Photon number density operator $i\hat{\mathbf{E}}\cdot\hat{\mathbf{A}}$

M. Hawton and T. Melde
Department of Physics, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1
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A photon number density operator proportional to the dot product of the electric field and the vector potential is introduced as an alternative to the Mandel operator. In the Lorentz gauge it is the time component of the four-vector obtained by contracting the electromagnetic field tensor with the vector potential. Its other components can be interpreted as a current-density vector, and these number and current-density operators satisfy a continuity equation. The photon density operators introduced here are all products of operators that satisfy Maxwell's equations and whose Lorentz and gauge transformation properties are well known.

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I. INTRODUCTION

There are many conceptual problems in quantum optics that require a description in terms of the motion of photons in physical space. If the electromagnetic radiation is polychromatic, then it is important to distinguish between photon density and its associated energy density. The counting rate of an ideal photon detector is usually taken to be proportional to the expectation value of the scalar product of the negative and positive frequency parts of the electric field and does not, strictly speaking, count photons. If photon numbers are required, say for a comparison with theoretical photon number statistics, then the fact that the number of counts is not exactly proportional to number density should be taken into account. Increasingly experiments are performed in which the distinction between number and energy density is significant. Short pulses must be described as a superposition of photon momenta. Nonlinear media result in frequency doubling and other sum and difference frequencies and also require a theoretical description appropriate to polychromatic light. Thus the practical need for a description of photon number dynamics has increased.

The photon density operator is usually constructed by first defining a photon detection operator, $\hat{\mathbf{a}}(\mathbf{x},t)$, with Fourier coefficients independent of $\mathbf{k}$ [1]. Thus all wave vectors are included with equal probability and the resulting operator acts like a Dirac δ function in $\mathbf{x}$ space. In this way it can describe a photon that is localized in a small region. The number density operator for detection of a photon at position $\mathbf{x}$ and time t is $\hat{\mathbf{a}}^\dagger(\mathbf{x},t)\cdot\hat{\mathbf{a}}(\mathbf{x},t)$ and the probability of finding a photon in the volume V is $\int_V d^3x\hat{\mathbf{a}}^\dagger(\mathbf{x},t)\cdot\hat{\mathbf{a}}(\mathbf{x},t)$. If this formalism is extended to include a photon current-density operator, then a continuity equation can be derived [2]. This description of photon dynamics can also be written in a Schrödinger-like form [3].

The expectation value of the photon density operator, $\langle\alpha\left|\hat{\mathbf{a}}^\dagger(\mathbf{x},t)\cdot\hat{\mathbf{a}}(\mathbf{x},t)\right|\alpha\rangle$, counts photons in the quantum mechanical state $|\,\alpha\rangle$. However, calculations using these operators are not entirely convenient. The source term in Maxwell's equations differs from the usual current density operator by a factor $\sqrt{k}$. It is thus necessary to convolute current density with $\sqrt{k}$ to obtain a differential equation in $\mathbf{x}$ space for the detection operator in the presence of matter. The photon number and current density can be represented as elements of a 4×4 antisymmetric matrix, but this entity is not a tensor. The action of Lorentz and gauge transformations on these photon operators is difficult to establish. A further difficulty lies in the fact that the components of the current-density operator defined in this way are part of a 4×4 matrix rather than a 4×1 vector. Charge and current densities of electrons and other charged particles form components of four-vectors. We will show that number and number-current density of photons should also form components of a four-vector. This allows the usual electromagnetic stress tensor to be constructed from the outer product of the photon density four-vector and an energy-momentum operator.

Particle density in x space can also be calculated from the wave function, that is, from scalar products between the particle's state vector and the eigenvectors of its position operator. However, exactly localized position eigenvectors of elementary systems with nonzero spin exist only if the particles have nonzero mass [4]. Wightman [5] introduced the projection operators of Mackey [6] that form a system of imprimitivity, but the zero mass photon was still excluded. Jauch and Piron [7] generalized these concepts to particles with zero mass by defining weak localizability and projection operators that are compatible in space domains which do not overlap. Zero mass nonzero spin particles that exist in a superposition of states with different helicities can be weakly localized to arbitrarily small volumes [8]. Pike and Sarkar [9] present a readable examination of this work and apply it to localized wave packets.

In the present paper an alternative photon density operator, $\hat{n}\equiv i\epsilon_0\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)}/\hbar+h.c.$, is proposed. Its desirable properties, which will be described in detail in Sec. II, are briefly as follows. (1) In $\mathbf{k}$ space it is trivial to see that $1/\sqrt{k}$ in $\hat{\mathbf{A}}$ and $\sqrt{k}$ in $\hat{\mathbf{E}}$ cancel, leaving a k-independent product that can be scaled to give the number of photons with wave vector $\mathbf{k}$. Thus, in calculations, it behaves very much like the Mandel operator. (2) Our photon density is written in terms of operators whose equations of motion and properties under Lorentz and gauge transformations are well known. (3) The expressions for the conserved linear and angular momentum resulting from symmetries of the Lagrangian are of the form of an expectation value where n plays the role of a probability density. (4) In the Lorentz gauge, $\hat{n}c$ is the zeroth component of a four-vector, $\hat{V}^\alpha = (\hat{n}c, \hat{\mathbf{v}})$, that is proportional to the inner product of the second-rank electromagnetic field tensor with the vector potential four-vector. Its other three components are those of the three-vector photon number-current-density operator, $\hat{\mathbf{v}}$. The continuity equation in matter free space reflects the fact that the four-divergence of $\hat{V}$ is a scalar.

The plan of the paper is as follows. In Sec. II the properties of $\hat{V}$ described above will be investigated. Then, in Sec. III, its relationship to the Mandel-Cook detection, number, and current operators will be discussed.

II. PHOTON OPERATORS

In this section the conserved quantities that result from symmetries of a general Lagrangian will first be described. Since the photon number densities discussed here are meant for applications in atomic and condensed matter physics, the number, energy and momentum densities will be written down for a nonrelativistic electron. The Dirac equation could be used in its place. By analogy, expressions for these physical variables will then be obtained for photons. The resulting photon current-density four-vector will be separated into positive and negative frequency parts, normally ordered, and then converted to a Hermitian operator, $\hat{V}^\alpha$. It will be shown that these photon number and current-density operators satisfy a continuity equation in matter free space. The Lorentz and gauge transformation properties of $\hat{V}^\alpha$ will then be examined. For consistency with the quantum field theory literature, covariant notation and natural units in which $\hbar = c = \epsilon_0 = 1$ will be used. The photon density operators will also be written in SI (*Système International*) units at the end.

Consider a Lagrangian density that depends on the field variables and their derivates with respect to the space and time coordinates. The contravariant space-time four-vector is $x^\mu = (t, \mathbf{x})$ and its covariant counterpart is $x_\mu = g_{\mu\nu}x^\nu$ where the metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1}$$

The gradient four-vector is $\partial_\mu = (\partial/\partial t, \boldsymbol{\nabla})$ so that differentiation with respect to contravariant x^μ results in a covariant tensor. The summation notation will be used, but only for four-vectors. The set of field variables do not necessarily make up a four-vector and will be denoted $\{\phi_r\}$. The Lagrangian density then depends on the field and its derivatives as $\mathcal{L}(\phi_r, \partial_\mu\phi_r)$. After second quantization the field variables can be replaced by operators, since their equations of motion are of the classical form in the Heisenberg picture.

Each symmetry of the Lagrangian generates a conservation law. If $\mathcal{L}$ has no explicit coordinate dependence then $\partial_\mu\mathcal{T}^{\mu\nu} = 0$ where [10]

$$\mathcal{T}^{\mu\nu} = \sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial^\nu\phi_r - g^{\mu\nu}\mathcal{L}. \tag{2}$$

The zeroth column of this second-rank tensor, $\mathcal{T}^{\mu 0}$, contains the energy density and the three components of linear momentum density. In addition, the Lagrangian may be invariant under phase transformations. This latter symmetry leads to

$$\partial_\mu V^\mu = 0, \tag{3}$$

where the charge-current four-vector is

$$V^\mu = -i\sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\lambda_{rs}\phi_s. \tag{4}$$

The invariance of $\mathcal{L}$ under $\phi_r \to \phi_r - i\epsilon\lambda_{rs}\phi_s$ defines the matrix λ_{rs}. It is diagonal if the symmetry operations are simple phase changes of the fields [10] and these diagonal elements are of opposite sign for a field and its complex conjugate.

Conservation laws follow from integration over the spatial coordinates of equations of the form (3), that is, $\partial V^0/\partial t + \boldsymbol{\nabla}\cdot\mathbf{v} = 0$. If $\int d^3x\boldsymbol{\nabla}\cdot\mathbf{v}$ reduces to a zero surface integral this leaves

$$\frac{d}{dt}\left(\int d^3xV^0\right) = 0. \tag{5}$$

Thus the net charge $\int d^3xV^0$, the total energy $\int d^3x\mathcal{T}^{00}$, and the total momentum components, $\int d^3x\mathcal{T}^{0i}$, are conserved.

A single nonrelativistic electron can be described by the Lagrangian density

$$\mathcal{L} = i\psi^*\dot{\psi} - \boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi/2m - U\psi^*\psi. \tag{6}$$

The equation of motion is the Schrödinger equation and $\mathcal{T}^{0\nu}$ is the energy-momentum density [11]. Because of the $g^{\mu\nu}\mathcal{L}$ term, the components of linear momentum are much easier to deal with than energy, and only linear momentum will be discussed here. From (2), the components of the momentum density three-vector are

$$\mathcal{T}^{0i} = \psi^*\left(-i\partial_i\psi\right). \tag{7}$$

The electron number and current densities obtained by substitution in (4),

$$V^\alpha = \left(\psi^*\psi, -i\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right)/2m\right), \tag{8}$$

satisfy the continuity equation, (3). [This is not strictly true in all Lorentz frames, since the Schrödinger equation and hence the Lagrangian, (6), are invalid for a relativistic electron.] The average linear momentum is

$$\mathbf{P} = \int d^3x \psi^*(-i\boldsymbol{\nabla}\psi). \quad (9)$$

The second quantized field operator can be expanded as

$$\hat{\psi} = \sum_n b_n \chi_n \quad (10)$$

for any complete set of eigenfunctions of the Hamiltonian, $\{\chi_n\}$. The coefficients b_n are electron destruction operators, and their adjoints, $b_n^\dagger$, are creation operators. The complex conjugate of the Schrödinger field will become the adjoint of (10), $\hat{\psi}^\dagger$. For an electron in the state $|\beta\rangle$ the average linear momentum becomes

$$\mathbf{P} = \left\langle \beta \left| \int d^3x \hat{\psi}^\dagger \left(-i\boldsymbol{\nabla}\hat{\psi}\right) \right| \beta \right\rangle. \quad (11)$$

The electron number operator is

$$\hat{N} = \int d^3x \hat{\psi}^\dagger \hat{\psi} \quad (12)$$

and will trivially count one electron. These expressions will be compared with the corresponding ones for photons.

For an electromagnetic field, $\{\phi_r\}$ are components of the scalar and vector potentials, ϕ and $\mathbf{A}$. The standard Lagrangian is

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) + \mathcal{L}_I, \quad (13)$$

where the interaction Lagrangian density is $\mathcal{L}_I = \mathbf{j}\cdot\mathbf{A} - \rho\phi$, $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial\mathbf{A}/\partial t$, and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. The momentum conjugate to a component of the vector potential is $\Pi_r = \partial\mathcal{L}/\partial\dot{A}_r = -E_r$. The energy and momentum density are again contained in the zeroth row of $\mathcal{T}$ as

$$\mathcal{T}_{0i} = \sum_r \Pi_r \frac{\partial A_r}{\partial x^i} - \delta_{0i}\mathcal{L}. \quad (14)$$

Its zeroth component is the energy density $\frac{1}{2}(E^2 + B^2) + \boldsymbol{\nabla}\cdot(\phi\mathbf{E}) - \mathbf{j}\cdot\mathbf{A}$ while the other three components form the momentum three-vector density, $\sum_{r=1}^3 E_r \boldsymbol{\nabla} A_r$. The linear momentum is the integral of this operator over $\mathbf{x}$ space,

$$\mathbf{P} = \int d^3x \sum_{r=1}^{3} iE_r(-i\boldsymbol{\nabla}A_r), \quad (15)$$

and is equivalent to the integral of $\mathbf{E}\times\mathbf{B}$ for a free field. In the presence of matter it equals $\mathbf{E}_\perp\times\mathbf{B}$ since $\int d^3x \sum_{r=1}^3 E_r \boldsymbol{\nabla} A_r = \int d^3x[\mathbf{E}\times\mathbf{B} - \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{E})]$ and the latter term equals $-\int d^3x \mathbf{E}_\parallel \times \mathbf{B}$ [12]. Thus it is purely transverse. A similar result is obtained for angular momentum with $-i\boldsymbol{\nabla}$ replaced by $-i\mathbf{r}\times\boldsymbol{\nabla}$ [13]. Thus $i\mathbf{E}\cdot\mathbf{A}$ plays the role of a probability density, analogous to $\psi^*\psi$ in (9).

If a photon number current were a direct consequence of a phase change symmetry of the Lagrangian density then (4) would yield, for $\lambda_{rs} = \delta_{rs}$,

$$V^\mu = i\left(\mathbf{E}\cdot\mathbf{A}, -\mathbf{B}\times\mathbf{A} + \mathbf{E}\phi\right). \quad (16)$$

In the Lorentz gauge this is just the four-vector formed by contraction of the electromagnetic field tensor with the vector potential, $\mathcal{F}^{\mu\nu}A_\nu$. A four-vector is required for the continuity equation to be satisfied in all Lorentz frames. The second-rank field tensor is $\mathcal{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ or

$$\mathcal{F}^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}. \quad (17)$$

However, there is no phase change symmetry for real fields and the corresponding Hermitian field operators. The above four-current density can only make sense if the fields are separated into their positive and negative frequency parts.

The Hermitian vector potential operator can be expanded in plane waves as

$$\hat{\mathbf{A}}(x) = \sum_{i=0}^{\infty} \left(\frac{1}{2Vk_i}\right)^{\frac{1}{2}} \hat{\mathbf{e}}_i[e^{i\mathbf{k}_i\cdot\mathbf{x}}a_i(t) + e^{-i\mathbf{k}_i\cdot\mathbf{x}}a_i^\dagger(t)]. \quad (18)$$

The sum is over all wave vectors, $\mathbf{k}_i$, and all polarizations λ_i for each $\mathbf{k}_i$. In the Coulomb gauge, $\hat{\mathbf{A}}$ is transverse. The operator $\hat{\mathbf{A}}(\mathbf{x},t)$ creates a photon at position $\mathbf{x}$ and time t. The above pseudophoton density, (16), will be converted to a counting operator by introducing positive and negative frequency field operators and normally ordering all its terms. In this form it can count only existing photons, and will not create new ones. Thus $i\hat{\mathbf{E}}\cdot\hat{\mathbf{A}}$ becomes $i\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)}$, signifying the negative and positive frequency operators, respectively. For a general operator $\hat{\mathbf{A}}$ its positive frequency part is

$$\hat{\mathbf{A}}^{(+)}(x) = \sum_{i=0}^{\infty} A_i \hat{\mathbf{e}}_i e^{i\mathbf{k}_i\cdot\mathbf{x}} a_i(0) e^{-i\omega_i t}. \quad (19)$$

The corresponding Hermitian operator is $\hat{\mathbf{A}} = \hat{\mathbf{A}}^{(+)} + \hat{\mathbf{A}}^{(-)}$ where the amplitudes of the vector potential, $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$, are $A_i = (1/2Vk_i)^{1/2}$, $E_i = B_i = (k_i/2V)^{1/2}$, and $\{a_i\}$ are destruction operators. The Mandel detection operator [1] has been referred to here as $\hat{\mathbf{a}}^{(+)}(x)$ and its Fourier amplitudes are $\alpha_i = (1/2V)^{1/2}$ so that the sum looks like a δ function in real or $\mathbf{x}$ space.

The Hermitian photon density operator which generalizes (16),

$$\hat{V}^\alpha = i(\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)} - \hat{\mathbf{A}}^{(-)}\cdot\hat{\mathbf{E}}^{(+)},\ -\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}^{(+)} - \hat{\mathbf{A}}^{(-)}\times\hat{\mathbf{B}}^{(+)} + \hat{\mathbf{E}}^{(-)}\phi^{(+)} - \phi^{(-)}\hat{\mathbf{E}}^{(+)})$$

can be written more compactly as

$$\hat{V}^{\alpha} = i\left(\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)},\ -\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}^{(+)} + \hat{\mathbf{E}}^{(-)}\phi^{(+)}\right) + \text{H.c.}, \tag{20}$$

where H.c. denotes the Hermitian conjugate or adjoint operator. A continuity equation can be found for the Hermitian operator four-vector, $\hat{V}^{\alpha}$, by substitution in (3). Using Maxwell's equations and substituting $\nabla\times\hat{\mathbf{B}}^{(-)}-\hat{\mathbf{j}}^{(-)}$ for $\partial\hat{\mathbf{E}}^{(-)}/\partial t$ arising from the time derivative cancels a term arising from the divergence. With $\nabla\cdot\hat{\mathbf{E}}^{(-)} = \rho^{(-)}$ and similar substitutions involving the positive frequencies one gets for the four-divergence of (20)

$$i\frac{\partial}{\partial t}\left(\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)}\right)+i\nabla\cdot\left(-\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}^{(+)} + \hat{\mathbf{E}}^{(-)}\phi^{(+)}\right)+\text{H.c.}$$
$$= ij_{\mu}^{(-)}A^{(+)\mu} + \text{H.c.} \tag{21}$$

The positive (negative) frequency current densities are approximately sources for the positive (negative) frequency fields if matter is not dense and the frequency shifts are small [14]. In the absence of matter (21) becomes $\partial_{\alpha}\hat{V}^{\alpha} = 0$ and is of the form of a continuity equation for the photon number and current-density operators. For photons in state $|\alpha\rangle$ the average momentum is

$$\mathbf{P} = \left\langle \alpha \left| \int d^3x \sum_{r=1}^{3} i\hat{E}_r^{(-)}\left(-i\nabla\hat{A}_r^{(+)}\right) + \text{H.c.}\right| \alpha \right\rangle. \tag{22}$$

The photon number operator

$$\hat{N} = i\int d^3x\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)} + \text{H.c.} \tag{23}$$

is time independent. It is identical to the usual number operator, $\sum_i a_i^{\dagger}a_i$, and to Mandel's total number operator [1] .

The above expressions involving electrons and those involving photons are analogous. The photon expansion, (18), can be compared to (10). In (18) the orthonormal states are the free particle states $e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ and the particle destruction operators b_n are called a_i. The sum over wave vectors represents a special case of the sum over a complete set of functions. The operators $i\hat{\mathbf{E}}$ and $\hat{\mathbf{A}}$ in (22) and (23) take the place of $\hat{\psi}^{\dagger}$ and $\hat{\psi}$ in (9) and (12). Thus there is a direct correspondence between the expressions involving photons and those involving electrons.

For working in the Lorentz gauge the Fermi Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}\left(\partial_{\mu}A^{\mu}\right)^2 \tag{24}$$

that gives a nonzero momentum conjugate to each component of A^{μ} should be used in place of (13). The photon current density then becomes

$$V^{\mu,Fermi} = V^{\mu} - i\left(\partial_{\nu}A^{\nu}\right)A^{\mu}. \tag{25}$$

This differs from (16) by its last term $-i\left(\partial_{\nu}A^{\nu}\right)A^{\mu}$ and this term is zero in the Lorentz gauge. Thus the current density (16) can be put in the form (25) in the Lorentz gauge by adding zero and (16) is consistent with (25). The photon density $V^{0,Fermi}$ in (25) takes the symmetric form $i(\dot{\phi}^{(-)}\phi^{(+)} - \dot{\mathbf{A}}^{(-)}\cdot\mathbf{A}^{(+)}) + \text{H.c.}$ if $i\nabla\cdot(\phi^{(-)}\mathbf{A}^{(+)}) +$ H.c. is added to it. This is the form that is obtained from the Lagrangian density used in [11]. Since the extra term is a divergence, the integrals over all space of the two forms are equal and the same total number operator is obtained.

In the absence of matter the photon number density operator in (20) is transverse, that is, $i\hat{\mathbf{E}}\cdot\hat{\mathbf{A}} = i\hat{\mathbf{E}}_{\perp}\cdot\hat{\mathbf{A}}_{\perp}$. The current density becomes

$$\hat{\mathbf{v}} = i\left(-\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}_{\perp}^{(+)} - \hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}_{\parallel}^{(+)} + \hat{\mathbf{E}}^{(-)}\phi^{(+)}\right) + \text{H.c.} \tag{26}$$

In the Coulomb gauge, only the first or transverse term is nonzero. In the Lorentz gauge, the last two terms cancel for a free photon of known wave vector. Thus the rather peculiar $\hat{\mathbf{E}}\phi$ terms are necessary to cancel $-\hat{\mathbf{B}}\times\hat{\mathbf{A}}_{\parallel}$, leaving only transverse terms in matter free space. These operators have only transverse components and thus only real physical photons contribute.

The properties of $\hat{V}^{\alpha}$ under Lorentz and gauge transformations will be examined next. We have no general proof that the longitudinal terms drop out if $\mathbf{k}' \neq \mathbf{k}$ in the product of two expansions of the form (18), so it is necessary to consider longitudinal and scalar photons. For a four-polarization e^{μ} the transversality condition, $e^{\mu}k_{\mu} = 0$, is not Lorentz invariant. Under a Lorentz transformation $e^{\mu} = (0, \mathbf{e})$ becomes $e'^{\mu} = (e'^{0}, \mathbf{e}' + e'^{0}\hat{\mathbf{k}}')$ where $\mathbf{e}$ is the transverse polarization vector [15]. Thus the polarization four-vector e^{μ} is no longer transverse and includes nonphysical scalar and longitudinal photons. This problem is always encountered in covariant quantization schemes and it can be dealt with in the usual way here. For a free field the Lorentz condition is replaced by the Gupta-Bleuler subsidiary condition [15]

$$\partial^{\mu}\mathbf{A}_{\mu}^{(-)}(x)\,|P_s\rangle = 0, \tag{27}$$

where P_s denotes physical state. For a photon with wave vector $\mathbf{k}$ the transverse photon creation operators can be chosen as $a_1^{\dagger}$ and $a_2^{\dagger}$. If $a_0^{\dagger}$ creates scalar photons and $a_3^{\dagger}$ creates longitudinal photons, the new set of photon creation operators $\alpha_0^{\dagger} = (a_0^{\dagger} - a_3^{\dagger})/\sqrt{2}$, $\alpha_1^{\dagger} = a_1^{\dagger}$, $\alpha_2^{\dagger} = a_2^{\dagger}$, and $\alpha_3^{\dagger} = (a_0^{\dagger} + a_3^{\dagger})/\sqrt{2}$ can be defined. For free photons, only the first three of these operators can create photon states that satisfy the subsidiary condition, and thus only these can be used in the creation of photon states. The first of these, the new zeroth operator, creates states with zero norm that are equivalent to the null vector. The nonphysical photons do not contribute to physical matrix elements and states containing free photons with arbitrary four-polarization are equivalent to states with only transverse photons.

In the Lorentz gauge, the photon current density $\hat{V}^{\alpha}$ given by (20) and the divergence ∂_{α} transform as four-vectors. Equation (21) is covariant and it is of the form of a continuity equation. If the Coulomb gauge is used then $\hat{V}^{\alpha}$ is not a four-vector. To transform it to a new frame of reference a gauge transformation to the Lorentz gauge can be performed, followed by a Lorentz transformation and then a gauge transformation to take it back to the Coulomb gauge.

III. DISCUSSION

Our photon current-density four-vector operator $\hat{V}^{\alpha} = (\hat{n}c, \hat{\mathbf{v}})$ in SI units is

$$\hat{V}^{\alpha} = \frac{i\epsilon_0 c\left(\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)},\ -c\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{A}}^{(+)} + \hat{\mathbf{E}}^{(-)}\phi^{(+)}/c\right)}{\hbar} + \text{H.c.} \tag{28}$$

In calculations performed in a single reference frame and in a single gauge (28) will give similar results to the Mandel operators. The main difference is that, when expanded in $\mathbf{k}$ space, the nondiagonal terms in (28) include a factor $\left(\sqrt{k/k'} + \sqrt{k'/k}\right)/2$ not present in Ref. [1]. If used to count the number of photons localized in a volume V smaller than the normalization volume, the commutation properties of $\hat{n}_{V,t} = \hat{V}^0/c$ will be essentially the same as those of the Mandel photon number operator since the nonzero contributions to the sum over $\mathbf{k}$ and $\mathbf{k}'$ requires $|\mathbf{k} - \mathbf{k}'| < 2/V^{1/3}$ [1]. It is the absence of these $\sqrt{k}$ factors that makes Cook's [2] operators nonlocal so that they do not form a four-vector exactly.

In the present description of photon dynamics, we do not define an operator in real space of the δ-function form representing the probability amplitude for finding a photon at x. Instead, state vectors in the position representation describe the electric or magnetic field or the vector potential due to a photon at x. Products such as $i\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{A}}^{(+)}$ describe photon number density and there is no photon probability amplitude operator which would be the counterpart of $\hat{\mathbf{a}}^{(+)}$ in Ref. [1].

The photon current-density operator defined in (28) and (20) is a four-vector in contrast to the generalized Mandel operators introduced by Cook. This represents a fundamental difference between these two descriptions of photon dynamics. To obtain the usual Maxwell stress tensor which includes the four-momentum density it is necessary to form the outer product between the photon current density and the momentum operators. This is usual in the description of the dynamics of particles and is evident in (9) and (15). For example, for a photon of known momentum, $\mathbf{k}_i$, and polarization, λ_i, $\hat{V}^{\mu} = a_i^{\dagger}a_i(1, \hat{\mathbf{k}}_i)/2V$ and $k_i^{\nu} = (k_i, \mathbf{k}_i)$. Here $\hat{\mathbf{k}}_i$ is a unit vector and natural units have been used. The second-rank tensor $\mathcal{T}^{\mu\nu} = \hat{V}^{\mu}k^{\nu}$ is the Maxwell stress tensor for a free photon. Its zeroth row, $\mathcal{T}^{0\mu} = (k_i, \mathbf{k}_i)a_i^{\dagger}a_i/2V$, is the photon energy-momentum density.

In summary, we have proposed a photon current-density four-vector, (28), that is an alternative to the extended [2] Mandel [1] detection operator formalism. It is defined in terms of the electric and magnetic field and vector potential operators whose transformation properties under Lorentz and gauge transformations are well known and which satisfy Maxwell's equations. This proposed photon current-density four-vector satisfies a continuity equation, $\partial_{\alpha}\hat{V}^{\alpha} = 0$, for free photons. Its zeroth component is a photon probability density operator multiplied by c, while its other three components form the photon current-density three-vector. Except possibly for cross terms in their $\mathbf{k}$-space expansions, both the photon number-density operator and the photon current-density vector operator include only transverse terms and thus describe only real physical photons in free space.

[1] L. Mandel, Phys. Rev. **144**, 1071 (1966).

[2] R. J. Cook, Phys. Rev. A **25**, 2164 (1982); **26**, 2754 (1982).

[3] T. Inagaki, Phys. Rev. A **49**, 2839 (1994).

[4] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); M. H. L. Pryce, Proc. R. Soc. London, Ser. A **195**, 62 (1948).

[5] A. S. Wightman, Rev. Mod. Phys. **34**, 845 (1962).

[6] G. W. Mackey, Proc. Natl. Acad. Sci. U.S.A. **35**, 537 (1949).

[7] J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).

[8] W. O. Amrein, Helv. Phys. Acta **42**, 149 (1969).

[9] E. R. Pike and S. Sarkar, *Frontiers in Quantum Optics* (Adam Hilger, Bristol, 1986), p. 282.

[10] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields (*McGraw-Hill, New York, 1965*).*

[11] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms* (John Wiley and Sons, New York, 1989), p. 157.

[12] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Ref. [11], p. 19.

[13] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Ref. [11], p. 153.

[14] M. Hawton, Phys. Rev. A **48**, 1824 (1993).

[15] O. Nachtmann, *Elementary Particle Physics, Concepts and Phenomena* (Springer-Verlag, New York, 1989); S. Gupta, Proc. R. Soc. London, Ser. A **63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).