2)} = 0$ $(j = 1,\ldots,N)$ from Eq. (9) the joint field-strength distribution $p(\{\mathcal{F}_j\},\{\varphi_{F_j}\})$ is given by

$$p(\{\mathcal{F}_j\},\{\varphi_{F_j}\}) = \langle\{\mathcal{F}_j\},\{\varphi_{F_j}\}|\hat{\varrho}|\{\mathcal{F}_j\},\{\varphi_{F_j}\}\rangle$$
$$= \frac{1}{(2\pi)^N}\int dy_1 \ldots \int dy_N \exp\left(-i\sum_{j=1}^{N} y_j\mathcal{F}_j\right) \Psi(\{y_j\},\{\varphi_{F_j}\}), \quad (12)$$

where the characteristic function $\Psi(\{y_j\},\{\varphi_{F_j}\})$ of the distribution $p(\{\mathcal{F}_j\},\{\varphi_{F_j}\})$ can be obtained from the expectation value of the (coherent) displacement operator as

$$\Psi(\{y_j\},\{\varphi_{F_j}\}) = \left\langle \hat{D}(\{iy_jF_j^*\})\right\rangle. \quad (13)$$

III. MULTIPORT HOMODYNING

In this section we show that the characteristic function of a N-mode-field density matrix as introduced in Eqs. (9) and (10) can directly be related to the characteristic functions of the $(N{+}R)$-fold joint count distributions measured in homodyne $2(N+R)$-port detection.

Let us suppose that the N modes of the field under consideration can be separated from each other so that each mode can be used as an input-signal mode in a multiport linear device. In particular, to detect a field that consists of modes of different frequencies ω_r $(r=1,\ldots,R)$, we assume that the set of modes can be subdivided into R groups, $N=\sum_{r=1}^{R} N_r$, where the N_r modes belonging to the rth group are equal in frequency. We further assume that the coherent reference field consists of R modes of frequencies ω_r $(r = 1,\ldots,R)$, so that each group of the signal-field modes can be assigned to a local oscillator (whose frequency is equal to the frequency of the modes of the group). This implies a detection scheme, where a linear lossless $2(N+R)$-port apparatus that consists of R subdevices is used as shown in Fig. 1. The N_r signal modes of the rth group and the associated local-oscillator mode are combined by the rth subdevice to give N_r+1 output modes. The $N{+}R$ output modes of all the subdevices fall on photodetectors and the $(N+R)$-joint count distribution is measured.

Since the rth subdevice is a linear lossless $2(N_r+1)$-port apparatus that transforms N_r+1 input modes into N_r+1 output modes, we may write

$$\hat{b}_k^{(r)} = \sum_{k'=1}^{N_r+1} U_{kk'}^{(r)}\hat{a}_{k'}^{(r)} \quad (r = 1,\ldots,R), \quad (14)$$

where $\hat{a}_k^{(r)}$ and $\hat{b}_k^{(r)}$ are the photon destruction operators

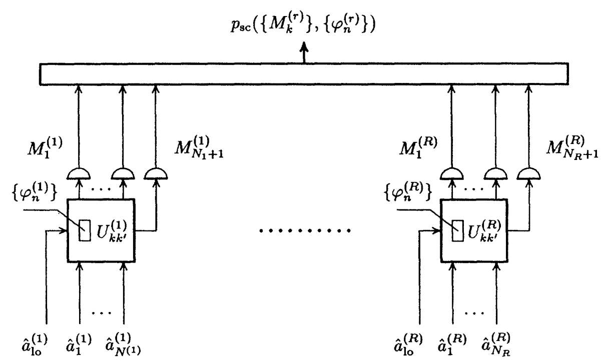

FIG. 1. Scheme of homodyne $2(N{+}R)$-port detection. For chosen r $(r=1,\ldots,R)$ the signal modes $\hat{a}_k^{(r)}$ $(k=1,\ldots,N_r)$ of frequency $\omega^{(r)}$ and a strong local oscillator $\hat{a}_{N_r+1}^{(r)} \equiv \hat{a}_{lo}^{(r)}$ of frequency $\omega^{(r)}$ are combined by a lossless linear $2(N_r{+}1)$-port device to give $N_r{+}1$ output modes $(U_{kk'}^{(r)}$, unitary transformation matrix of the rth device; $\{\varphi_n^{(r)}\}$, phase control of the rth device). Simultaneous detection of the $N+R$ output modes $(N=\sum_{r=1}^{R})$ yields the $(N+R)$-fold joint (scaled) count distributions $p_{sc}(\{M_k^{(r)}\},\{\varphi_n^{(r)}\})$.

of the input and output modes, respectively, and $U_{kk'}^{(r)}$ is a unitary matrix,

$$\sum_{k''=1}^{N_r+1} U_{kk''}^{(r)} (U_{k'k''}^{(r)})^* = \sum_{k''=1}^{N_r+1} U_{k''k}^{(r)} (U_{k''k'}^{(r)})^* = \delta_{kk'}. \tag{15}$$

Note that any discrete finite-dimensional unitary matrix can be constructed in the laboratory using devices, such as beam splitters, phase shifters, and mirrors [27].

Altogether, $N+R$ input modes are transformed into $N+R$ output modes, which can be detected using standard photoelectric detectors to obtain the joint count distribution

$$p(\{m_k^{(r)}\}) \equiv p\left(m_1^{(1)}, \ldots, m_{N_1+1}^{(1)}, \ldots, m_1^{(R)}, \ldots, m_{N_R+1}^{(R)}\right). \tag{16}$$

From photodetection theory [2,3] it is well known that the characteristic function of $p(\{m_k^{(r)}\})$,

$$\Omega(\{x_k^{(r)}\}) = \sum_{\{m_k^{(r)}\}} p(\{m_k^{(r)}\}) \exp\left(i \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} x_k^{(r)} m_k^{(r)}\right), \tag{17}$$

can be represented as

$$\Omega(\{x_k^{(r)}\}) = \left\langle : \exp\left[\sum_{r=1}^{R} \sum_{k=1}^{N_r+1} \left(e^{ix_k^{(r)}} - 1\right) \eta_k^{(r)} \hat{n}_k^{(r)}\right] : \right\rangle. \tag{18}$$

Here $\hat{n}_k^{(r)}$ are the photon-number operators of the output modes,

$$\hat{n}_k^{(r)} = (\hat{b}_k^{(r)})^\dagger \hat{b}_k^{(r)}, \tag{19}$$

and $\eta_k^{(r)}$ are the efficiencies of the associated photodetectors. The notation : : indicates normal order.

We now assume that the (N_r+1)th input mode in the rth subdevice $(r=1,\ldots,R)$ is prepared in a coherent state $|\alpha_r\rangle$ $(\alpha_r = |\alpha_r| e^{i\varphi_{\alpha_r}})$ with large values of $|\alpha_r|$ (strong local oscillators). To obtain the asymptotic behavior of the joint count distribution for large values of $|\alpha_r|$, it is convenient to introduce scaled counts

$$M_k^{(r)} = \frac{m_k^{(r)} - \eta_k^{(r)} |U_{k\,N_r+1}^{(r)}|^2 |\alpha_r|^2}{\eta_k^{(r)} |\alpha_r|} \tag{20}$$

$(k=1,\ldots,N_r+1,\ r=1,\ldots,R)$ and to consider the joint scaled count distribution

$$p_{\rm sc}\left(\{M_k^{(r)}\}\right) = p\left(\left\{\eta_k^{(r)} |\alpha_r| M_k^{(r)} + \eta_k^{(r)} |U_{k\,N_r+1}^{(r)}|^2 |\alpha_r|^2\right\}\right). \tag{21}$$

Its characteristic function is given by

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \sum_{\{M_k^{(r)}\}} p_{\rm sc}\left(\{M_k^{(r)}\}\right) \times \exp\left(i \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} x_k^{(r)} M_k^{(r)}\right), \tag{22}$$

where for large values of $|\alpha_r|$ the Fourier sums may be performed as Fourier integrals. Using Eqs. (17) and (20), the characteristic functions $\Omega_{\rm sc}(\{x_k^{(r)}\})$ and $\Omega(\{x_k^{(r)}\})$ are related to each other as

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \Omega\left(\left\{\frac{x_k^{(r)}}{\eta_k^{(r)} |\alpha_r|}\right\}\right) \times \exp\left\{-i \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} |U_{k\,N_r+1}^{(r)}|^2 |\alpha_r| x_k^{(r)}\right\}. \tag{23}$$

Note that the $x_k^{(r)}$ are real variables. We now combine Eqs. (23) and (18) and use Eq. (19) together with (14). Following Ref. [7], we expand in the resulting expression for $\Omega_{\rm sc}(\{x_k^{(r)}\})$ the exponential $\exp[ix_k^{(r)}/\eta_k^{(r)}|\alpha_r|]$ into a power series $(|x_k^{(r)}|/(\eta_k^{(r)}|\alpha_r|) \ll 1)$ and omit terms of higher than second order in $|x_k^{(r)}|/(\eta_k^{(r)}|\alpha_r|)$. After some calculation we obtain, on keeping only the leading terms,

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \exp\left[-\frac{1}{2} \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} |U_{k\,N_r+1}^{(r)}|^2 \frac{(x_k^{(r)})^2}{\eta_k^{(r)}}\right] \times \left\langle : \exp\left\{\sum_{r=1}^{R} \sum_{n=1}^{N_r} \sum_{k=1}^{N_r+1} \left[ie^{i\varphi_{\alpha_r}} x_k^{(r)} U_{k\,N_r+1}^{(r)} (U_{kn}^{(r)})^* (\hat{a}_n^{(r)})^\dagger - \text{H.c.}\right]\right\} : \right\rangle, \tag{24}$$

which may be rewritten as

$$\Omega_{\rm sc}(\{x_k^{(r)}\}) = \exp\left[-\frac{1}{2} \sum_{r=1}^{R} \sum_{k=1}^{N_r+1} |U_{k\,N_r+1}^{(r)}|^2 \frac{(x_k^{(r)})^2}{\eta_k^{(r)}}\right] \times \exp\left[\frac{1}{2} \sum_{r=1}^{R} \sum_{n=1}^{N_r} \left|\sum_{k=1}^{N_r+1} U_{k\,N_r+1}^{(r)} (U_{kn}^{(r)})^* x_k^{(r)}\right|^2\right] \left\langle \hat{D}\left(\{ie^{i\varphi_{\alpha_r}} z_n^{(r)}\}\right)\right\rangle, \tag{25}$$

where $\hat{D}(\{\beta_n^{(r)} = ie^{i\varphi_{\alpha_r}} z_n^{(r)}\})$ is the (coherent) displacement operator of the N-mode signal field,

$$\hat{D}(\{\beta_n^{(r)}\}) = \exp\left[\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left(\beta_n^{(r)}(\hat{a}_n^{(r)})^\dagger - \mathrm{H.c.}\right)\right] \tag{26}$$

[cf. Eq. (10)], and

$$z_k^{(r)} = \sum_{k'=1}^{N_r+1} U_{k'\,N_r+1}^{(r)}(U_{k'k}^{(r)})^* x_{k'}^{(r)} \tag{27}$$

$(k=1,\ldots,N_r+1,\ r=1,\ldots,R)$.

Recalling Eqs. (9) and (10), from Eq. (25) we see that the characteristic function of the joint (scaled) count distribution, $\Omega_{\rm sc}(\{x_k^{(r)}\})$, can directly be obtained from the characteristic function of the density matrix of the N-mode signal field, $\langle\hat{D}(\{\beta_n^{(r)}\})\rangle$.

To obtain the characteristic function of the density matrix from the characteristic function of the joint count distribution, we invert Eq. (27), on using the relations (15),

$$x_k^{(r)} = \sum_{k'=1}^{N_r+1} \frac{U_{kk'}^{(r)}}{U_{k\,N_r+1}^{(r)}} z_{k'}^{(r)}, \tag{28}$$

and rewrite Eq. (25) as

$$\left\langle\hat{D}\left(\left\{ie^{i\varphi_{\alpha_r}} z_n^{(r)}\right\}\right)\right\rangle = \Omega_{\rm sc}\left(\left\{x_k^{(r)} = \sum_{k'=1}^{N_r+1}\frac{U_{kk'}^{(r)}}{U_{k\,N_r+1}^{(r)}} z_{k'}^{(r)}\right\}\right)\exp\left[-\frac{1}{2}\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left|z_n^{(r)}\right|^2\right]$$
$$\times\exp\left[\frac{1}{2}\sum_{r=1}^{R}\sum_{k',k''=1}^{N_r+1}\left(\sum_{k=1}^{N_r+1}\frac{U_{kk'}^{(r)}(U_{kk''}^{(r)})^*}{\eta_k^{(r)}}\right) z_{k'}^{(r)}\left(z_{k''}^{(r)}\right)^*\right] \tag{29}$$

$(n=1,\ldots,N_r)$. From inspection of Eq. (29) we see that the left-hand side of this equation does not depend on the $z_{N_r+1}^{(r)}$ $(r=1,\ldots,R)$. With regard to high-precision state determination we, therefore, let

$$z_{N_r+1}^{(r)} = 0 \tag{30}$$

$(r=1,\ldots,R)$, otherwise the $z_{N_r+1}^{(r)}$ would give rise to a growing exponential leading to an artificial enhancement of experimental inaccuracies even in perfect detection $(\eta_k^{(r)}=1)$. Note that the conditions (30) correspond to

$$\sum_{k=1}^{N_r+1}\left|U_{k\,N_r+1}^{(r)}\right|^2 x_k^{(r)} = 0 \tag{31}$$

[see Eq. (27)], which means that for each r one of the $x_k^{(r)}$ is given by the remaining ones, e.g.,

$$x_{k_r}^{(r)} = -\sum_{\substack{l=1\\ l\neq k_r}}^{N_r+1}\frac{\left|U_{l\,N_r+1}^{(r)}\right|^2}{\left|U_{k_r\,N_r+1}^{(r)}\right|^2} x_l^{(r)} \tag{32}$$

$(1\leq k_r\leq N_r+1)$, and Eq. (28) reduces to

$$\sum_{n=1}^{N_r}\frac{U_{ln}^{(r)}}{U_{l\,N_r+1}^{(r)}} z_n^{(r)} = x_l^{(r)} \quad (l\neq k_r). \tag{33}$$

In other words, to obtain the characteristic function of the N-mode signal field, we may restrict attention to a N-dimensional hypersurface in the space of the $N+R$ variables of the characteristic function of the (scaled) joint count distribution $\Omega_{\rm sc}(\{x_k^{(r)}\})$.

Since the $x_k^{(r)}$ are real, for chosen unitary matrices $U_{kk'}^{(r)}$ the allowed complex values of the $z_n^{(r)}$ in Eq. (29) are restricted to those satisfying the condition that the expression on the left-hand side in Eq. (33) is real. This means, from a single measurement performed by an apparatus with chosen mode-mixing properties the characteristic function of the state of the signal field, $\langle\hat{D}(\{\beta_n^{(r)} = ie^{i\varphi_{\alpha_r}} z_n^{(r)}\})\rangle$, can only be obtained for certain complex values of its arguments $\beta_n^{(r)}$.

To obtain the characteristic function for arbitrary arguments, the $z_n^{(r)}$ must of course be allowed to attain arbitrary complex values. This can be achieved by appropriately varying the transformation matrices $U_{kk'}^{(r)}$, which implies measurement of a set of count distributions, for example, by using phase shifters in the apparatus and varying the phase parameters from measurement to measurement. As mentioned, any discrete finite-dimensional unitary matrix can be constructed in the laboratory [27]. In this scheme we may, therefore, regard the $U_{kk'}^{(r)}$ as being given in a parametrized form $U_{kk'}^{(r)}(\{\varphi_n^{(r)}\})$ $(n=1,\ldots,N_r)$ and choose, for a given set of complex values of $z_n^{(r)}$, the parameters $\varphi_n^{(r)}$ in such a way that the left-hand side in Eq. (33) is real for all values of l $(l\neq k_r)$.

In particular, starting with real $U_{kk'}^{(r)}$, the situation is rather simple. In this case, insertion of phase shifters in the input channels of the signal modes, e.g.,

$$U_{kk'}^{(r)}\left(\{\varphi_n^{(r)}\}\right) = e^{i\varphi_{k'}^{(r)}} U_{kk'}^{(r)} \tag{34}$$

$(\varphi_{N_r+1}^{(r)}=0)$, obviously implies that the expression on the left-hand side in Eq. (33) is always real, provided that

$$z_n^{(r)} = \exp\left[-i\varphi_n^{(r)}\right] y_n^{(r)} \qquad (y_n^{(r)},\ \text{real}). \tag{35}$$

Hence, if the characteristic functions of the joint (scaled) count distributions are known for all values of the phase parameters $\varphi_n^{(r)}$ within π intervals, the characteristic function of the density matrix of the signal-radiation field is known as well.

For simplicity we assume that the photodetectors used have equal efficiencies ($\eta_k^{(r)} = \eta_{k'}^{(r')} \equiv \eta$). From Eq. (29) $[U_{kk'}^{(r)} \to U_{kk'}^{(r)}(\{\varphi_l^{(r)}\})]$ together with Eq. (30) and the relations (15) we see that

$$\left\langle \hat{D}\left(\left\{ie^{i\varphi_{\alpha_r}} z_n^{(r)}\right\}\right)\right\rangle = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left|z_n^{(r)}\right|^2\right]\Omega_{\rm sc}\left(\left\{x_k^{(r)} = \sum_{n=1}^{N_r}\frac{U_{kn}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)}{U_{k\,N_r+1}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)} z_n^{(r)}\right\}, \{\varphi_n^{(r)}\}\right), \tag{36}$$

which in the case of perfect detection ($\eta = 1$) reduces to

$$\left\langle \hat{D}\left(\left\{ie^{i\varphi_{\alpha_r}} z_n^{(r)}\right\}\right)\right\rangle = \Omega_{\rm sc}\left(\left\{x_k^{(r)} = \sum_{n=1}^{N_r}\frac{U_{kn}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)}{U_{k\,N_r+1}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)} z_n^{(r)}\right\}, \{\varphi_n^{(r)}\}\right). \tag{37}$$

Here the notation $\Omega_{\rm sc}(\{x_k^{(r)}\}, \{\varphi_n^{(r)}\})$ is used to explicitly indicate the dependence on the $\varphi_n^{(r)}$ of the measured distributions and their characteristic functions.

Equation (37) shows that the characteristic function of the density matrix of the N-mode signal field can directly be obtained, for appropriately chosen arguments, from the characteristic functions of the joint (scaled) count distributions measured in perfect homodyne $2(N+R)$-port detection with phase control. Hence, the quantum state of the signal field expressed in terms of the density matrix can be reconstructed by straightforward Fourier transformation [Eq. (9)].

When the count distributions are measured by photodetectors whose efficiencies are less than unity, the characteristic functions of the count distributions must be corrected by an inverse Gaussian [according to Eq. (36)], which obviously corresponds to a deconvolution in the Fourier space. It should be pointed out that the appearance of the inverse Gaussian requires a very careful consideration of the experimental inaccuracies.

IV. BALANCED MULTIPORT HOMODYNING

From Sec. III we know that when the characteristic function of the density matrix of the N-mode signal field is related to the characteristic functions of the joint count distributions, $\Omega(\{x_k^{(r)}\})$, the originally $(N+R)$-dimensional space of independent variables $x_k^{(r)}$ can be reduced to a N-dimensional hypersurface in this space. To reconstruct the quantum state of the N-mode signal field, it should, therefore, be sufficient to measure N-fold joint count distributions in place of $(N+R)$-fold ones. As it is shown below, the reduction conditions (30) used in Sec. III just imply that the characteristic functions of N-fold joint difference-count distributions are picked out. Since the joint difference-count distributions can be measured in balanced homodyning, both a reduction of data and, similarly to balanced four-port homodyning [28–31], an elimination of classical excess noise of the local oscillators are feasible.

Let us return to Eq. (20) and introduce $N = \sum_{r=1}^{R} N_r$ scaled difference counts

$$D_{lk_r}^{(r)} = M_l^{(r)} - \frac{\left|U_{l\,N_r+1}^{(r)}\right|^2}{\left|U_{k_r\,N_r+1}^{(r)}\right|^2} M_{k_r}^{(r)} \tag{38}$$

($l = 1, \ldots, k_r - 1, k_r + 1, \ldots, N_r + 1$, $r = 1, \ldots, R$) and consider the joint scaled difference-count distribution

$$p_{\rm sdc}\left(\{D_{lk_r}^{(r)}\}\right) = \sum_{\{M_{k_r}^{(r)}\}} p_{\rm sc}\left(\left\{D_{lk_r}^{(r)} + \frac{\left|U_{l\,N_r+1}^{(r)}\right|^2}{\left|U_{k_r\,N_r+1}^{(r)}\right|^2} M_{k_r}^{(r)}, M_{k_r}^{(r)}\right\}\right), \tag{39}$$

with the characteristic function

$$\Omega_{\rm sdc}\left(\{x_l^{(r)}\}\right) = \sum_{\{D_{lk_r}^{(r)}\}} p_{\rm sdc}\left(\{D_{lk_r}^{(r)}\}\right) \exp\left(i\sum_{r=1}^{R}\sum_{\substack{l=1\\ l\neq k_r}}^{N_r+1} x_l^{(r)} D_{lk_r}^{(r)}\right). \tag{40}$$

In Eq. (39) the notation $\{\ldots, M_{k_r}^{(r)}\}$ is introduced to indicate that (for given r) the k_rth argument in $p_{\rm sc}$ is not changed. Combining Eqs. (40), (39), and (22), we easily see that

$$\Omega_{\rm sdc}\left(\{x_l^{(r)}\}\right) = \Omega_{\rm sc}\left(\{x_k^{(r)}\}\right), \tag{41}$$

with

$$x_{k_r}^{(r)} = -\sum_{\substack{l=1\\ l\neq k_r}}^{N_r+1}\frac{\left|U_{l\,N_r+1}^{(r)}\right|^2}{\left|U_{k_r\,N_r+1}^{(r)}\right|^2} x_l^{(r)} \tag{42}$$

(recall that $l = 1, \ldots, k_r - 1, k_r + 1, \ldots N_r + 1$ whereas $k = 1, \ldots, N_r + 1$).

Since the relations (42) are just the conditions (30) given in the form (32), Eq. (41) implies that the results obtained in Sec. III using the conditions (30) remain valid when the characteristic functions of the joint (scaled) count distributions, $\Omega_{\rm sc}$, are simply replaced by the characteristic functions of the joint (scaled) difference-count distributions, $\Omega_{\rm sdc}$. In particular, in place of Eq. (36) we obtain

$$\left\langle \hat{D}\left(\{ie^{i\varphi_{\alpha_r}} z_n^{(r)}\}\right)\right\rangle = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left|z_n^{(r)}\right|^2\right]\Omega_{\rm sdc}\left(\left\{x_l^{(r)} = \sum_{n=1}^{N_r}\frac{U_{ln}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)}{U_{l\,N_r+1}^{(r)}\left(\{\varphi_{n'}^{(r)}\}\right)}z_n^{(r)}\right\}, \{\varphi_n^{(r)}\}\right), \tag{43}$$

which reveals that the characteristic function of the density matrix of the N-mode signal field can directly be obtained from the characteristic functions of the N-fold joint (scaled) difference-count distributions measured in phase-controlled balanced homodyne $2(N+R)$-port detection. Knowing the joint difference-count distributions, two Fourier integrals per mode must be performed to obtain the density matrix of the signal field. For example, starting with an apparatus that performs real transformations $U_{kk'}^{(r)}$, Eq. (43) reads as, on using Eqs. (34) and (35),

$$\left\langle \hat{D}\left(\{\beta_n^{(r)} = ie^{i(\varphi_{\alpha_r}-\varphi_n^{(r)})}y_n^{(r)}\}\right)\right\rangle = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{R}\sum_{n=1}^{N_r}\left|y_n^{(r)}\right|^2\right]\Omega_{\rm sdc}\left(\left\{x_l^{(r)} = \sum_{n=1}^{N_r}\frac{U_{ln}^{(r)}}{U_{l\,N_r+1}^{(r)}}y_n^{(r)}\right\}, \{\varphi_n^{(r)}\}\right) \tag{44}$$

($y_n^{(r)}$, real). For each chosen set of input-phase parameters $\varphi_n^{(r)}$ the joint difference-count distribution can be measured and its characteristic function can be obtained by an N-fold Fourier transformation. Varying the $y_n^{(r)}$ ($-\infty \le y_n^{(r)} \le +\infty$), the arguments of the characteristic function of the count distribution can be assigned to the arguments of the characteristic function of the density matrix as indicated in Eq. (44). In this way $\langle \hat{D}(\{\beta_n^{(r)}\})\rangle$ is obtained for the arguments $\beta_n^{(r)}$ along the lines $\beta_n^{(r)} = ie^{i(\varphi_{\alpha_r}-\varphi_n^{(r)})}y_n^{(r)}$ in the complex $\beta_n^{(r)}$-planes. Varying the phase parameters $\varphi_n^{(r)}$ within π intervals and repeating the procedure outlined above, $\langle \hat{D}(\{\beta_n^{(r)}\})\rangle$ can be obtained for all arguments. Now the N-fold Fourier transformation of $\langle \hat{D}(\{\beta_n^{(r)}\})\rangle$ may be performed to obtain the signal-field density matrix [cf. Eqs. (9) and (10)].

Although at least one local oscillator per group of signal-field modes of equal frequencies is needed to reconstruct the quantum state of the signal field, in practice it may be advantageous to increase the number of local oscillators and accordingly reduce the dimensions of the transformation matrices $U_{kk'}^{(r)}$ ($r=1,\ldots,R$).

To illustrate the theory developed let us consider a scheme where each signal mode is assigned to a local oscillator, so that $N_r = 1$ and $R = N$. In this case the overall detection scheme simply consists of N four-port devices (N unitary 2×2 matrices), which can be realized by single beam splitters (Fig. 2). Supposing that a phase shifter is inserted in the second input channel (local-oscillator) of each four-port device, we may write

$$U_{kk'}^{(r)}\left(\varphi_2^{(r)}\right) = e^{i\varphi_{k'}^{(r)}}U_{kk'}^{(r)} \quad \left(\varphi_{k'}^{(r)} = \delta_{k'2}\,\varphi_2^{(r)}\right), \tag{45}$$

$$U_{kk'}^{(r)} = e^{i\varphi_{kk'}^{(r)}}\left|U_{kk'}^{(r)}\right| \tag{46}$$

($r=1,\ldots,N$; $k,k'=1,2,$). The condition that the $U_{kk'}^{(r)}$ are unitary matrices obviously implies that

$$\left|U_{12}^{(r)}\right|^2 + \left|U_{22}^{(r)}\right|^2 = 1, \tag{47}$$

$$\left|U_{11}^{(r)}\right| = \left|U_{22}^{(r)}\right|, \quad \left|U_{21}^{(r)}\right| = \left|U_{12}^{(r)}\right|, \tag{48}$$

$$\varphi_{11}^{(r)} - \varphi_{21}^{(r)} + \varphi_{22}^{(r)} - \varphi_{12}^{(r)} = \pm\pi. \tag{49}$$

In particular, when the photodetectors have equal efficiencies ($\eta_k^{(r)} = \eta_{k'}^{(r')} \equiv \eta$) and the four-port devices are 50%:50% ($|U_{12}^{(r)}|^2 = |U_{22}^{(r)}|^2 = 1/2$), application of Eq. (43) yields ($k_r = 2$)

$$\left\langle \hat{D}\left(\{ie^{i\varphi_{\alpha_r}} z^{(r)}\}\right)\right\rangle = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{N}\left|z^{(r)}\right|^2\right]\Omega_{\rm sdc}\left(\left\{e^{i(\varphi_{11}^{(r)}-\varphi_{12}^{(r)}-\varphi_2^{(r)})}z^{(r)}\right\}, \{\varphi_2^{(r)}\}\right). \tag{50}$$

Choosing

$$z^{(r)} = x^{(r)}e^{i\phi^{(r)}} \tag{51}$$

($x^{(r)}$, real), where

$$\phi^{(r)} = -\varphi_{11}^{(r)} + \varphi_{12}^{(r)} + \varphi_2^{(r)}, \tag{52}$$

we easily see that [according to the condition (33)] the arguments of $\Omega_{\rm sdc}$ are real. Hence, Eq. (50) may be rewritten as

$$\left\langle \hat{D}\left(\{\beta^{(r)}\}\right)\right\rangle = \exp\left[\frac{1-\eta}{2\eta}\sum_{r=1}^{N}\left(x^{(r)}\right)^2\right]\Omega_{\rm sdc}\left(\{x^{(r)}\}, \{\varphi_2^{(r)}\}\right), \tag{53}$$

where

$$\beta^{(r)} = ix^{(r)}\exp\left[i\left(\varphi_{\alpha_r} - \varphi_{11}^{(r)} + \varphi_{12}^{(r)} + \varphi_2^{(r)}\right)\right], \tag{54}$$

from which we see that the phase control by the phase shifters (which corresponds to a change of the local-oscillator phases) effectively reduces to a variation of the $\varphi_2^{(r)}$ within π intervals. Recalling Eqs. (20) and (38), in the case under consideration the scaled difference counts $D_{12}^{(r)}$ are related to the observed (unscaled) difference counts $d_{12}^{(r)} = m_1^{(r)} - m_2^{(r)}$ as

$$D_{12}^{(r)} = \frac{d_{12}^{(r)}}{\eta|\alpha_r|}\,, \tag{55}$$

which implies that the corresponding distributions are related to each other as

$$p_{\rm sdc}\Big(\{D_{12}^{(r)}\},\{\varphi_2^{(r)}\}\Big) = p_{\rm dc}\Big(\{\eta|\alpha_r|\,D_{12}^{(r)}\},\{\varphi_2^{(r)}\}\Big)\,. \tag{56}$$

Hence, the characteristic functions of the joint scaled difference-count distributions are given by

$$\begin{aligned}\Omega_{\rm sdc}\Big(\{x^{(r)}\},\{\varphi_2^{(r)}\}\Big) &= \sum_{\{D_{12}^{(r)}\}} p_{\rm sdc}\Big(\{D_{12}^{(r)}\},\{\varphi_2^{(r)}\}\Big)\\ &\quad\times \exp\left(i\sum_{r=1}^{R} x^{(r)}D_{12}^{(r)}\right)\\ &= \sum_{\{d_{12}^{(r)}\}} p_{\rm dc}\Big(\{d_{12}^{(r)}\},\{\varphi_2^{(r)}\}\Big)\\ &\quad\times \exp\left(i\sum_{r=1}^{R}\frac{x^{(r)}}{\eta|\alpha_r|}\,d_{12}^{(r)}\right). \end{aligned}\tag{57}$$

Equation (53) together with Eqs. (54) and (57) shows that when the joint difference-count distributions are known for all values of the phases $\varphi_2^{(r)}$ $(r = 1,\ldots,N)$ within π intervals, then the characteristic function of the density matrix of the N-mode signal field is known for all complex values of its arguments. Note that the Fourier sums in Eq. (57) are effectively Fourier integrals, because of the large values of $|\alpha_r|$.

To explicitly obtain the signal-field density matrix in the form (9), in Eq. (53) together with Eq. (54) we may let $(r\to j,\ \beta^{(r)}\to\beta_j)$,

$$\begin{aligned}\beta_j &= -\frac{\mathcal{F}_j^{(2)}F_j^*}{|F_j|^2} + iy_jF_j^*\\ &= ix^{(j)}\exp\Big[i\big(\varphi_{\alpha_j} - \varphi_{11}^{(j)} + \varphi_{12}^{(j)} + \varphi_2^{(j)}\big)\Big]\end{aligned}\tag{58}$$

$(j=1,\ldots,N)$, so that

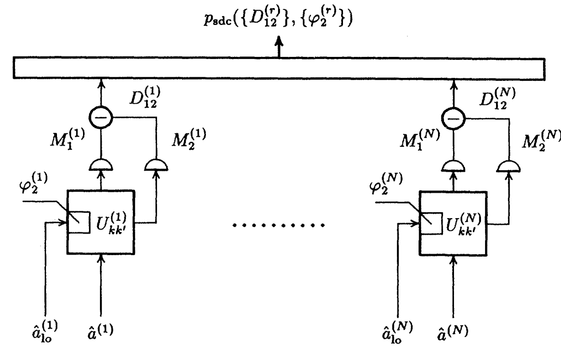

FIG. 2. Scheme of balanced homodyne $(N\times 4)$-port detection. Each signal mode $\hat{a}^{(r)}$ $(r=1,\ldots,N)$ and an associated strong local oscillator $\hat{a}_{\rm lo}^{(r)}$ of the same frequency are combined by a lossless linear four-port device to give two output modes $[U_{kk'}^{(r)}$, unitary (2×2) transformation matrix of the rth four-port device; $\varphi_2^{(r)}$, input-phase control of the rth local oscillator]. Simultaneous detection of the N (scaled) difference counts $D_{12}^{(r)} = M_1^{(r)} - M_2^{(r)}$ yields the N-fold joint (scaled) difference-count distributions $p_{\rm sdc}(\{D_{12}^{(r)}\},\varphi_2^{(r)})$.

$$x^{(j)} = \left[\left(-\frac{\mathcal{F}_j^{(2)}}{|F_j|}\right)^2 + \big(y_j|F_j|\big)^2\right]^{1/2}, \tag{59}$$

$$\begin{aligned}\varphi_2^{(j)} &= \arg\left(-\frac{\mathcal{F}_j^{(2)}}{|F_j|} + iy_j|F_j|\right)\\ &\qquad -\varphi_{F_j} - \varphi_{\alpha_j} + \varphi_{11}^{(j)} - \varphi_{12}^{(j)} - \tfrac{1}{2}\pi.\end{aligned}\tag{60}$$

It is worth noting that the joint difference-count distributions measured in the $4N$-port scheme under consideration can be used to characterize the multimode signal field in a physically very illustrative way. To show this, we recall that according to Eq. (13) the characteristic function Ψ of a joint field-strength distribution is given, for appropriately chosen arguments, by the expectation value of the corresponding multimode-field displacement operator. Comparing Eqs. (53) and (54) with Eq. (13), we find that the characteristic functions of the joint (scaled) difference-count and joint field-strength distributions are related to each other as

$$\Omega_{\rm sdc}\Big(\{x^{(r)}\},\{\varphi_2^{(r)}\}\Big) = \exp\left[-\frac{1-\eta}{2\eta}\sum_{r=1}^{N}\big(x^{(r)}\big)^2\right]\Psi\left(\left\{\frac{x^{(r)}}{|F^{(r)}|}\right\},\{-\varphi_{\alpha_r} + \varphi_{11}^{(r)} - \varphi_{12}^{(r)} - \varphi_2^{(r)}\}\right), \tag{61}$$

which in the case of perfect detection $(\eta=1)$ reduces to

$$\Omega_{\rm sdc}\Big(\{x^{(r)}\},\{\varphi_2^{(r)}\}\Big) = \Psi\left(\left\{\frac{x^{(r)}}{|F^{(r)}|}\right\},\{-\varphi_{\alpha_r} + \varphi_{11}^{(r)} - \varphi_{12}^{(r)} - \varphi_2^{(r)}\}\right). \tag{62}$$

In this case the characteristic function of each joint (scaled) difference-count distribution is equal to the characteristic function of a joint field-strength distribution of the signal field. Hence we obtain the result that the measured joint difference-count distributions are joint field-strength distributions of the N-mode signal field:

$$p_{\rm sdc}\Big(\{D_{12}^{(r)}\},\{\varphi_2^{(r)}\}\Big) = \left[\prod_{r=1}^{N}|F^{(r)}|\right] p\big(\{\mathcal{F}^{(r)} = |F^{(r)}|D_{12}^{(r)}\},\{-\varphi_{\alpha_r} + \varphi_{11}^{(r)} - \varphi_{12}^{(r)} - \varphi_2^{(r)}\}\big). \tag{63}$$

Equation (61) clearly shows that for $\eta < 1$ the measured joint difference-count distribution is a convolution of the true joint field-strength distribution with a Gaussian. This is a consequence of the additional noise introduced by the detectors.

V. SUMMARY AND CONCLUSIONS

In the present paper we have studied the reconstruction of the quantum state of a radiation field consisting of N (correlated) modes of R different frequencies ($R \leq N$) from the data that can be recorded in appropriate multiport homodyning. Because of the complexity of the problem, let us summarize the main aspects of the theory and refer to some basic equations.

The full information on a (correlated) multimode radiation field is contained in the mean value of the multimode-field displacement operator $\hat{D}(\{\beta_j\})$ defined in Eq. (10). For example, any moment of the annihilation and creation operators of the modes can be derived from $\langle\hat{D}(\{\beta_j\})\rangle$ by appropriate differentiation with respect to the variables β_j. In this manner also the cross correlations of the modes are specified. Generally speaking, the density matrix of the N-mode field is uniquely related to $\langle\hat{D}(\{\beta_j\})\rangle$. In particular, the density matrix in a field-strength basis can be obtained from $\langle\hat{D}(\{\beta_j\})\rangle$ by N-fold Fourier transformation [Eq. (9)]. Thus the problem of the reconstruction of the quantum state of the field may be reduced to that of the determination of its characteristic function $\langle\hat{D}(\{\beta_j\})\rangle$.

This characteristic function can be obtained from the count distributions measured in multiport homodyne detection. If there are R groups of modes of different frequencies, for the rth group ($r=1,\ldots,R$) a linear multiport device can be used to combine N_r signal-field modes of equal frequencies ($\sum_r N_r = N$) with a local oscillator mode of the same frequency to obtain N_r+1 output modes [Eq. (14)]. Simultaneous detection of the $N+R$ output modes of the R devices then yields a $(N+R)$-fold joint count distribution. Its characteristic function is obtained by Fourier transformation of the count distribution [Eqs. (17) and (23)] and can be related to the characteristic function $\langle\hat{D}(\{\beta_j\})\rangle$ of the signal field at certain values of the arguments β_j [Eq. (29)]. To obtain $\langle\hat{D}(\{\beta_j\})\rangle$ for all (relevant) values of the β_j, a succession of measurements with appropriate phase control is required, for example by inserting phase shifters in the input channels and varying the phase parameters from measurement to measurement.

To obtain the characteristic function of the state of a N-mode signal field, R reduction conditions can be imposed on the $(N+R)$-fold joint count distributions. In particular, it is sufficient to measure N-fold joint difference-count distributions. In this way both a reduction of data and an elimination of the disturbing effects of classical excess noise of the local oscillators can be achieved. Fourier transformation of the (phase-controlled) joint difference-count distributions yields again their characteristic functions [Eq. (40)], from which the sought characteristic function of the state of the signal field can directly be obtained [Eqs. (43) or (44)].

In the special case of all mode frequencies being different from each other the determination of the quantum state needs as many local oscillator modes as signal modes are present. Thus a combination of N balanced four-port homodyne detection schemes can be used. Clearly, such an approach is also possible when some of the mode frequencies are equal, but it then requires more local oscillator modes than would be necessary. However, such a measurement scheme is of interest since it allows a direct physical interpretation of the measured joint difference-count distribution as a (scaled) joint field-strength distribution [cf. Eq. (63)] of the N-mode signal field under study.

Finally, it should be pointed out that the theory also applies when some of the input ports are unused, which simply means that the corresponding modes can be regarded as being signal-field modes in the vacuum state. In particular, unused input ports may be useful to obtain the count distributions in terms of pseudodistributions of the signal field similar to the (smoothed) single-mode Q function observed in balanced homodyne eight-port detection when two input ports are unused [19–22].

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