Moving mirrors and time-varying dielectrics

Harry Johnston and Sarben Sarkar
Wheatstone Physics Laboratory, King's College London, Strand, London WC2R 2LS, England

(Received 1 July 1994)

The spectral distribution of light generated by a finite cavity with one moving mirror is compared with that produced by a fixed cavity containing a time-varying dielectric. In both cases a motion over a finite time interval is considered. Although the moving mirror is usually considered to be an idealization for the time-varying dielectric, there are qualitative differences in the spectra produced. The spectral distribution for the moving mirror case behaves as $1/n^3$, while that for the time-varying dielectric behaves on average as $1/n^4$ but is rapidly oscillating.

PACS number(s): 42.50.Ar, 42.50.Dv, 04.70.Dy

I. INTRODUCTION

It was noticed by Unruh [1] and Davies [2] that the radiation from black holes [3] can be more generally considered as an effect of acceleration. There has recently been a proposal for producing in the laboratory an equivalent to accelerating mirrors with sufficiently fast effective acceleration for Unruh radiation to be experimentally detected. In this proposal [4] the refractive index of a dielectric is changed rapidly. The optical path length increases or decreases with the refractive index.

Accelerations of a mirror can then intuitively be modeled by the corresponding "accelerations" in the time dependence of the refractive index. The moving mirror problem at one end of a Fabry-Pérot cavity has also been rigorously formulated [7] for analytic motions, which are slow in an asymptotic series in a parameter related to the scale of time variation.

The similarity of a time-varying dielectric to a moving mirror is intuitively appealing, and it is often implied [5,6] that the most important factor is the variation of the optical path lengths involved. In the case of an optical cavity, many physically different systems can produce the same time-dependent optical path lengths, and it is not clear *a priori* whether there are any qualitative differences in the light generated by such systems. In this work we will give explicit calculations for the spectral distribution of the mean number of particles for both systems, and we find such qualitative differences.

II. MOVING MIRRORS

Let us consider a Fabry-Pérot cavity with a mirror that is moving as indicated in Fig. 1. We will take the motion $q(t)$ to be given by

$$q(t)=\begin{cases} q^{(1)}(t)=s_0 \quad \text{for } t\le 0 \\ q^{(2)}(t)=s_0+s_2t^2 \quad \text{for } t>0,\ t \text{ small} \\ q^{(2)}(t)=b_0+b_2(t-t_0)^2 \quad \text{for } t<t_0\ ,\ |t-t_0| \text{ small} \\ q^{(3)}(t)=b_0 \quad \text{for } t\ge t_0\ . \end{cases} \tag{1}$$

The accelerations are finite at $t=0$ and $t=t_0$.

Moreover, the cavity will be taken to be effectively infinite in the transverse y-z direction. Consequently, if the vector potential is linearly polarized in the z direction, then ignoring transverse mode structure we have

$$\frac{\partial^2}{\partial t^2}A(x,t)=\frac{\partial^2}{\partial x^2}A(x,t)\ , \tag{2}$$

where the vector potential $\mathbf{A}(x,t)$, with polarization $\mathbf{e}_z$, is given by

$$\mathbf{A}(x,t)=A(x,t)\mathbf{e}_z\ . \tag{3}$$

For perfect mirrors the boundary conditions are

$$A(q(t),t)=0 \tag{4a}$$

and

$$A(0,t)=0\ . \tag{4b}$$

Physically this boundary condition holds for all time, due to charge and current fluctuations in the mirror.

It can be shown [7] that the corresponding operator $\hat{A}$ has the expansion

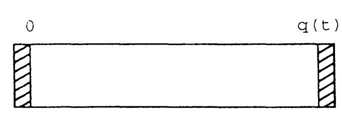

FIG. 1. A Fabry-Pérot cavity with a moving mirror.

 ©1995 The American Physical Society

$$\hat{A}(t,x)=\sum_n[\hat{p}_n v_n(t,x)-\hat{q}_n u_n(t,x)] , \quad (5)$$

where

$$u_n(t,x)=\frac{1}{(2n\pi)^{1/2}}\{\cos[n\pi R(t-x)]-\cos[n\pi R(t+x)]\} \quad (6)$$

and

$$v_n(t,x)=\frac{1}{(2n\pi)^{1/2}}\{\sin[n\pi R(t+x)]-\sin[n\pi R(t-x)]\} . \quad (7)$$

R is a differentiable and invertible function that satisfies

$$R(t-q(t))=R(t+q(t))-2 . \quad (8)$$

The operators $\hat{p}_n$ and $\hat{q}_n$ can be determined in terms of $\hat{A}$ and the functions u_n and v_n. It is first necessary to introduce the bilinear form ω on the function space of solutions of (2):

$$\omega(A^{(1)},A^{(2)})=\int_0^{q(t)}dx\left\{A^{(2)}(t,x)\frac{\partial}{\partial t}A^{(1)}(t,x)-A^{(1)}(t,x)\frac{\partial}{\partial t}A^{(2)}(t,x)\right\} \quad (9)$$

for any two solutions $A^{(1)}$ and $A^{(2)}$. ω is independent of t. It is possible to verify the relations [7]

$$\omega(u_m,u_n)=\omega(v_m,v_n)=0 , \quad (10)$$

$$\omega(u_m,v_n)=\delta_{mn} . \quad (11)$$

Consequently, from (5),

$$\hat{p}_n=-\omega(\hat{A},u_n) , \quad (12)$$

$$\hat{q}_m=-\omega(\hat{A},v_m) . \quad (13)$$

The motion $q(t)$ is analytic, except at the points $t=0$ and $t=t_0$, and so there are three distinct regimes of motion $q^{(1)}(t)$, $q^{(2)}(t)$, and $q^{(3)}(t)$. Corresponding to these regimes there are functions $R^{(i)}$, $i=1,2,3$ satisfying (8), and in turn functions $v_n^{(i)}(x,t)$ and $u_n^{(i)}(x,t)$. From (12) and (13) we have the relations [7] for $i=1,2$,

$$\hat{p}_n^{(i+1)}=-\sum_{m=1}^{\infty}[\omega(v_m^{(i)},u_n^{(i+1)})\hat{p}_m^{(i)}-\omega(u_m^{(i)},u_n^{(i+1)})\hat{q}_m^{(i)}] \quad (14)$$

and

$$\hat{q}_n^{(i+1)}=-\sum_{m=1}^{\infty}[\omega(v_m^{(i)},v_n^{(i+1)})\hat{p}_m^{(i)}-\omega(u_m^{(i)},v_n^{(i+1)})\hat{q}_m^{(i)}] . \quad (15)$$

These equations are equivalent to Bogoliubov transformations for the usual creation and annihilation operators ($\hat{a}_n^\dagger$ and $\hat{a}_n$) defined by

$$\hat{a}_n^{(i)}=\frac{\hat{q}_n^{(i)}+i\hat{p}_n^{(i)}}{\sqrt{2}} , \qquad \hat{a}_n^{(i)\dagger}=\frac{\hat{q}_n^{(i)}-i\hat{p}_n^{(i)}}{\sqrt{2}} . \quad (16)$$

As shown in the Appendix we find that

$$\hat{a}_n^{(3)}=\tfrac{1}{2}e^{-in\pi t_0/b_0}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\{-[S_{lm}(s_0,s_2)R_{nm}(b_0,b_2)+R_{lm}(s_0,s_2)S_{nm}(b_0,b_2)]\hat{a}_l^{(1)}+[-S_{lm}(s_0,s_2)R_{nm}(b_0,b_2)+R_{lm}(s_0,s_2)S_{nm}(b_0,b_2)]\hat{a}_l^{\dagger(1)}\} , \quad (17)$$

where

$$S_{lm}(s_0,s_2)=\begin{cases}-1+\dfrac{s_0s_2}{4m^2\pi^2}+O(s_2^2) & \text{when } l=m\\[2ex] \dfrac{4(-1)^{(l+m)}l(lm)^{1/2}(l^2+3m^2)s_0s_2}{(l^2-m^2)^3\pi^2}+O(s_2^2) & \text{when } l\neq m\end{cases} \quad (18)$$

and

$$R_{lm}(s_0,s_2)=S_{ml}(s_0,s_2)+2\delta_{ml}+O(s_2^2) . \quad (19)$$

If for $t<0$ the cavity was in a vacuum state, then from (17) we can derive the average number of photons in mode n, $\bar{N}_n$, produced in the cavity for $t>t_0$. We find

$$\bar{N}_n=\frac{-2n}{3\pi^4}\left[\psi^{(4)}(1+n)+\frac{n}{5}\psi^{(5)}(1+n)\right]\times(s_0s_2-b_0b_2)^2+O((s_2,b_2)^3) , \quad (20)$$

where $\psi^{(j)}(x)=(d^{j+1})/(dx^{j+1})\ln\Gamma(x)$. For large n,

$$\bar{N}_n\sim\frac{4(s_0s_2-b_0b_2)^2}{5\pi^4n^3}+O\left[\frac{1}{n^5},(s_2,b_2)^3\right] . \quad (21)$$

III. TIME-VARYING DIELECTRIC

In order to mimic a moving mirror with a time-varying dielectric we will now consider a cavity of fixed length L with a dielectric material of length d fixed at one end (see Fig. 2).

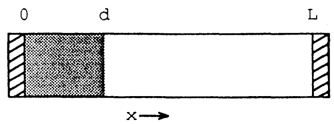

FIG. 2. A Fabry-Pérot cavity with a time-varying dielectric.

The relative dielectric constant of the dielectric can be written as

$$\epsilon_m = 1 + p(t) \ . \tag{22}$$

We will consider the dielectric from time $t=0$ to $t=T$ and set

$$p(0) = \dot{p}(0) = p(T) = \dot{p}(T) = 0 \ . \tag{23}$$

This corresponds to a mirror moving from rest at the initial position and then returning to the same position, with a finite acceleration at both ends of the motion.

The equation of motion is

$$\epsilon(x,t)\ddot{A}(x,t) + \dot{\epsilon}(x,t)\dot{A}(x,t) = A''(x,t) \ , \tag{24}$$

where the dot indicates a time derivative and the prime indicates a spatial derivative. $\epsilon(x,t)$ coincides with ϵ_m in $0<x<d$ and is 1 outside this interval. The boundary conditions are $A(0,t)=A(L,t)=0$. These equations are valid for the quantum field operator $\hat{A}$, as well as for the classical field. The commutation relation is

$$[\hat{A}(x,t), \hat{\Pi}(x't)] = i\delta(x-x') \ ,$$

where $\hat{\Pi}(x,t) = \epsilon(x,t)(\partial/\partial t)\hat{A}(x,t)$.

We can rewrite the equation of motion as

$$\ddot{A} - A'' = (1-\epsilon)\ddot{A} - \dot{\epsilon}\dot{A} = -\theta(d-x)\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}A\right] \ . \tag{25}$$

This leads to the integral equation

$$A(x,t) - A_0(x,t) = -\int\int dx_1 dt_1 G(x,x_1,t,t_1) \times \theta(d-x_1)\frac{\partial}{\partial t_1}\left[p(t_1) \times \frac{\partial}{\partial t_1}A(x_1,t_1)\right] \ , \tag{26}$$

where

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]A_0(x,t) = 0 \tag{27}$$

and

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]G(x,x_0,t,t_0) = \delta(x-x_0)\delta(t-t_0) \ . \tag{28}$$

G is a retarded Green's function and is found to be

$$G(x,t,x_0,t_0) = \sum_{n=1}^{\infty} G_n(t-t_0)(\sin\omega_n x)(\sin w_n x_0) \ , \tag{29}$$

where

$$G_n(t-t_0) = \frac{2}{n\pi}\theta(t-t_0)[\sin\omega_n(t-t_0)] \tag{30}$$

and w_n denotes $n\pi/L$.

From (26) it is straightforward to define a perturbation expansion in p. In particular, to first order,

$$A(x,t) - A_0(x,t) \equiv A_1(x,t) - A_0(x,t) = -\int\int dx_1 dt_1 G(x,t,x_1,t_1) \times \theta(d-x_1)\frac{\partial}{\partial t_1}\left[p(t_1) \times \frac{\partial}{\partial t_1}A_0(x_1,t_1)\right] \ , \tag{31}$$

and this order will suffice for our calculation. It will be convenient to define the operator

$$\hat{a}_n(t) = \frac{1}{\sqrt{2}}\int_0^L dx\left[\frac{2}{L}\right]^{1/2}(\sin\omega_n x)\left\{i\sqrt{\omega_n}\hat{A}(x,t) - \frac{1}{\sqrt{\omega_n}}\hat{\Pi}(x,t)\right\} . \tag{32}$$

In terms of $\hat{a}_n$ we can verify that

$$\hat{A}(x,t) = \sum_{n=1}^{\infty}\frac{i}{\sqrt{L\omega_n}}(\sin\omega_n x)[\hat{a}_n^{\dagger}(t) - \hat{a}_n(t)] \ . \tag{33}$$

In zeroth order,

$$\hat{a}_n(t) = \hat{a}_n(0)e^{-i\omega_n t} \ . \tag{34}$$

We should note that $\hat{a}_n(0)$ and $\hat{a}_n(T)$ are the standard harmonic-oscillator destruction operators (up to a phase factor) for cavity mode n, but in general $\hat{a}_n(t)$ is not.

We define coefficients a_{nm} and b_{nm} by

$$\hat{a}_n(T) = \sum_{m=1}^{\infty}\{a_{nm}^* e^{-i\omega_m T}\hat{a}_m(0) + b_{nm}^* e^{i\omega_m T}\hat{a}_m^{\dagger}(0)\} \ . \tag{35}$$

The first-order perturbation expansion gives

$$a_{nj} - \delta_{nj} = \frac{i}{\pi n}\omega_n\sqrt{\omega_n\omega_j}T_{nj}e^{i(\omega_n-\omega_j)T} \times \int_0^T dt_1 p(t_1)e^{-i(\omega_n-\omega_j)t_1} \ , \tag{36a}$$

$$b_{nj} = \frac{i}{\pi n}\omega_n\sqrt{\omega_n\omega_j}T_{nj}e^{i(\omega_n+\omega_j)T} \times \int_0^T dt_1 p(t_1)e^{-i(\omega_n+\omega_j)t_1} \ , \tag{36b}$$

where

$$T_{nj} \equiv \int_0^d dx(\sin\omega_n x)(\sin\omega_m x) = \frac{1}{2}\left[\frac{\sin d(\omega_m-\omega_n)}{\omega_m-\omega_n} - \frac{\sin d(\omega_m+\omega_n)}{\omega_m+\omega_n}\right] . \tag{36c}$$

Now

$$\int_0^T dt_1 p(t_1) e^{-i(\omega_n \pm \omega_j)t_1} = -\sum_{k=0}^{\infty} \left[\frac{-i}{\omega_n \pm \omega_j} \right]^{k+1} \left[\frac{\partial^k p}{\partial t^k} e^{-i(\omega_n \pm \omega_j)t_1} \right]_{t_1=0}^{T}, \tag{37}$$

(integration by parts, recursively) assuming this series converges. For a sufficiently slow, smooth motion this can be approximated by

$$\int_0^T dt_1 p(t_1) e^{-i(\omega_n \pm \omega_j)t_1} \approx \frac{-i}{(\omega_n \pm \omega_j)^3} [\ddot{p}(T) e^{-i(\omega_n \pm \omega_j)T} - \ddot{p}(0)] . \tag{38}$$

This gives

$$\begin{aligned} \bar{N}_n &= \sum_m |b_{nm}|^2 \\ &= \sum_m \frac{L^4}{4\pi^6} \frac{nm}{(n+m)^6} \left[\frac{\sin f(m-n)}{m-n} - \frac{\sin f(m+n)}{m+n} \right]^2 \\ &\quad \times \{ \ddot{p}(T)^2 + \ddot{p}(0)^2 - 2\ddot{p}(0)\ddot{p}(T)[\cos(n+m)\tau] \} , \end{aligned} \tag{39}$$

where $f = d\pi/L$ and $\tau = T\pi/L$.

It is possible to approximate this by means of an asymptotic expansion in n. Only the first term can be written in a simple form. We find

$$\bar{N}_n \approx \frac{fL^4}{256\pi^5 n^4} \{ \ddot{p}(0)^2 + \ddot{p}(T)^2 + 2\ddot{p}(0)\ddot{p}(T)\beta(f,\tau)\cos[2n\tau + \phi(f,\tau)] \} \tag{40}$$

where

$$\beta(f,\tau) e^{i\phi(f,\tau)} \equiv \frac{-1}{2\pi f} [2f^2 + 2\mathrm{Li}_2(e^{i\tau}) - \mathrm{Li}_2(e^{i(\tau-2f)}) - \mathrm{Li}_2(e^{i(\tau+2f)})] \tag{41}$$

and Li_2 is the dilogarithm function

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_z^0 \frac{\ln(1-t)}{t} dt .$$

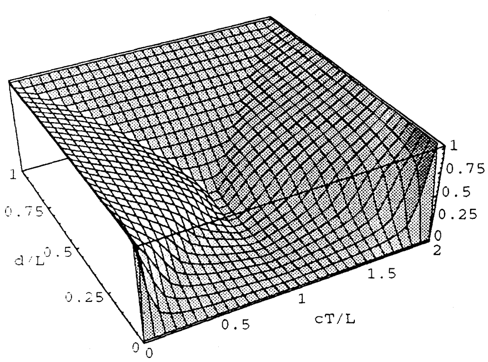

FIG. 3. Magnitude $\beta(f,\tau)$ of the spectral oscillations.

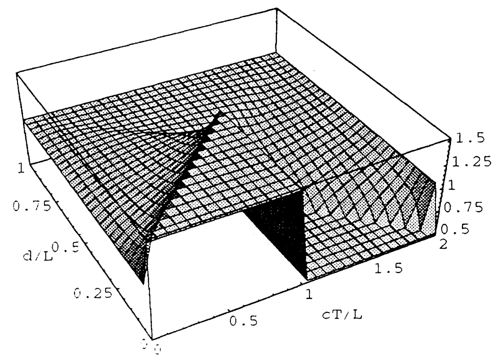

FIG. 4. Phase $\phi(f,\tau)/\pi$ of the spectral oscillations.

See Figs. 3 and 4. It can be seen that $\bar{N}_n$ varies rapidly with n between

$$\frac{fL^4}{256\pi^5 n^4} [\ddot{p}(0) + \ddot{p}(T)]^2$$

and

$$\frac{fL^4}{256\pi^5 n^4} [\ddot{p}(0) - \ddot{p}(T)]^2 .$$

IV. COMPARISON OF THE TIME-VARYING DIELECTRIC TO THE MOVING MIRROR

We will now consider two systems, one a moving mirror and the other a time-varying dielectric, with identical time-dependent optical path lengths.

The optical path length of the moving mirror cavity from Sec. II is simply $q(t)$. The optical path length of the dielectric cavity in Sec. III is $q(t) = (L-d) + \epsilon_m d = L + p(t)d$.

Clearly

$$\begin{aligned} s_0 &= L , \\ s_2 &= \frac{d^2}{dt^2} [L + p(t)d]_{t=0} = \ddot{p}(0)d = \ddot{p}(0)\frac{fL}{\pi} , \\ b_0 &= L , \\ b_2 &= -\ddot{p}(T)\frac{fL}{\pi} \end{aligned} \tag{42}$$

and so, on using the moving mirror formulation (cf. Sec. II),

$$\begin{aligned} \bar{N}_n &= \frac{4}{5\pi^4 n^3} \left[L\ddot{p}(0)\frac{fL}{\pi} + L\ddot{p}(T)\frac{fL}{\pi} \right]^2 \\ &= \frac{4f^2L^4}{5\pi^6 n^3} [\ddot{p}(0) + \ddot{p}(T)]^2 . \end{aligned} \tag{43}$$

On comparing this to the dielectric,

$$\bar{N}_n = \frac{fL^4}{256\pi^5 n^4} \{ \ddot{p}(0)^2 + \ddot{p}(T)^2 + 2\ddot{p}(0)\ddot{p}(T)\beta(f,\tau)\cos[2n\tau + \phi(f,\tau)] \} , \tag{44}$$

we see clear quantitative and qualitative differences. In particular, we should note the sinusoidal term whose magnitude can be large in certain parameter regimes (see Fig. 3). There is also a large difference in the magnitude of photon production due to the additional factor of $fn/320\pi$. Unless the dielectric strip is substantially thinner than the wavelengths of interest, there will be far fewer photons produced than in the equivalent moving mirror cavity.

APPENDIX

It has been proved [7] that to a good approximation

$$R(t\mp x)=a(t,x)+c^{\pm}(t,x)\ , \tag{A1}$$

where

$$a(t,x)=\int^{t}\frac{dt'}{q(t')}+\tfrac{1}{6}\int^{t}dt'\left[-2\frac{[\dot q(t')]^2}{q(t)}+\ddot q(t')\right] \tag{A2}$$

and

$$c(t,x)=\pm\left[\frac{x}{q(t)}+\frac{1}{6}\left[\frac{x}{q(t)}-\frac{x^3}{q(t)^3}\right]\times[-2\dot q(t)^2+q(t)\ddot q(t)]\right] \tag{A3}$$

for a sufficiently slow motion $q(t)$.

From (9), (8), (6), and (7) it is straightforward to show that

$$\omega(v_m^{(1)},u_n^{(2)})=\frac{1}{2\pi(mn)^{1/2}}\int_{-q(t)}^{q(t)}dx\,\sin[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\cos[n\pi R^{(2)}(t+x)]\ , \tag{A4}$$

$$\omega(u_m^{(1)},u_n^{(2)})=\frac{1}{2\pi(mn)^{1/2}}\int_{-q(t)}^{q(t)}dx\,\cos[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\cos[n\pi R^{(2)}(t+x)]\ , \tag{A5}$$

$$\omega(v_m^{(1)},v_n^{(2)})=\frac{1}{2\pi(mn)^{1/2}}\int_{-q(t)}^{q(t)}dx\,\sin[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\sin[n\pi R^{(2)}(t+x)]\ , \tag{A6}$$

$$\omega(u_m^{(1)},v_n^{(2)})=\frac{1}{2\pi(mn)^{1/2}}\int_{-q(t)}^{q(t)}dx\,\cos[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\sin[n\pi R^{(2)}(t+x)]\ . \tag{A7}$$

Now, on using (A1), (A2), and (A3) we find that

$$\begin{aligned}\sin[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\cos[n\pi R^{(2)}(t+x)]\Big|_{t=0}&=\frac{-m\pi}{2s_0}\{\cos[a(m,-n,x)]+\cos[a(m,n,x)]\}\\&\quad+\frac{n\pi}{6s_0^2}(3s_0+s_0^2s_2-3s_2x^2)\{\cos[a(m,n,x)]-\cos[a(m,-n,x)]\}\ ,\end{aligned} \tag{A8}$$

where

$$a(m,n,x)=\frac{\pi}{s_0}(m+n)x+n\pi\frac{s_0s_2}{3}\left[\frac{x}{s_0}-\frac{x^3}{s_0^3}\right]\ . \tag{A9}$$

Similarly

$$\begin{aligned}\cos[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\cos[n\pi R^{(2)}(t+x)]\Big|_{t=0}&=\frac{m\pi}{2s_0}\{\sin[a(m,n,x)]+\sin[a(m,-n,x)]\}\\&\quad-\frac{n\pi}{6s_0^2}(3s_0+s_0^2s_2-3s_2x^2)\{\sin[a(m,n,x)]-\sin[a(m,-n,x)]\}\ ,\end{aligned} \tag{A10}$$

$$\begin{aligned}\sin[m\pi R^{(1)}(t+x)]\overleftrightarrow{\frac{\partial}{\partial x}}\sin[n\pi R^{(2)}(t+x)]\Big|_{t=0}&=\frac{-m\pi}{2s_0}\{\sin[a(m,n,x)]-\sin[a(m,-n,x)]\}\\&\quad+\frac{n\pi}{6s_0^2}(3s_0+s_0^2s_2-3s_2x^2)\{\sin[a(m,n,x)]+\sin[a(m,-n,x)]\}\ ,\end{aligned} \tag{A11}$$

and

$$\cos[m\pi R^{(1)}(t+x)]\frac{\overleftrightarrow{\partial}}{\partial x}\sin[n\pi R^{(2)}(t+x)]\bigg|_{t=0}=\frac{m\pi}{2s_0}\{-\cos[a(m,n,x)]+\cos[a(m,-n,x)]\}$$
$$+\frac{n\pi}{6s_0^2}(3s_0+s_0^2s_2-3s_2x^2)\{\cos[a(m,n,x)]+\cos[a(m,-n,x)]\}\ . \tag{A12}$$

Consequently, in order to evaluate (A4)–(A7), we need the canonical integrals

$$\mathcal{I}_{mn}^{\mp}(s_0,s_2)=\int_{-s_0}^{s_0}dx\,\cos[a(m,\mp n,x)] \tag{A13}$$

and

$$\mathcal{L}_{mn}^{\mp}(s_0,s_2)=\int_{-s_0}^{s_0}dx\,x^2\cos[a(m,\mp n,x)]\ . \tag{A14}$$

To lowest order in s_2

$$\mathcal{I}_{mn}^{\mp}(s_0,s_2)=\int_{-s_0}^{s_0}dx\,\sin\left[\frac{\pi x}{s_0}(m\mp n)\right]\frac{\pi x}{3s_0^2}(\pm ns_0^2\mp nx^2)s_2+\int_{-s_0}^{s_0}dx\,\cos\left[\frac{\pi x}{s_0}(m\mp n)\right]$$
$$=2s_0\delta_{m(\pm n)}+\frac{\pi n s_2}{3s_0^2}\int_{-s_0}^{s_0}dx\,\sin\left[\frac{\pi x}{s_0}(m\mp n)\right](\pm s_0^2x\mp x^3)\ . \tag{A15}$$

Now

$$\int_{-s_0}^{s_0}dx\,x\sin\left[\frac{\pi x}{s_0}(m\mp n)\right]=s_0^2\int_{-1}^{1}dx\,x\sin[\pi x(m\mp n)]$$
$$=2s_0^2\frac{\sin[\pi(m\mp n)]-\pi(m\mp n)\cos[\pi(m\mp n)]}{\pi^2(m\mp n)^2}$$
$$=\begin{cases}\dfrac{-2s_0^2(-1)^{m\mp n}}{\pi(m\mp n)} & \text{for } m\mp n\neq 0\\ 0 \quad \text{for } m\mp n=0\end{cases} \tag{A16}$$

and

$$\int_{-s_0}^{s_0}dx\,x^3\sin\left[\frac{\pi x}{s_0}(m\mp n)\right]=s_0^4\int_{-1}^{1}dx\,x^3\sin[\pi x(m\mp n)]$$
$$=\frac{2s_0^4}{[\pi(m\mp n)]^4}\{6[\pi(m\mp n)]\cos[\pi(m\mp n)]-[\pi(m\mp n)]^3\cos[\pi(m\mp n)]$$
$$-6\sin[\pi(m\mp n)]+3[\pi(m\mp n)]^2\sin[\pi(m\mp n)]\}$$
$$=\begin{cases}\dfrac{12s_0^4(-1)^{m\mp n}}{[\pi(m\mp n)]^3}-\dfrac{2s_0^4(-1)^{m\mp n}}{\pi(m\mp n)} & \text{for } m\mp n\neq 0\\ 0 \quad \text{for } m\mp n=0\end{cases} \tag{A17}$$

so

$$\mathcal{I}_{mn}^{\mp}(s_0,s_2)=\begin{cases}\dfrac{\mp 4\pi n s_0^2 s_2}{\pi^3(m\mp n)^3}(-1)^{m\mp n} & \text{for } m\mp n\neq 0\\ 2s_0 \quad \text{for } m\mp n=0\ .\end{cases} \tag{A18}$$

Similarly we can show

$$\mathcal{L}_{mn}^{\mp}(s_0,0)=\begin{cases}\dfrac{4s_0^3}{\pi^2(m\mp n)^2}(-1)^{m\mp n} & \text{for } m\mp n\neq 0\\ \dfrac{2s_0^3}{3} \quad \text{for } m\mp n=0\ .\end{cases} \tag{A19}$$

Note that (A19) only includes the terms of zero order in

s_2. It turns out that only these terms are necessary to determine the average number of particles to second order in s_2.

It is useful to define

$$R_{nm}(s_0,s_2)=f_{nm}(s_0,s_2)[\mathcal{L}^+_{nm}(s_0,s_2)-\mathcal{L}^-_{nm}(s_0,s_2)]+d^-_{nm}(s_0,s_2)\mathcal{I}^+_{nm}(s_0,s_2)+d^+_{nm}(s_0,s_2)\mathcal{I}^-_{nm}(s_0,s_2) \quad (A20)$$

and

$$S_{nm}(s_0,s_2)=f_{nm}(s_0,s_2)[\mathcal{L}^+_{nm}(s_0,s_2)+\mathcal{L}^-_{nm}(s_0,s_2)]+d^-_{nm}(s_0,s_2)\mathcal{I}^+_{nm}(s_0,s_2)-d^+_{nm}(s_0,s_2)\mathcal{I}^-_{nm}(s_0,s_2)\ , \quad (A21)$$

where

$$f_{mn}(s_0,s_2)=\frac{1}{4}\left[\frac{n}{m}\right]^{1/2}\frac{s_2}{s_0^2} \quad (A22)$$

and

$$d^{\pm}_{mn}(s_0,s_2)=\frac{m\pm n(1+s_0s_2/3)}{4s_0(mn)^{1/2}}\ . \quad (A23)$$

These definitions are equivalent to Eqs. (18) and (19). This simplification is a long but straightforward process, best suited to computer programs such as MATHEMATICA [8].

It is now straightforward to show that

$$\omega(u_m^{(2)},u_n^{(3)})=-\sin\left[\frac{n\pi t_0}{b_0}\right]R_{nm}(b_0,b_2)\ , \quad (A24a)$$

$$\omega(v_m^{(2)},v_n^{(3)})=\sin\left[\frac{n\pi t_0}{b_0}\right]S_{nm}(b_0,b_2)\ , \quad (A24b)$$

$$\omega(u_m^{(2)},v_n^{(3)})=\cos\left[\frac{n\pi t_0}{b_0}\right]R_{nm}(b_0,b_2)\ , \quad (A24c)$$

$$\omega(v_m^{(2)},u_n^{(3)})=\cos\left[\frac{n\pi t_0}{b_0}\right]S_{nm}(b_0,b_2)\ , \quad (A24d)$$

and

$$\omega(u_l^{(1)},u_m^{(2)})=0\ , \quad (A25a)$$

$$\omega(v_l^{(1)},v_m^{(2)})=0\ , \quad (A25b)$$

$$\omega(u_l^{(1)},v_m^{(2)})=-S_{lm}(s_0,s_2)\ , \quad (A25c)$$

$$\omega(v_l^{(1)},u_m^{(2)})=-R_{lm}(s_0,s_2)\ . \quad (A25d)$$

From (14) and (15) we deduce that

$$\hat{p}_n^{(3)}=\sum_{m,l}[\hat{p}_m^{(1)}\omega(v_m^{(1)},u_l^{(2)})\omega(v_l^{(2)},u_n^{(3)})+\hat{q}_m^{(1)}\omega(u_m^{(1)},v_l^{(2)})\omega(u_l^{(2)},u_n^{(3)})] \quad (A26a)$$

and

$$\hat{q}_n^{(3)}=\sum_{m,l}[\hat{p}_m^{(1)}\omega(v_m^{(1)},u_l^{(2)})\omega(v_l^{(2)},v_n^{(3)})+\hat{q}_m^{(1)}\omega(u_m^{(1)},v_l^{(2)})\omega(u_l^{(2)},v_n^{(3)})]\ . \quad (A26b)$$

Consequently from (16) we find that

$$\hat{a}_n^{(3)}=\tfrac{1}{2}\sum_{l,m}(\hat{a}_l^{(1)}\{\omega(u_l^{(1)},v_m^{(2)})[\omega(u_m^{(2)},v_n^{(3)})+i\omega(u_m^{(2)},u_n^{(3)})]+\omega(v_l^{(1)},u_m^{(2)})[\omega(v_m^{(2)},u_n^{(3)})-i\omega(v_m^{(2)},v_n^{(3)})]\}$$
$$+\hat{a}_l^{(1)\dagger}\{\omega(u_l^{(1)},v_m^{(2)})[\omega(u_m^{(2)},v_n^{(3)})+i\omega(u_m^{(2)},u_n^{(3)})]+\omega(v_l^{(1)},u_m^{(2)})[\omega(v_m^{(2)},u_n^{(3)})-i\omega(v_m^{(2)},v_n^{(3)})]\})\ . \quad (A27)$$

On using Equations (A24a)–(A25d), we arrive at (17).

[1] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[2] P. C. W. Davies, J. Phys. A **8**, 609 (1975).
[3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[4] E. Yablonovitch, Phys. Rev. Lett. **58**, 2059 (1987); **62**, 1742 (1989).
[5] E. Yablonovitch, in *Resonances,* edited by M. D. Levenson, E. Mazur, P. S. Pershan, and Y. R. Shen (World Scientific, Singapore, 1990), p. 112.
[6] H. A. Haus (unpublished).
[7] G. T. Moore, J. Math. Phys. **11**, 2679 (1970); S. Sarkar, J. Phys. A **21**, 971 (1988); S. Sarkar, Quantum Opt. **4**, 345 (1992).
[8] S. Wolfram, *Mathematica—A System for Doing Mathematics By Computer,* 2nd ed. (Addison-Wesley, Redwood City, CA, 1991).