Spectral geometry and the N-body problem

Richard L. Hall
Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Boulevard West, Montréal, Québec, Canada H3G 1M8
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Exact N-body solutions for the harmonic oscillator and lower bounds provided by the equivalent two-body method are used as basis results for an application of spectral geometry. It is proved that the relative energy $\mathcal{E}$ of a system of N identical particles interacting by the attractive pair potential $V_0 f(r/a)$ and obeying nonrelativistic quantum mechanics is given approximately by the expression $E \approx \min_{r>0}\{K(N)/r^2 + v f(r)\}$, where $E = (m\mathcal{E}a^2)/(N-1)\hbar^2$ and $v = NmV_0 a^2/2\hbar^2$. A table of K numbers is provided for bosons and fermions (with arbitrary spin) in one and three dimensions. If the potential shape $f(r)$ is a convex function of $-1/r$ and a concave function of r^2, then the approximations yield both upper and lower energy bounds. Detailed results are given for the case of gravitating fermion systems.

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I. INTRODUCTION

The principal subject of this paper is the ground-state energy of a quantum-mechanical system composed of N identical particles bound together by attractive pair potentials. Such systems always collapse in the large-N limit, even if the particles are fermions. That is to say, the binding energy per particle rises with N to infinity [1,2]. Some physical systems such as stars *do,* under suitable circumstances, have this collapsing tendency and other systems such as atoms never have very large values for N. In an earlier paper [3] we studied gravitating boson systems. We now expand the discussion to allow a wider class of interaction potentials and also to include the much more difficult case of fermions. As in the earlier work, the goal is to find energy *bounds.* Because the fermion problem is more complicated, there are fewer opportunities to sharpen any energy bounds we may devise. Hence some of the fine tuning, such as the sharpening of the variational upper energy bounds for bosons [3], will not be pursued here. Rather the emphasis of this article will be on the simplicity and generality of the results. For a given class of potential shapes, we try to find recipes for upper and lower energy bounds, as functions of N and all the other parameters of the problem. When sharpening of results is known to be possible, this will be noted; but the immediate goal is simplicity.

There are three main ingredients to our approach: (i) the "equivalent two-body problem"; (ii) "spectral geometry," which was created principally as a technique for expressing the results of (i); and (iii) exact solutions for harmonic oscillators. The equivalent two-body method (to be discussed in Sec. IV) depends on the symmetrization postulate for systems composed of identical particles: the permutation symmetry of the N-body state induces a relationship between the N-body energy and the spectrum of a specially constructed two-body system in which there is an overall factor of $N-1$ and a strengthened coupling, enhanced by the factor $N/2$; this relationship leads to energy lower bounds. The most unique aspects of this paper are to do with the application of (i) to fermion system and the use of (ii) to exploit (iii) for fermion systems. We restate and simplify some of the earlier results for bosons since they can be expressed, along with the fermion results, in the same framework provided by spectral geometry.

Since spectral geometry is a central theme of this work we describe immediately the main tool we shall need, the "potential-envelope method." In suitable units of Hamiltonian for a single particle moving in $\mathbb{R}^3$ in an attractive central potential with shape $f(r)$ is given by

$$H = -\Delta + v f(r) , \tag{1.1}$$

where v is a positive coupling parameter. For each potential shape $f(r)$ there will be a corresponding family $\{F_{nl}(v)\}$ of energy trajectories that describe how the Schrödinger eigenvalues of H depend on the coupling parameter. Here l is the usual orbital angular momentum quantum number and $n = 1,2,3,\ldots$ labels the discrete eigenvalues in each angular momentum subspace. Thus in our notation the "principal quantum number" of the elementary gravitational problem is given by $(n+l)$.

For the "Coulomb-harmonic" class of potentials, which includes, for example, the following "mixture" of powers and logarithms

$$f(r) = -\frac{1}{r} + \ln(r) + r^{3/2} , \tag{1.2}$$

it is evident that the potential shape can be written simultaneously in the forms

$$g^{(1)}\left[-\frac{1}{r}\right] = f(r) = g^{(2)}(r^2) , \tag{1.3}$$

where the transformation functions $g^{(1)}$ and $g^{(2)}$ are respectively convex and concave in their arguments. For power-law (or logarithmic) *bases,* the energies are given [4,6] by the semiclassical formula

$$F_{nl}(v)\approx \min_{r>0}\left\{\frac{K}{r^2}+vf(r)\right\}. \tag{1.4}$$

For the Coulomb-harmonic class of potential shapes $f(r)$ we have $K=(n+l)^2$ for a lower bound and $K=(2n+l-\frac{1}{2})^2$ for an upper bound; the only restriction for these bounds to be valid is that $f(r)$ be a convex function of $-1/r$ and a concave function of r^2. It is perhaps remarkable that we shall be able to express many of the results of this paper for the N-body problem in the same simple form (1.4): all we shall have to do is provide the appropriate expressions for K in each case. For $f(r)$ in the form of a pure power (1.4) is *exact* and we have

$$f(r)=\mathrm{sgn}(q)r^q\rightarrow F(v)=\mathrm{sgn}(q)cv^{2/(2+q)}$$
$$\Longrightarrow K=\frac{|q|}{2}\left[\frac{2c}{2+q}\right]^{1+2/q}. \tag{1.5}$$

In Sec. II we define the N-body problem and discuss briefly the question of relative coordinates. It turns out that the quality of our general fermion lower bound (derived in Sec. IV) depends on the choice of such coordinates and we are led to find an optimal set; that is to say, a set that yields the best lower bound. In Sec. III we introduce the exact N-body solutions for harmonic oscillators, which we shall employ to construct some general N-fermion energy bounds (both upper and lower) using envelope theory. In Sec. V we apply the general lower bound theory of Sec. IV to the case of gravitating fermion systems: the result of this study provides the Coulomb basis for the complementary *lower* bound formulas generated by envelope theory. In Secs. VI and VII we use spectral geometry to "spread" the Coulomb and harmonic oscillator results to "nearby" potentials, in the sense of smooth transformations. In Sec. VIII we study the Coulomb-harmonic class and express all the energy bounds for this class of potentials in an N-body version of (1.4).

II. THE N-BODY PROBLEM IN THE CENTER-OF-MASS FRAME

The Hamiltonian, with the center-of-mass removed, for a system of N identical particles each of mass m interacting via central pair potentials may be written

$$\mathcal{H}=\frac{1}{2m}\sum_{i=1}^{N}\mathbf{p}_i^2-\frac{1}{2mN}\left[\sum_{i=1}^{N}\mathbf{p}_i\right]^2 + \sum_{j>i=1}^{N}V_0 f\left[\frac{|\mathbf{r}_i-\mathbf{r}_j|}{a}\right], \tag{2.1}$$

where V_0 and a are respectively the depth and the range parameters of the potential with shape f. By algebraic rearrangement (2.1) may be rewritten in the more symmetrical form

$$\mathcal{H}=\sum_{j>i=1}^{N}\left\{\frac{1}{2mN}(\mathbf{p}_i-\mathbf{p}_j)^2+V_0 f\left[\frac{|\mathbf{r}_i-\mathbf{r}_j|}{a}\right]\right\}. \tag{2.2}$$

We now define new coordinates by $\rho=BR$, where $\rho=[\boldsymbol{\rho}_i]$ and $R=[\mathbf{r}_i]$ are column vectors of the new and old coordinates, respectively, and B is a real constant $N\times N$ matrix. For convenience we require all the rows of B to be unit vectors and we let the elements of the first row all be equal to $1/\sqrt{N}$ so that $\boldsymbol{\rho}_1$ is proportional to the center-of-mass coordinate; we also require that the remaining $N-1$ rows of B be orthogonal to the first row so that they define a set of $N-1$ relative coordinates. One more row is also fixed so that we have at least one pair distance at our disposal, namely,

$$\boldsymbol{\rho}_2=\frac{\mathbf{r}_1-\mathbf{r}_2}{\sqrt{2}}. \tag{2.3}$$

For boson systems, we have found that Jacobi relative coordinates, for which B is orthogonal, are the most efficacious; but for systems of fermions, we shall need a wider choice. In general, corresponding to the transformation $\rho=BR$ of the coordinates, it follows that the column vector P of the associated momenta transforms to the new momenta $\Pi=[\pi_i]$ by the relation $\Pi=(B^T)^{-1}P$. We now define the parameter $\lambda\geq 1$, which is almost a coefficient of orthogonality of B, by the relation

$$\lambda=\sum_{j=1}^{N}[(B^T)^{-1}]_{2j}^2. \tag{2.4}$$

The point of this is that if Ψ is any translation-invariant wave function for the N-body system composed of identical bosons or fermions, then we can write [5] the following mean energy relation between the N-body and the two-body systems:

$$(\Psi,\mathcal{H}\Psi)=(\Psi,\mathbb{H}\Psi), \tag{2.5}$$

where the "reduced" two-particle Hamiltonian $\mathbb{H}$ is given by

$$\mathbb{H}=(N-1)\left[\frac{1}{2m\lambda}\pi_2^2+\frac{N}{2}V_0 f\left[\frac{\sqrt{2}|\boldsymbol{\rho}_2|}{a}\right]\right]. \tag{2.6}$$

Further simplifications can be achieved if work with dimensionless quantities. We suppose that the translation-invariant N-body energy is $\mathcal{E}$ and we define the dimensionless energy and coupling parameters E and v by the expressions

$$E=\frac{m\mathcal{E}a^2}{(N-1)\hbar^2},\quad v=\frac{NmV_0a^2}{2\hbar^2}. \tag{2.7}$$

It is then natural to define a dimensionless version of the reduced two-body Hamiltonian $\mathbb{H}$ and the relative coordinate $\boldsymbol{\rho}_2$ by the relations

$$H=\frac{m\mathbb{H}a^2}{(N-1)\hbar^2}=-\frac{1}{\lambda}\Delta+vf(r),$$
$$\mathbf{r}=\sqrt{2}\boldsymbol{\rho}_2/a=\mathbf{r}_1-\mathbf{r}_1, \tag{2.8}$$
$$r=\|\mathbf{r}\|.$$

We note that the Hamiltonian H depends on N through the coupling parameter v and possibly through λ. By the Rayleigh-Ritz (min-max) principle [7–9], we have the following characterization of the N-body ground-state ener-

gy parameter E in terms of H:

$$E=\min_{\Psi}\frac{(\Psi,H\Psi)}{(\Psi,\Psi)}=F^{(N)}(v)\ , \tag{2.9}$$

where Ψ is a translation-invariant function of the $N-1$ relative coordinates (and spin variables, if any), which is either symmetric or anti-symmetric under the permutation of the N individual-particle indices. The N-body energy $\mathcal{E}$ is recovered from E by inverting (2.7). Thus we have explicitly

$$\mathcal{E}=\frac{(N-1)\hbar^2}{ma^2}F^{(N)}\left[\frac{NmV_0a^2}{2\hbar^2}\right]\ . \tag{2.10}$$

However, for the remainder of this paper we shall work with the simple dimensionless form $E=F^{(N)}(v)$: the problem is to find $F^{(N)}$. In each situation we shall have to specify the dimension of the space (usually 3), the type of particle, and its spin.

III. EXACT SOLUTIONS FOR HARMONIC OSCILLATORS

There is a long history of interest in exact N-body solutions for the harmonic oscillator. We shall need essentially four exact solutions: those for bosons and fermions in one and three dimensions. In three dimensions, the question of the spin of the fermions adds another minor variation. The earliest reference that we could find is to a nuclear model by Houston [10] in 1935. The most useful reference for our purposes is a paper by Post [11], who in 1953 found three of the solutions we need. Some of these results were rediscovered later, for example, by Almström [12] in 1969. Applications to nuclear and quark physics may be found in the book by Moshinsky [13], in a paper by Horgan [14], and in a review article by Mitra [15]. Lim [16] solved the three-body problem with exchange forces present; more general solutions of this type were found by Schwesinger [17]. Harmonic "atoms" were considered by Calles and Moshinsky [18] and by Hall [19] and "harmonic matter" was discussed by Hall [20]. Very general solutions allowing for (almost) arbitrary masses and couplings were given by Tikochinsky [21] and by Hall and Schwesinger [22]. Variations on this theme (and some rediscoveries) are still to be found in recent papers, for example, in those by Fan [23] and Michelot [24].

We now turn to the exact solutions required in this paper, each of which will be expressed by giving the appropriate trajectory function. Instead of introducing a distinguishing notation for the various trajectory functions, we shall call them all $F^{(N)}(v)$ and separate the cases by textual comment. For spinless bosons in $\mathbb{R}^D$ we have by the argument (for example) of Post [11]

$$E=F^{(N)}(v)=Dv^{1/2}\ . \tag{3.1}$$

For fermions, the imposition of the necessary permutation antisymmetry is not very comfortable in the center-of-mass frame. Hence, for fermions, following Post [11], we use individual-particle coordinates and then correct for the center-of-mass kinetic energy afterwards. We first consider the case of N spinless fermions in one dimension, that is to say, we find the lowest energy corresponding to a spatially antisymmetric state. In view of the identity

$$\sum_{j>i=1}^{N}(x_i-x_j)^2+N\frac{\left[\sum_{i=1}^{N}x_i\right]^2}{\sqrt{N}}=N\sum_{i=1}^{N}x_i^2\ , \tag{3.2}$$

we see that the dimensionless Hamiltonian *including* the center-of-mass motion is given by

$$H_0=\sum_{i=1}^{N}\{-\Delta_i+Nx_i^2\}\ . \tag{3.3}$$

The energy of the lowest spatially antisymmetric state of this system is given by summing the first N energies of the corresponding single-particle subsystem, that is to say,

$$E_0=\sum_{i=0}^{N-1}(2i+1)\sqrt{N}=N^2\sqrt{N}\ . \tag{3.4}$$

But this includes also the energy of the center of mass $\sqrt{N}$, which we must subtract from E_0. Hence, in terms of the notation (2.7) in which the energy is divided by $N-1$ and the coupling is multiplied by $N/2$, we have the following exact energy for the translation-invariant problem:

$$E=(N+1)v^{1/2}\ . \tag{3.5}$$

Now we shall consider the fermion problem in three dimensions and also allow for spin. We simply extend the argument given above for the problem in one dimension. We suppose that the spin factors span a space of dimension d. Thus, for spin-$\frac{1}{2}$ fermions, $d=2$. Essentially it all comes down to counting linearly independent polynomials in $\mathbb{R}^3$. If we have, say, k quanta, then we have a degeneracy "shell" with size equal to the number of partitions

$$\frac{(k+2)!}{2!k!}=\tfrac{1}{2}(k+1)(k+2)$$

of k over the three dimensions; this must then be multiplied by d to allow for spin. Meanwhile the energy corresponding to each instance of the k quanta is $2k+3$. In order to obtain a simple energy formula we choose N so that exactly ν shells are filled, $\nu=1,2,\ldots$. This is to say, we consider only values of N given by

$$N=\sum_{k=0}^{\nu-1}\frac{d}{2}(k+1)(k+2)=\frac{d}{6}\nu(\nu+1)(\nu+2)\ . \tag{3.6}$$

The corresponding energy (including the center-of-mass kinetic energy of $3\sqrt{N}$) is therefore

$$\begin{aligned}E_0&=\sum_{k=0}^{\nu-1}\frac{d}{2}(k+1)(k+2)(2k+3)\sqrt{N}\\&=\frac{d}{4}\nu(\nu+1)^2(\nu+2)\sqrt{N}\ .\end{aligned} \tag{3.7}$$

Finally, subtracting off the center-of-mass energy and expressing the result in our standard notation, we have for

N fermions in $\mathbb{R}^3$ with spin degeneracy d

$$E=\frac{\frac{d}{4}\nu(\nu+1)^2(\nu+2)-3}{N-1}v^{1/2}. \tag{3.8}$$

where N is given by (3.6). If we consider "spinless fermions" ($d=1$), then we must exclude the case $\nu=1\Longrightarrow N=1$. However, for spin-$\frac{1}{2}$ fermions, we have

$$\nu=1,\ d=2\Longrightarrow N=2\Longrightarrow E=3v^{1/2},$$

which is the correct ground-state energy for the two-body problem in a singlet spin state. It is convenient for our later applications to rewrite (3.8) in the form

$$E=C(N)v^{1/2} \tag{3.9}$$

and to establish the following inequalities for the function $C(N)$ determined implicitly by (3.6) and (3.8):

$$\frac{3}{2}\left\{\frac{6N}{d}\right\}^{1/3}<C(N)<\frac{3N}{2(N-1)}\left\{\left\{\frac{6N}{d}\right\}^{1/3}+\frac{N-2}{N}\right\}$$
$$<\frac{3}{2}\left\{\left\{\frac{6N}{d}\right\}^{1/3}+2\right\}. \tag{3.10}$$

The simpler upper bound on the extreme right-hand side of (3.10) is valid whenever $N\geq\nu+2$, that is to say, for all $\nu\geq 1$ if $d\geq 2$. In all cases, the left-hand side of (3.10) provides the asymptotic form of $C(N)$ for large N.

IV. EQUIVALENT TWO-BODY LOWER ENERGY BOUNDS

The history of the equivalent two-body method for boson systems has been described in recent articles [25,3]. The main result is a general energy lower bound. For boson systems we use Jacobi orthogonal coordinates ($\lambda=1$); with this choice, the lower bound in our notation may be written as

$$E=F^{(N)}(v)\geq e(v), \tag{4.1}$$

where $e(v)$ is the lowest eigenvalue of the one-particle Hamiltonian $H=-\Delta+vf(r)$. As we mentioned in Sec. III, for the harmonic oscillator $f(r)=r^2$ in $\mathbb{R}^D$, this bound yields the exact energy $Dv^{1/2}$. The same reasoning that leads to (4.1) establishes a similar lower bound for fermion systems: the bound is the energy of the lowest antisymmetric state of the two-fermion problem for which H is the reduced Hamiltonian. However, as N increases, the quality of this bound becomes very poor because the two-particle problem keeps so "little" of the full antisymmetry: the bound has the wrong N dependence.

In order to deal with this weakness, another bound had to be devised. Bopp [26] found an approximation for the N-fermion energy as a *sum* of energies of a certain two-fermion problem, but he failed to obtain an energy *bound*. Coleman [27], in a study of density-matrix methods in quantum chemistry, showed how more of the necessary permutation symmetry for atomlike fermion systems could be rigorously incorporated into a lower bound formula. In a later paper Hall [5] applied similar reasoning to generate a lower bound for the more complicated translation-invariant case. Nonorthogonal relative coordinates ($\lambda>1$) are required because it turns out [5] that and optimal lower bound of this type is obtained when the individual-particle exchange (23) converts ρ_2 into ρ_3, thus

$$\rho_3=\frac{\mathbf{r}_1-\mathbf{r}_3}{\sqrt{2}} \tag{4.2}$$

and all the remaining rows of the transformation matrix B are orthogonal to the first row. For such a matrix we find $\lambda=\frac{4}{3}$. The energy formula is derived [5] by first Fourier analyzing the N-body wave function partially in terms of the eigenfunctions of the two-body problem with Hamiltonian (4.4) and then showing that the necessary fermion permutation symmetry implies that the squares of the weights in this expansion are bounded above by $(N-1)^{-1}$. The lower-bound energy formula obtained in this way is given by

$$E>E^L=\frac{1}{\lambda(N-1)}\sum_{i=1}^{N-1}e_i(\lambda v),\quad \lambda=\frac{4}{3}, \tag{4.3}$$

where $e_i(v)$ is the ith eigenvalue of the two-fermion system whose reduced Hamiltonian is given by $H=-\Delta+vf(r)$. In this sum over energies we must count *all* the states of the two-body system from the bottom of the spectrum, *including* spin. A similar energy estimate, which is weaker for large N, may be found in the work of Carr and Post [28].

In order to fix ideas, we now apply (4.3) to the harmonic oscillator for which we have the exact solutions in Sec. III at our disposal, to check the results. In the case of spinless fermions (the lowest spatially antisymmetric states) in one dimension, we have

$$H=-\Delta+vx^2\rightarrow e_i(v)=(2i-1)v^{1/2}, \tag{4.4}$$

where, in order to be consistent with (4.3), we count unconventionally (for the harmonic oscillator) from $i=1$. Hence the exact value [from (3.5)] and the lower bound (4.3) in this case are given by

$$(N+1)v^{1/2}=E>\frac{\sqrt{3}}{2\lambda(N-1)}\sum_{i=1}^{N-1}(2i-1)v^{1/2}$$
$$=\frac{\sqrt{3}}{2}(N-1)v^{1/2}. \tag{4.5}$$

Thus, for large N, the lower bound yields about 86% of the exact energy. It would be interesting to try to do better than this. Obviously the lower bound does not incorporate sufficient antisymmetry. We note that it is easy to falsify the simple conjecture that one should only count the *fermion* two-body states.

We now consider the harmonic oscillator again, but for N fermions in $\mathbb{R}^3$. We suppose that the spin space for a two-particle system has dimension δ: for example, in the case of spin-$\frac{1}{2}$ fermions, we would have $\delta=4$. By reasoning similar to that used in Sec. III, we choose to fill ν shells and, for convenience in the present application of (4.3), we choose to consider those values of N satisfying

$$N-1=\sum_{k=0}^{\nu-1}\frac{\delta}{2}(k+1)(k+2)=\frac{\delta}{6}\nu(\nu+1)(\nu+2)\ . \tag{4.6}$$

With these special values of N the lower bound (4.3) becomes a sum over ν shells

$$E>\frac{\sqrt{3}}{\lambda(N-1)}\sum_{k=0}^{\nu-1}\delta(k+1)(k+2)(2k+3)v^{1/2}\ . \tag{4.7}$$

If we perform this sum and then substitute for $N-1$ from (4.6) we find

$$E>E^L=\frac{3\sqrt{3}}{4}(\nu+1)v^{1/2}\ . \tag{4.8}$$

Thus, in the case of spin-$\frac{1}{2}$ fermions for which $d=2$ and $\delta=4$, the equivalent two-body bound (4.8) and the result (3.10) for the exact energy E imply, in the large-N limit, $E^L/E\to 3^{1/2}2^{-4/3}$; consequently the lower bound yields about 69% of the exact energy in this case. It is an interesting unsolved problem to devise a general lower bound formula which, for the harmonic oscillator in the large-N limit, would yield a better fermion lower energy estimate than this.

V. A LOWER ENERGY BOUND FOR GRAVITATING FERMION SYSTEMS

The problem we consider now is the fermion counterpart of the boson problem we studied in Ref. [3]. We suppose that a system of N identical fermions interacts in $\mathbb{R}^3$ via an attractive Coulomb potential $f(r)=-1/r$. The problem is to find a lower bound to the ground-state energy. The two-body Hamiltonian has the eigenvalues

$$H=-\frac{1}{\lambda}\Delta-\frac{v}{r}\to-\frac{\lambda v^2}{4(n+l)^2}\ . \tag{5.1}$$

For a given value of $k=n+l$, the number of degenerate two-body states is given by the well-known expression δk^2, where δ is the dimension of the spin space of a *two*-particle system. Thus, for convenience, we consider ν closed two-particle shells and set

$$N-1=\sum_{k=1}^{\nu}\delta k^2+\frac{\delta}{6}\nu(\nu+1)(2\nu+1)\ . \tag{5.2}$$

Meanwhile, the lower-bound energy formula (4.3) in this case becomes

$$E>-\frac{1}{\lambda(N-1)}\sum_{k=1}^{\nu}\delta k^2\frac{(\lambda v)^2}{4k^2}=-\frac{\delta\lambda\nu v^2}{4(N-1)}\ . \tag{5.3}$$

If we substitute $\lambda=\frac{4}{3}$ and the expression (5.2) for $N-1$, then we obtain

$$E>-\frac{2v^2}{(\nu+1)(2\nu+1)}\ . \tag{5.4}$$

From (5.2) and (5.4) we can derive the following slightly weaker lower energy bound, which will be very useful in practice:

$$E>E^L=-C(N)v^2=-\left[\frac{\delta}{3(N-1)}\right]^{2/3}v^2\ . \tag{5.5}$$

We shall express this result in energy units when we adjoint an upper bound in Sec. VIII.

VI. UPPER AND LOWER ENERGY BOUNDS FOR $f(r)=g(r^2)$

We now return to the N-body problem and suppose that the potential shape may be expressed as a smooth transformation g of the harmonic oscillator. Thus

$$f(r)=g(r^2)\ . \tag{6.1}$$

We shall obtain lower bounds when g is convex and upper bounds when g is concave. All the exact N-body solutions mentioned in Sec. III have the form

$$E=C(N)v^{1/2}\ . \tag{6.2}$$

Hence by (1.4) and (1.5) we know that the N-body energy bounds are provided by the general expression

$$E\approx\min_{r>0}\left\{\frac{K}{r^2}+vf(r)\right\},\quad K=\left[\frac{C(N)}{2}\right]^2. \tag{6.3}$$

Thus one expression covers the case of bosons or fermions various dimensions: all one has to do is to make sure about convexity and use the correct $C(N)$. If, for convenience, the approximations (3.10) are used for $C(N)$, then, since the right-hand side of (6.3) is monotone increasing in K, the lower estimate for $C(N)$ corresponds to the case of g convex and an energy lower bound and vice versa when g is concave.

VII. LOWER ENERGY BOUNDS FOR $f(r)=g(-r^{-1})$ AND g CONVEX

We now suppose that the potential shape may be expressed as a smooth *convex* transformation g of the Coulomb potential. Thus

$$f(r)=g(-r^{-1}),\quad g\ \text{convex}\ . \tag{7.1}$$

We restrict our considerations to convex transformations since, for the Coulomb potential, we only have N-body *lower* energy bounds at our disposal. Apart from these restrictions, the situation is *mutatis mutandis* the same as for the harmonic transformations. Our Coulomb lower bounds from Sec. V all have the form

$$E>-C(N)v^2 \tag{7.2}$$

and the more general lower bound corresponding to the potential $f(r)$ is therefore provided by

$$E>\min_{r>0}\left\{\frac{K}{r^2}+vf(r)\right\},\quad K=\frac{1}{4C(N)}\ . \tag{7.3}$$

VIII. THE COULOMB-HARMONIC CLASS

The general formulas we provide for the energy bounds are so simple that it is unnecessary to work out many special cases in detail. In this section we collect together the results that apply to the Coulomb-harmonic class of potentials defined by (1.3) in which $g^{(1)}$ is convex and $g^{(2)}$ is

TABLE I. N-body K numbers for Coulomb and harmonic-oscillator envelopes. If the potential shape $f(r)$ is a convex function of $-1/r$ and a concave function of r^2, then by using the appropriate K numbers in expression (8.1) or (8.8), we obtain, respectively, lower and upper energy bounds for the N-body problem.

	Bosons		Fermions	
$h(r)$	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}$	$\mathbb{R}^3$
$-\frac{1}{r}$		1		$\frac{1}{4}\left[\frac{3(N-1)}{\delta}\right]^{2/3}$
r^2	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{1}{4}(N+1)^2$	$\frac{9}{4}\left[\left[\frac{3N}{4d}\right]^{1/3}+1\right]^2$

concave. Thus, where necessary, we must choose upper-bound K numbers for the harmonic oscillator and lower-bound K numbers for the Coulomb potential. From Secs. VI and VII and (1.5) we collect the appropriate K numbers and display them in Table I. In an application, one would first verify that (1.3) is satisfied by the potential shape $f(r)$; then, with the appropriate K numbers from Table I, one would find the corresponding energy bounds using the universal expression

$$E \approx \min_{r>0}\left\{\frac{K}{r^2}+vf(r)\right\}. \tag{8.1}$$

For the special case of pure powers

$$f(r)=\mathrm{sgn}(q)r^q, \quad -1\le q\le 2, \quad q\neq 0, \tag{8.2}$$

we obtain from (8.1) the convenient formula

$$E \approx \mathrm{sgn}(q)\left[1+\frac{2}{q}\right]\left[\frac{2K}{|q|}\right]^{q/(2+q)} v^{2/(2+q)}. \tag{8.3}$$

Thus, returning from (2.10) all the details of the final energy result, we have, in particular, for the gravitating fermion system

$$-\frac{m\gamma^2}{\hbar^2}AN^2(N-1)^{1/3} < \mathcal{E} < -\frac{m\gamma^2}{\hbar^2}BN^{7/3}\left[1+\left[\frac{4d}{3N}\right]^{1/3}\right]^2, \tag{8.4}$$

where $\gamma=V_0a$ is the gravitational coupling parameter, the numerical factors A and B are given by

$$\frac{1}{4}\left[\frac{\delta}{3}\right]^{2/3} = A < B = \frac{1}{9}\left[\frac{d}{6}\right]^{2/3}, \tag{8.5}$$

and d and δ are the dimensions, respectively, of the one- and two-particle spin spaces. Hence, for spinless fermions ($d=\delta=1$), we obtain the values given by the *inner* pair of the following inequalities:

$$-\tfrac{1}{2} < -0.1202 < A < B < -0.0336 < -0.001\,06\,; \tag{8.6}$$

the outer pair are from the paper of Levy-Leblond [29]. Thus, after 25 years, the upper and lower bounds that we offer for gravitating fermions, although better, still differ by a factor of 3. For gravitating spin-$\frac{1}{2}$ fermions $d=2$, $\delta=4$, the corresponding numerical factors become

$$-0.3029 < A < B < -0.0534\,. \tag{8.7}$$

Bounds of similar quality may easily be obtained for any member of the entire Coulomb-harmonic class of potentials. For potentials other than pure powers, such as (1.2), the dependence of the N-body energy bounds on v may be expressed parametrically, in terms of $r>0$. The appropriate formulas follow immediately from (8.1), by differentiation. Thus

$$E \approx K\left\{\frac{1}{r^2}+\frac{2f(r)}{r^3f'(r)}\right\}, \quad v=\frac{2K}{r^3f'(r)}\,. \tag{8.8}$$

It is interesting that, for a given potential shape $f(r)$, all these N-body energy trajectories for various systems of bosons or fermions are magnifications of a single curve, with magnification K.

IX. CONCLUSION

The main purpose of this paper is to derive the simple recipe (8.8) for bounds on the energy of the many-body problem. We have considered this possibility earlier, for the case of bosons; but for fermions it is different. By eschewing the opportunities for refinement that are available in the case of bosons, our results for the two classes of many-body problem can be expressed in terms of a single formula.

Our lower bound is based on the Coulomb potential for two reasons: first, many potentials of interest are convex functions of $-1/r$ and convexity is required by the envelope method in order to yield a lower energy bound; second, the Coulomb potential is convenient because the sum over the two-body energies needed by the equivalent two-body method may be expressed in closed form. Clearly other choices of lower-bound basis are possible: thus Table I is extensible.

As physics embraces more fully the nonlinear alternative, the importance of precious known exact solutions, in any situation, has become more emphasized. Even though quantum mechanics is a linear theory, the many-body problem remains seemingly intractable in general. In this article we have used geometric technique to "spread" the gravitational lower bound and the oscillator solubility, approximately, to interesting classes of potentials obtained from $-1/r$ and r^2 by smooth transformations. Whenever these transformations have definite convexity, the approximations become energy *bounds*.

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