Harmonic oscillator with the radiation reaction interaction

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Classical and quantum (semiclassical) theory of the radiation reaction interaction is applied to the harmonic oscillator. Spontaneous emission is analyzed and it is shown that both approaches give the same result. The energy shift of bound states is described.

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I. INTRODUCTION

Two effects in the interaction of radiation with matter have been beyond the reach of theoretical description for a long time. One is the spontaneous decay and the other is the frequency shift of radiation which is emitted by a charge when interacting with an electromagnetic (EM) field (often referred to as the Lamb shift, and not to be confused with Compton's frequency shift). The first is a very important effect because without its proper theoretical description the dynamics equations which include both the EM field and charges would not obey the energy conservation law. In contrast, the Lamb effect does not have such a dramatic importance—it is a minor effect—and yet its description requires a major theoretical effort. Both effects are the result of the radiation reaction interaction (RRI), the interaction by which the charge's EM field affects its own dynamics. One of the earliest efforts to formulate the RRI is nearly a century old [1,2]. When quantum theory was established the pursuit took a different line, but even there the effort within semiclassical radiation theory (charge is treated by quantum theory while the EM field is treated classically) was abandoned for various (albeit very debatable) reasons [3,4]. Today, it is believed, the theory which describes the RRI properly, and hence the spontaneous decay and the Lamb shift, is quantum electrodynamics (QED). The success of QED is indisputable, but it is based primarily on the excellent agreement between the theoretical and the experimental value for, say, Lamb's shift rather than its theoretical foundation [5–7].

In this work we will investigate the RRI in the formulation of the semiclassical radiation theory. As it was pointed out earlier, serious objections were raised against this approach, but in defense of it the last word was not said, and in order to do this the limits of semiclassical theory should be thoroughly investigated. This work has the aim of investigating these limits with the emphasis on showing how spontaneous emission and the frequency shift can be described, and then applied to the harmonic oscillator problem.

The efforts to formulate the semiclassical theory of the RRI are not recent. One of the earliest is due to Fermi [8], but it was far from being a systematic study and a a serious investigation of the problem. Semiclassical arguments were also used in one of the first studies of the Lamb shift [9], but again the use of the theory was far from being systematic and mathematically sound. More systematic semiclassical study of the RRI was made at a much later date [10], but several crucial aspects of the theory failed to be noted. In particular, the roles of the Coulomb and the magnetic terms were not investigated. Recently, the theory of the RRI was developed starting from a general principle, the uncertainty principle [11–13]. This approach prevents any ad hoc assumptions, a typical feature of the previous semiclassical analysis of the RRI. One of these assumptions is the neglect of the Coulomb interaction for which it can be shown to be nonphysical [11–13]. Furthermore, by starting from this general principle it is possible to formulate the classical theory of the RRI, and in this way bridge the gap between classical and quantum theory which exists if within the former one keeps the deterministic view.

The basic idea in the formulation of the RRI is to note that the position and the momentum of a charge are given by the probability distributions $P(\vec{r}, t)$ and $Q(\vec{p}, t)$, respectively. If the uncertainty principle is introduced, then P and Q are related, and as a consequence all the observables have average values. This also applies to the EM field produced by a charge, and so the average value of, say, the scalar potential is (retardation is neglected)

$$\Phi_{\rm av} = \int d^3r' \frac{P(\vec{r}', t)}{|\vec{r} - \vec{r}'|} \tag{1}$$

which is exactly the potential if $P(\vec{r}, t)$ is treated as a charge density. Therefore $P(\vec{r}, t)$ plays the role of the charge density, and similarly one can show that the probability current plays the role of the charge current. The EM field is then derived from the scalar and the vector potentials, which obey the equations

$$\Delta\Phi - \frac{\partial^2\Phi}{\partial t^2} = -4\pi\alpha P(\vec{r}, t) , \tag{2}$$

$$\Delta\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} = -4\pi\alpha\vec{J}(\vec{r}, t) ,$$

where we use the scaled coordinates: $\vec{r}$ and t designate the products $\kappa\vec{r}$ and $c\kappa t$, respectively, where $\kappa = mc/\hbar$ is Compton's wave number. The fine structure constant is defined as $\alpha = e^2/(c\hbar)$. In this scaling Φ and $\vec{A}$ represent the true scalar and vector potentials multiplied by $e/(mc^2)$. In quantum theory $P(\vec{r}, t) = |\psi(\vec{r}, t)|^2$ and

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$$\vec{J} = \mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right) - \vec{A}|\psi|^2 \tag{3}$$

while in classical theory one defines the phase space density $\rho(\vec{r},\vec{p},t)$ from which

$$P(\vec{r},t) = \int d^3p\,\rho(\vec{r},\vec{p},t)\ , \tag{4}$$
$$\vec{J}(\vec{r},t) = \int d^3p\,\frac{\vec{p}}{m}\rho(\vec{r},\vec{p},t)\ .$$

The basic idea in the formulation of the RRI is to use this EM field in the equations of motion for charge: Schrödinger's equation in quantum theory and Newton's equation in classical theory. Schrödinger's equation modifies into

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\left(\vec{\nabla} - i\vec{A}\right)^2\psi + \Phi\psi + V\psi \tag{5}$$

while the classical equation is

$$\ddot{\vec{r}} = -\vec{\nabla}V + \vec{E} + \vec{v}\times\vec{H}\ , \tag{6}$$

where $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t$ and $\vec{H} = \vec{\nabla}\times\vec{A}$. In further discussion we will discuss Schrödinger's equation, since the same arguments apply to the classical equation of motion.

This straightforward formulation of the RRI produces a nonphysical equation of motion for the charge. The source of the nonphysical term is the assumption that $P(\vec{r},t)$ represents the charge density, which is really not true. The probability density can only interact with itself if the retardation effect is taken into account, and hence the instantaneous Coulomb term has to be omitted from the equation of motion. This, however, is not sufficient because the instantaneous Coulomb field in one coordinate system may produce additional instantaneous magnetic field in another. For example, two charges in the coordinate system where they are both at rest interact through the static Coulomb interaction, but in the frame where they move they also interact through the magnetic force. Obviously, this magnetic force is not an essentially new interaction, but the result of moving from one coordinate system into the other. If the instantaneous Coulomb interaction is to be neglected, then the same should be applied to the magnetic term which arises by moving from one coordinate system into the other. Changing the coordinate system means that the wave function acquires the phase $\vec{v}(t)\cdot\vec{r}$, where $\vec{v}(t)$ is the translation velocity (note the scaled variables). This means that the current acquires an additional term of the form $\vec{v}(t)\,P(\vec{r},t)$, and the vector potential (the part which is instantaneous) acquires the term

$$\vec{A}_{\mathrm{inst}} = \alpha\int d^3r'\,\frac{\vec{v}(t)P(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\ . \tag{7}$$

Therefore, if the instantaneous scalar potential is omitted, then the vector potential should also be modified by the term (7). From the quantum definition of the current, and writing the wave function as $\psi = \sqrt{P}\,\exp(i\delta)$, then

$$\vec{J} = P(\vec{r},t)\vec{\nabla}\delta(\vec{r},t) = P\left[\vec{v}(t) + \left(\vec{r}\cdot\vec{\nabla}\right)\vec{\nabla}\delta + \cdots\right]\ , \tag{8}$$

where $\vec{v}(t)$ represents translation velocity of the probability distribution as the whole. It follows that

$$\vec{v}(t) = \frac{\vec{J}(0,t)}{P(0,t)} = \frac{\vec{J}_0(t)}{P_0(t)} \tag{9}$$

which determines $\vec{A}_{\mathrm{inst}}$ in terms of the probability distribution and the current.

The scalar and the vector potentials are

$$\Phi = \alpha\int d^3r'\,\frac{P(\vec{r}',t-R)}{R}\ , \tag{10}$$
$$\vec{A} = \alpha\int d^3r'\,\frac{\vec{J}(\vec{r}',t-R)}{R}\ ,$$

where $R = |\vec{r}-\vec{r}'|$. If the retardation effect is small, then both P and $\vec{J}$ can be expanded in the series of R, in which case the scalar potential is approximately

$$\Phi \approx \alpha\int d^3r'\frac{P(\vec{r}',t)}{R} + \frac{\alpha}{2}\int d^3r'\,R\,\ddot{P}(\vec{r}',t) - \frac{\alpha}{6}\int d^3r'\,R^2\,\frac{d}{dt}\ddot{P}(\vec{r}',t)\ , \tag{11}$$

where the first term is recognized as the instantaneous scalar potential. In further discussion this term is omitted (the next term of the order $\dot{P}$ is exactly equal to zero).

Similarly we expand the vector potential and the first few terms are

$$\vec{A} = \alpha\int d^3r'\frac{\vec{J}(\vec{r}',t)}{R} - \alpha\int d^3r'\,\dot{\vec{J}}(\vec{r}',t) + \cdots\ . \tag{12}$$

The neglect of the instantaneous Coulomb term in (11) has as the consequence, according to the previous discussion, modification of the vector potential with $\vec{A}_{\mathrm{inst}}$. Therefore, the vector potential which describes the RRI is

$$\vec{A} = \alpha\int\frac{d^3r'}{R}\left[\vec{J}(\vec{r}',t) - \frac{\vec{J}_0(t)}{P_0(t)}P(\vec{r}',t)\right] - \alpha\int d^3r'\,\dot{\vec{J}}(\vec{r}',t) + \cdots \tag{13}$$

and Schrödinger's equation (5) is

$$i\dot{w} = -\frac{1}{2}\left(\vec{\nabla} - i\vec{A}_0\right)^2 w + Vw + \frac{\alpha w}{2}\int d^3r'R\ddot{P} - \frac{2\alpha w}{3}\vec{r}\int d^3r'\dddot{\vec{J}}\ , \tag{14}$$

where we made replacement

$$\psi = w(\vec{r},t)\,\exp\left[-i\alpha\vec{r}\cdot\int d^3r'\dot{\vec{J}} - i\frac{\alpha}{3}\int d^3r'\,\vec{r}'\cdot\dot{\vec{J}}\right]\ . \tag{15}$$

$\vec{A}_0$ is the first term in the expansion (13).

In this way we formulated the dynamics equation in quantum (semiclassical) theory, which includes the RRI. The classical dynamics equation is obtained from (6) and will be discussed in a separate section. The RRI theory will be applied to the harmonic oscillator problem: charge is bound by the harmonic force and it is interacting with the plane EM wave. This is also known as the "forced harmonic oscillator with damping" problem, which has been treated on a number of occasions. The relationship of those studies to ours will be reviewed at the end of the paper.

II. QUANTUM THEORY

The interaction of a harmonic oscillator with a time varying, linearly polarized, electric field will be analyzed. The quantum equation, without the radiation reaction interaction included, is in this case

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + \frac{\omega^2}{2}r^2\psi + \epsilon z f_0(t)\psi , \tag{16}$$

where we use the scaled coordinates and time, defined in Sec. I, and ω stands for $\omega/(c\kappa)$. The function $f_0(t)$ determines the time dependence of the electric field. The solution of (16) can be written as the product

$$\psi = \psi_x(x,t)\psi_y(y,t)\psi_z(z,t) \tag{17}$$

and the functions which depend only on the coordinates x and y satisfy the unperturbed equation for harmonic oscillator, while ψ_z satisfies

$$i\dot{\psi}_z = -\frac{1}{2}\frac{\partial^2}{\partial z^2}\psi_z + \frac{\omega^2}{2}z^2\psi_z + \epsilon z f_0(t)\psi_z . \tag{18}$$

The solution of (18) can be parametrized as

$$\psi_z = \phi[z - c(t), t]\, e^{a(t)+z\,b(t)} , \tag{19}$$

where the parameters satisfy the equations

$$\dot{b}(t) = -i\epsilon f_0(t) - i\omega^2 c(t) , \quad \dot{c}(t) = -i\,b(t) , \tag{20}$$
$$\dot{a}(t) = \frac{i}{2}\left[b^2(t) + \omega^2 c^2(t)\right] ,$$

and ϕ satisfies

$$i\dot{\phi} = -\frac{1}{2}\frac{\partial^2}{\partial \underline{z}^2}\phi + \frac{\omega^2}{2}\underline{z}^2\phi , \tag{21}$$

where the new coordinate is $\underline{z} = z - c(t)$. The most general solution of Eqs. (20) is

$$b = C_1 e^{-it\omega} + C_2 e^{it\omega} - i\epsilon\int_0^t dt' \cos[\omega(t-t')]\, f_0(t'), \tag{22}$$

$$c = \frac{1}{\omega}\left(C_1 e^{-it\omega} - C_2\, e^{it\omega}\right) - \frac{\epsilon}{\omega}\int_0^t dt' \sin[\omega(t-t')]\, f_0(t') ,$$

where C_1 and C_2 are constants which need to be determined from the initial conditions. In our studies it will be assumed that at $t = 0$ the initial ψ corresponds to one of the eigenfunctions of the harmonic oscillator, say the ground state. In this case the value of the parameters a, b, and c in (19) is zero at $t = 0$, which implies that $C_1 = C_2 = 0$ in (22).

Equation (21) is the same as for the unperturbed harmonic oscillator, and the time evolution of the solution is given by

$$\phi(\underline{z}, t) = \int dz'\, G(\underline{z} - z', t)\, \kappa_0(z') , \tag{23}$$

where κ_0 is the initial wave function at $t = 0$. The Green's function $G(z,t)$ is given by

$$G(\underline{z} - z', t) = -i\Theta(t)\sum_n \phi_n(\underline{z})\phi_n(z')\, e^{-itE_n} , \tag{24}$$

where ϕ_n are the eigenfunctions of the harmonic oscillator, and the corresponding eigenvalues are E_n. The step function $\Theta(t)$ is zero for $t < 0$ and equal to 1 otherwise. The same solution applies for the other coordinates, the only difference being that the coordinate $\underline{z}$ in (23) is time dependent. In this way we obtained the solution of the time dependent equation (16), in the form of the product (17), where each component has the time evolution in the form (23).

When the RRI is included in the dynamics of the forced harmonic oscillator, Eq. (16) modifies, according to (14), into

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + \frac{\omega^2}{2}r^2\psi + \epsilon z f_0(t)\psi + \vec{r}\cdot\dot{\vec{A}}_{\rm ret}\psi , \tag{25}$$

where

$$\vec{A}_{\rm ret} = -\frac{2\alpha}{3}\int d^3r\dot{\vec{J}} , \tag{26}$$

where we included only that part of the RRI which describes the spontaneous emission. As will be shown, the other terms contribute to the energy shift of states. Equation (25) is nonlinear, but for a harmonic oscillator it has an analytic solution. It is obtained by noting that when the solution of the unperturbed oscillator (RRI is neglected) is used for calculating $\vec{A}_{\rm ret}$, we obtain

$$\vec{r}\cdot\dot{\vec{A}}_{\rm ret} = qz\epsilon\left[\dot{f}_0(t) - \omega^2\int_0^t dt' \cos[\omega(t-t')]\, f_0(t')\right] , \tag{27}$$

where $q = 2\alpha/3$. In this derivation we used the relationship

$$\int d^3r\, \vec{J}(\vec{r},t) = \int d^3r\, \mathrm{Im}[\psi^*\nabla\psi] = \mathrm{Im}[b]\,\hat{z}$$
$$= -\hat{z}\,\epsilon\int_0^t dt' \cos[\omega(t-t')]\, f_0(t') . \tag{28}$$

The interaction term (27) is very similar in structure to the external force in (16), and the two together in

(25) form the interaction term which can be written as $\epsilon z f_1(t)$. The iteration is repeated, but now the starting function is f_1, and it is continued until the iterations converge. This will happen when the equation

$$f(t) = f_0(t) + q\left[\dot{f}(t) - \omega^2 \int_0^t dt' \cos[\omega(t-t')]\, f(t')\right] \tag{29}$$

is satisfied. The function $f(t)$ is the solution of the iteration procedure, and replaces the original function $f_0(t)$ in (22) for the solution of the RRI.

The function $f(t)$ satisfies an integro-differential equation, which is solved by differentiating (29) twice, and then forming a linear combination of the resulting equation with the original one to get

$$-q\dddot{f} + \ddot{f} + \omega^2 f = \ddot{f}_0 + \omega^2 f_0 . \tag{30}$$

The most general solution of the last equation can be written in the form

$$f = \sum_{n=1}^{3}\left[a_n - \frac{\gamma_n + \omega^2}{q(\gamma_n - \gamma_{n+1})(\gamma_n - \gamma_{n+2})} \times \int_0^t dt'\, e^{-\gamma_n t'} f_0(t')\right] e^{\gamma_n t} , \tag{31}$$

where the indices $n+1$ and $n+2$ of γ mean that the index other than n is taken, e.g., if $n=2$, then $n+1=3$ and $n+2=1$. The parameters a_n are constants which need to be determined, while γ_n satisfies the equation

$$-q\gamma_n^3 + \gamma_n^2 + \omega^2 = 0. \tag{32}$$

The roots of this equation are complicated, and their precise value will not be given. However, two of them are complex, with their real part negative, while the third is real and positive. Their expansion in the power series in q is

$$\begin{aligned}
\gamma_1 &= i\omega - q\frac{\omega^2}{2} - \frac{5i}{8}q^2\omega^3 + q^3\omega^4 + O(q^4) , \\
\gamma_2 &= -i\omega - q\frac{\omega^2}{2} + \frac{5i}{8}q^2\omega^3 + q^3\omega^4 + O(q^4) , \\
\gamma_3 &= \frac{1}{q} + q\omega^2 - 2q^3\omega^4 + O(q^5) .
\end{aligned} \tag{33}$$

The real part of γ_3 is positive and hence the solution for $f(t)$ diverges when $t \to \infty$. There are two remedies to this problem: either one dismisses the term $n=3$ in the sum for f or imposes requirement on the parameter a_3 so that the solution is finite. It turns out, as it will be shown shortly, that the second option is the only choice, i.e., dismissing the term $n=3$ produces a nonphysical result. From this requirement a_3 is not arbitrary, and only the other two need to be determined. The function (31) is the most general solution of Eq. (30), but that does not mean that it satisfies the original equation (29). If this requirement is imposed, then we get a condition which the constants a_1 and a_2 should satisfy. After a somewhat lengthy algebra one gets two equations for these two constants, and when they are solved, the final solution for $f(t)$ is

$$\begin{aligned}
f(t) = {} & 2\,\mathrm{Re}\left\{\frac{\gamma_1^2 + \omega^2}{q(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}\left[I_0 - \int_0^t dt'\, e^{-\gamma_1 t'} f_0(t')\right] e^{\gamma_1 t}\right\} \\
& + \frac{\gamma_3^2 + \omega^2}{q(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_3)} \int_t^\infty dt'\, e^{\gamma_3(t-t')} f_0(t') \equiv I_1 + I_2 ,
\end{aligned} \tag{34}$$

where

$$I_0 = \int_0^\infty dt'\, e^{-\gamma_3 t'} f_0(t') . \tag{35}$$

Therefore, we obtained the solution for $f(t)$, which satisfies (29) and is finite in the limit $t \to \infty$. The solution has no arbitrary parameters. It would now be of interest to expand the solution in the powers series in q, and check whether the original iteration procedure is recovered (the iteration procedure is in fact a power series in q). After somewhat lengthy algebra the two terms in (34) have expansion

$$\begin{aligned}
I_1 &\sim -q\omega^2 \int_0^t dt'\, f_0(t') \cos[\omega(t-t')] , \\
I_2 &\sim f_0(t) + q\dot{f}_0(t) ,
\end{aligned} \tag{36}$$

and together they indeed reproduce the first iteration step in solving Eq. (25). The most dominant contribution in the series comes from the term I_2 which could have been dismissed as nonphysical. As we see, it was essential not to dismiss it and instead require that the solution for $f(t)$ is finite for infinite time.

III. CLASSICAL THEORY

Classical theory, which includes the RRI, cannot be formulated without an amendment. It involves abandoning the deterministic basis of classical theory and introducing in it the principle of uncertainty. This means that instead of the usual question—given the initial position and velocity of a particle what is its position at some later time?—one should formulate classical theory to answer the question—given initial probability densities of finding a particle at a certain position and with certain velocity, what are these probability densities at some later time? The uncertainty principle requires that the

probability densities are related, which prevents making both of them arbitrarily narrow, thus reducing classical theory to the usual deterministic basis. The relationship between the two densities is not self-evident, but quantum theory offers one possibility, which is adopted in our analysis.

Once one works with the probability densities in classical theory, the equation of motion which includes the RRI is formulated along the lines of the quantum dynamics equation. Thus Newton's equation for the same system as in (16) is (in the scaled variables)

$$\ddot{\vec{r}} = -\omega^2 \vec{r} + \vec{E} + \vec{v} \times \vec{H} - \epsilon \hat{z} f_0(t) , \tag{37}$$

where $\vec{E}$ and $\vec{H}$ depend on the probability density and the probability current. Therefore, this equation has no meaning for a "pointlike" particle, but for a particle which is represented by the probability density in the phase space. Specifically, if the energy shift is neglected, then the only component of the RRI potential is scalar,

$$\Phi = \vec{r} \cdot \dot{\vec{A}}_{\rm ret}, \tag{38}$$

where we used definition (25). It follows that $\vec{H}$ is zero and $\vec{E}$ is

$$\vec{E} = -\vec{\nabla}\Phi = -\dot{\vec{A}}_{\rm ret} = \frac{2\alpha}{3}\frac{d^2}{dt^2}\int d^3r \, \vec{J} , \tag{39}$$

where now the current is

$$\vec{J} = \int d^3v \, \vec{v} \, \rho(\vec{r}, \vec{v}, t) , \tag{40}$$

where ρ is the phase space probability density. It has the property

$$P(\vec{r}, t) = \int d^3v \, \rho(\vec{r}, \vec{v}, t) , \quad Q(\vec{v}, t) = \int d^3r \, \rho(\vec{r}, \vec{v}, t) , \tag{41}$$

where P is the probability density in the coordinate space and Q is an analogous quantity in the velocity space. The uncertainty principle requires that P and Q are related, and so ρ must reflect this relationship.

The RRI term (39) in the classical equation of motion can be transformed by using an important property of the phase space density. This property says that the total amount of phase space probability in a volume element $d^3r d^3v$ stays constant in time, in which case the interaction term (39) becomes

$$\begin{aligned} \vec{E} &= \frac{2\alpha}{3}\frac{d^2}{dt^2}\int d^3r \, \vec{J} \\ &= \frac{2\alpha}{3}\frac{d^2}{dt^2}\int d^3r_0 d^3v_0 \, \vec{v}(t) \, \rho_0(\vec{r}_0, \vec{v}_0) , \end{aligned} \tag{42}$$

where ρ_0 is the initial phase space probability density, and $\vec{v}$ is the velocity of a particle at time t if its initial coordinate was $\vec{r}_0$ and its velocity $\vec{v}_0$. Initial ρ_0 is determined from the initial conditions, and in the example of Sec. II this is the ground state of the harmonic oscillator. In this case ρ_0 is given simply as the product of P_0 and Q_0, or explicitly

$$\rho_0(\vec{r}, \vec{v}) = \frac{1}{\pi^3} e^{-\omega r^2 - v^2/\omega} . \tag{43}$$

The classical equation of motion (37) is very complicated, because at each instant of time the force on the particle is given by the contribution from all the other trajectories which are sampled into the phase space density ρ. Nevertheless, the equation can be solved analytically for the harmonic oscillator, following the same procedure as in the case of the quantum solution. In the first step we neglect the RRI and solve the resulting equation. Since the RRI and the external field affect only the z coordinate, we only have to consider the equation

$$\ddot{z} = -\omega^2 z - \epsilon f_0(t) \tag{44}$$

which has the solution

$$\begin{aligned} z = z_0 \cos(\omega t) + \frac{\dot{z}_0}{\omega}\sin(\omega t) \\ -\frac{\epsilon}{\omega}\int dt' \, \sin[\omega(t - t')] \, f_0(t') , \end{aligned} \tag{45}$$

where z_0 and $\dot{z}_0$ are initial values for the coordinate and the velocity, respectively. The electric field (42), which has only the z component, can now be calculated. First, we need the third time derivative of z, which is given by

$$\frac{d^3 z}{dt^3} = -\omega^2 \dot{z} - \epsilon \dot{f}_0(t) \tag{46}$$

and when this is replaced in (42) the electric field is

$$\vec{E} = \frac{2\alpha\epsilon}{3}\left[-\dot{f}_0 + \omega^2 \int dt' \, \cos[\omega(t - t')] \, f_0(t')\right] \hat{z} . \tag{47}$$

The structure of the electric field is the same as that of the external field, and together they form an effective external field with the time dependent factor $f_1(t)$. We proceed as in Sec. II until the iterations converge. The equation which is obtained for the iterated function $f(t)$ is the same as (29), and so its solution is (34). Therefore, both classical and quantum theory predict the same function $f(t)$ which describes the combined effect of the external force on the harmonic oscillator and the RRI. The question is now whether classical theory gives the same answer for the time evolution of the probability distribution and the current.

The probability distribution P is defined in (41), and when one uses the property of the phase space density

$$\rho(\vec{r}, \vec{v}, t) d^3v = \frac{1}{D}\rho_0(\vec{r}_0, \vec{v}_0) d^3v_0, \tag{48}$$

where $D = d^3r/d^3r_0$ is the Jacobian, then the z dependence of the probability density is

$$P(z, t) = \int d\dot{z}_0 \frac{1}{\frac{dz}{dz_0}} \rho_0(z_0, \dot{z}_0) . \tag{49}$$

From the solution for z we find $dz/dz_0 = \cos(wt)$, and also the relationship of z_0 to z and $\dot{z}_0$. When all these are replaced in the expression for $P(z,t)$, the integration over the variable $\dot{z}_0$ produces the same result as the squared module of (19). Therefore we have shown that classical theory gives the same answer for the dynamics of the probability distribution as quantum theory.

Similarly, one can calculate the current. Its z component is

$$J_z(z,t) = \int d\dot{z}_0 \frac{1}{\frac{dz}{dz_0}} , \dot{z}\, \rho_0(z_0, \dot{z}_0), \tag{50}$$

where $\dot{z}$ is calculated from the solution (45). By following the steps as in calculating $P(z,t)$ we obtain the same result as the quantum in (28). From the current and the probability densities we can calculate the "classical" wave function by noting that

$$\psi(z,t) = \sqrt{P(z,t)}\, e^{i\delta(z,t)} , \tag{51}$$

where

$$\delta(z,t) = \int^z dz' \frac{J(z',t)}{P(z',t)} \tag{52}$$

and the result is the same as (19), except for a nonessential z independent phase.

We have shown that classical theory gives exactly the same result as quantum theory for the effect of the RRI on the motion of a charge. One could consider it a surprising result, on the other hand, that the harmonic oscillator is a rather special case for which it was shown that this equality exists even when the empirical damping term is included [14]. However, in our case the equations are nonlinear and it is not self-evident that this equality should exist.

IV. ENERGY SHIFT

In the study so far we have only considered a single term from among those which describe the effect of the RRI on the dynamics of a charge. The meaning of the others will be discussed in this section. Schrödinger's equation, which includes these terms, and when the term for the spontaneous emission is neglected, is

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + \frac{\omega^2}{2} r^2 \psi + \frac{i}{2}\psi \vec{\nabla}\vec{A}_0 + i\vec{A}_0 \cdot \vec{\nabla}\psi + \frac{\alpha\psi}{2}\int d^3r'\, R\, \ddot{P} + \epsilon\, z\, f_0(t)\, \psi , \tag{53}$$

where we neglected A_0^2. If the spontaneous emission is included, then as an approximation we can replace $f_0(t)$ by $f(t)$, given by (34). At $t = 0$ it is assumed that ψ is one of the eigenstates (ground state) of the harmonic oscillator, which means that the RRI is exactly zero.

There are two extreme cases which can be analyzed, and the first is the weak coupling limit. In this limit $f(t)$ is small so that ψ is only slightly different from the initial stationary state. We can write in this case

$$\psi = e^{-iE_{n_0}t}\phi_{n_0}(\vec{r}) + \psi^{(1)} = \psi^{(0)} + \psi^{(1)} , \tag{54}$$

where now $\psi^{(1)}$ satisfies the equation

$$i\dot{\psi}^{(1)} = -\frac{1}{2}\Delta\psi^{(1)} + \frac{\omega^2}{2} r^2 \psi^{(1)} + \frac{i}{2}\psi^{(0)}\vec{\nabla}\cdot\vec{A}_0^{(1)} + i\vec{A}_0^{(1)}\cdot\vec{\nabla}\psi^{(0)} + \frac{\alpha}{2}\psi^{(0)}\int d^3r'\, R\, \ddot{P}^{(1)} , \tag{55}$$

where we neglected the terms of the order $\left(\psi^{(1)}\right)^2$, and higher. In the last equation we use definitions

$$P^{(1)} = \psi^{(0)^*}\psi^{(1)} + \psi^{(0)}\psi^{(1)^*} ,$$

$$\vec{A}_0{}^{(1)} = \alpha \int \frac{d^3r'}{R}\left(\vec{J}^{(1)} - \frac{\vec{J}_0{}^{(1)}}{\phi_{n_0}^2(0)}\phi_{n_0}^2(\vec{r}')\right) , \tag{56}$$

$$\vec{J}^{(1)} = \mathrm{Im}\left(\psi^{(0)^*}\vec{\nabla}\psi^{(1)} + \psi^{(1)^*}\vec{\nabla}\psi^{(0)}\right) .$$

Although the equation is now linear in the unknown $\psi^{(1)}$, it is still an integro-differential equation, and can be solved by expanding the solution in the complete set of eigenfunctions for harmonic oscillator

$$\psi^{(1)} = \sum_n C_n(t)\, \phi_n(\vec{r}) , \tag{57}$$

where the index n stands for three quantum numbers. In general, the equations which C_n satisfy are quite complicated, but in the weak coupling limit the dominant coefficients are C_0 (for the ground state) and the one which goes with the eigenfunction $z\exp(-\omega r^2/2)$. The last one will be labeled with the index $n = 1$. The equations for C_n simplify considerably, and for C_1 it is

$$i\dot{C}_1 = E_1 C_1 + \frac{i}{2}\int d^3r \phi_1\phi_0\vec{\nabla}\cdot\vec{a}_{(1)} + iC_1\int d^3r\vec{a}_{(1)}\cdot\phi_1\vec{\nabla}\phi_0 + \frac{\alpha}{2}e^{-iE_0t}\frac{d^2}{dt^2}\left(e^{iE_0t}C_1\right)\int d^3r\phi_1(\vec{r})\phi_0(\vec{r})\int d^3r'\, R\,\phi_0(\vec{r}')\phi_1(\vec{r}'), \tag{58}$$

where

$$\vec{a}_{(1)} = \frac{\alpha}{2i}\int \frac{d^3r'}{R}\left(\phi_0\vec{\nabla}\phi_1 - \phi_1\vec{\nabla}\phi_0\right) - \frac{\alpha\hat{z}}{2i}\frac{\frac{\partial}{\partial z}\phi_1(0)}{\phi_0(0)}\int \frac{d^3r'}{R}\phi_0^2 . \quad (59)$$

In the derivation of the equation for C_1 we neglected the terms of the order $\exp(-i2E_0t)$, and $\vec{\nabla}\phi_0 = \vec{0}$. For the harmonic oscillator $\vec{a}_{(1)} = \vec{0}$, and hence the equation for C_1 is

$$i\dot{C}_1 = E_1C_1 - \frac{\alpha}{2}(E_0 - E_1)^2 C_1 \int d^3r\,\phi_0\phi_1 \times \int d^3r'\,R\phi_0\phi_1 , \quad (60)$$

where we used the approximation $\dot{C}_1 = -iE_1C_1$. The easiest way to calculate the double integral over the space is to use as the intermediate step the Fourier transform of $\phi_0\phi_1$, in which case one finds its analytic expression. The equation for C_1 is finally

$$i\dot{C}_1 = \left[E_1 + \frac{\alpha}{6}(E_0 - E_1)^2\sqrt{\frac{2}{\pi w}}\right]C_1 . \quad (61)$$

The additional term with E_1 is the energy shift of the eigenstate 1. A note of warning: the double integral (60) has a negative value, despite the fact that the integrand appears to be positive definite.

At the other extreme is the strong coupling limit, when the RRI term is small compared to the external field. The solution without the RRI is given by (19), and when the RRI terms are calculated we find that $\vec{A}_0$ in (53) is exactly zero. The reason for this is that, as the solution (19) suggests, the ground state probability density moves back and forth in the harmonic potential and, therefore, according to the discussion in Introduction, it cannot interact with itself. The remaining term, involving $\ddot{P}$, in (53) is nonzero, and the approximate equation which describes the influence of the RRI is

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + \frac{\omega^2}{2}r^2\psi + \frac{\alpha\psi}{2}\int d^3r'\,R\,\ddot{P}^{(0)} + \epsilon\, z\, f_0(t)\,\psi , \quad (62)$$

where

$$P^{(0)} = \left(\frac{\omega}{\pi}\right)^{3/2}\exp\{-\omega(x^2 + y^2 + [z - c(t)]^2)\}. \quad (63)$$

The integral over $P^{(0)}$ is a complicated function of r, which has expansion in the even powers of r. When the second time derivative is calculated the first few terms in this expansion are

$$\int d^3r'\,R\,\ddot{P}^{(0)} \approx \frac{16}{3}\sqrt{\frac{w}{\pi}}\left[\dot{c}^2 - \underline{z}\ddot{c} + \frac{6}{5}w\left(x^2 + y^2 + 3\underline{z}^2\right)\dot{c}^2 - \frac{6}{5}w\underline{z}\left(x^2 + y^2 + \underline{z}^2\right)\ddot{c}\right] + O\left(r^6\right) , \quad (64)$$

where $\underline{z}$ was defined in (21). We notice that the RRI is now both external field and time dependent. The dominant contribution is an overall shift of the energy scale, and the next contribution is a modification of the external field. The harmonic potential is modified in the third order contribution, which is the source of the energy shifts of the bound states. The terms which follow are of a similar nature, but their characteristic feature is to couple the vibration modes in all three space directions, and therefore the spatial separability is lost.

V. EXAMPLE AND DISCUSSION

Analysis of one example will illustrate typical dynamics when the effect of the spontaneous emission of radiation is included in the interaction of a plane EM wave with a harmonic oscillator. It is assumed that initially the harmonic oscillator is in its ground state, and the temporal dependence of the field is

$$f_0(t) = g(t)\cos(\omega t) , \quad (65)$$

where the modulation $g(t)$ represents a square pulse of the form

$$g(t) = \frac{1}{\left[1 + e^{-\eta(t - t_{mn})}\right]\left[1 + e^{\eta(t - t_{mx})}\right]} , \quad (66)$$

where t_{mn} is time of the onset of interaction and t_{mx} is the end of the pulse.

The effect of the spontaneous emission on the dynamics of the harmonic oscillator is very weak, but it is negligible if the EM field is very strong. Therefore in our study we should try to keep the EM field weak but enhance the effect of the spontaneous emission. The only possible way to achieve this is to study harmonic oscillators of high frequency. This can be shown if we use additional scaling, where the time variable is replaced by tw and the coordinates by $\sqrt{w}\vec{r}$, in which case Eq. (25) becomes

$$i\dot{\psi} = -\frac{1}{2}\Delta\psi + \frac{1}{2}r^2\psi + \epsilon\, z\, f_0(t)\,\psi + \omega\vec{r}\cdot\dot{\vec{A}}_{\rm ret}\,\psi , \quad (67)$$

where $\vec{A}$ is given by (26), but now in the scaled variables. From the last equation it is clear that the effect of the spontaneous emission indeed depends on the frequency of the harmonic oscillator.

The parameters, in the units of the latest scaling, which we used in the example, are the following: $t_{mn} = 30$, $t_{mx} = 6.5 \times 10^5$, and $\eta = 0.3$, while the strength of coupling is $\epsilon = 0.000\,12$. Time dependence of the energy of the harmonic oscillator for these parameters is shown in Fig. 1, where the broken line represents the shape of the pulse $g(t)$. The energy starts to increase as if there is no effect of the spontaneous emission (ground state energy is $3/2$), but after a certain time it starts to level off, reaching a steady value. At this point there is a balance between the intake of energy from the EM wave and the energy of the emitted radiation. The energy of this plateau depends on the coupling strength of the field. When the field is turned off, the energy of the oscillator starts decreasing due to the spontaneous emission until the ground state is reached (stationary state).

We have also calculated the spatial integral over the current since this quantity is directly related to the intensity of the radiated energy. Time dependence of this quantity is shown in Fig. 2, where its oscillations are only symbolically shown (they are so rapid that on the time scale shown they would not be resolved), while the thick lines show their envelope. The broken line shows the pulse $g(t)$. The amplitude of the integrated current directly correlates with what we obtained for the energy of the oscillator. It starts from zero because the initial state is stationary, and then it increases to reach a stationary value. At this point the maximum of the amplitude indicates that the intensity of radiation is at its peak and that a lot of energy is emitted in the form of EM radiation. After the pulse is over the amplitude of the current decreases because energy is emitted at the expense of the energy of the oscillator.

From the time evolution of the probability distribution (and the wave function) we can obtain the transition probability into a particular stationary state of the harmonic oscillator. This requires calculation of the "overlap" integral between the "exact" wave function and the wave function for this state, and its square modulus is the required transition probability. Each stationary state of the harmonic oscillator is characterized by three quantum numbers; n_r, l, and m (quantum numbers for the radial motion, angular momentum, and the "magnetic" term, respectively); however, because of the symmetry of the problem the transitions which change the m quantum numbers are not allowed. Therefore we only have to consider two quantum numbers, n_r and l, and fix $m = 0$. A typical distribution of the transition probabilities among these states is shown in Fig. 3 as a two-dimensional topographical plot. The three-dimensional inset shows how

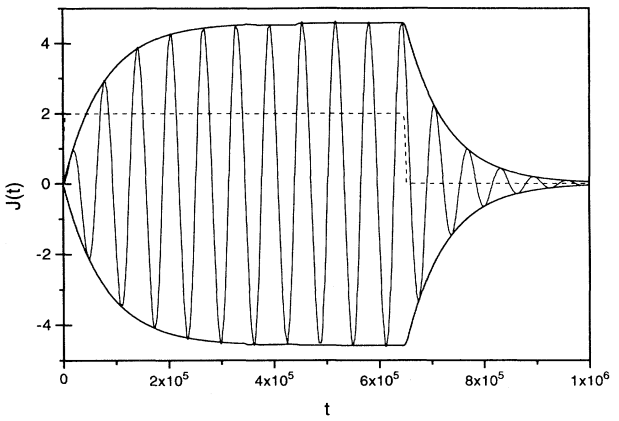


FIG. 2. Time dependence of the spatially integrated current $J(t)$ for the harmonic oscillator in Fig. 1. Envelope of the oscillations of $J(t)$ is shown by a thick solid line, while they are show symbolically by an oscillating line. The envelope of the EM field is shown by a broken line.

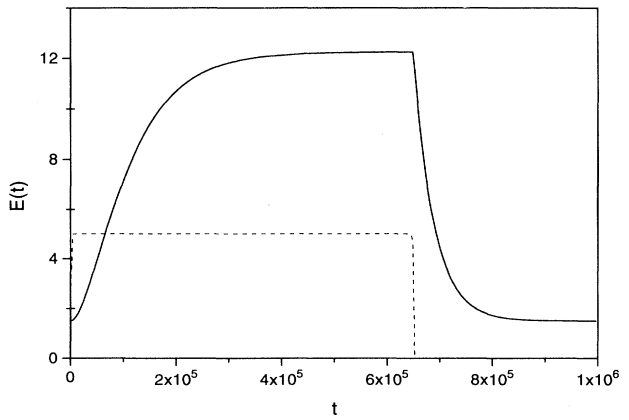


FIG. 1. Time dependence of energy of harmonic oscillator which interacts with the EM field at resonance frequency. The envelope of the EM field is shown by the broken line.

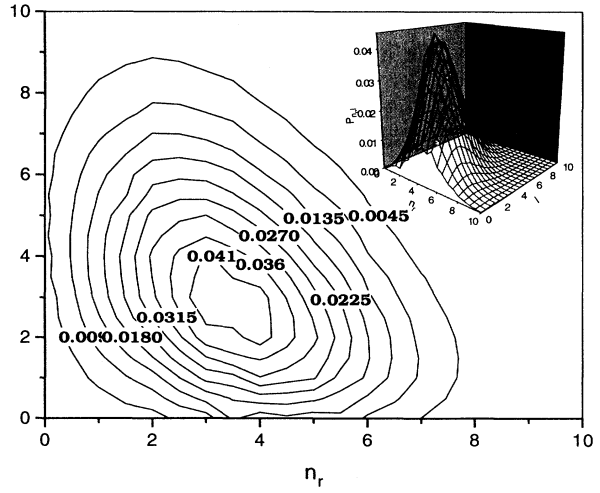


FIG. 3. Topological view of the transition probabilities into the states (n_r, l) of harmonic oscillator, at $t = 4000$. The three-dimensional inset shows the same distribution from another perspective.

this distribution looks from a different perspective. Calculation of these transition probabilities was done at the time when the energy of the harmonic oscillator is at its maximum ($t = 5 \times 10^5$). The maximum transition probability is into the $n_r = 3$ and $l = 3$ state, but this is a misleading result which implies that the harmonic oscillator has certain angular momentum. In fact, it is quite the contrary: the probability distribution in the Cartesian x and y coordinates is stationary, while it is oscillating along the z axis. This result suggests a more general conclusion: individual states of a system, i.e., the stationary states, should be treated with great caution when used as the basis set in the expansion of the exact solution. What matters is the entire wave function (or the probability distribution) while the basis functions are a useful mathematical tool for obtaining it. To attach a much deeper meaning to these functions may often lead to a misleading conclusion. This result is well known but repeated here in order to emphasize the difference between the overall probability distribution and its components. Thus, for example, from the transition probabilities into various states one could conclude that the harmonic oscillator can emit radiation of the frequencies which are multiples of the basic one. In our semiclassical treatment we did not find any evidence of that, because the spectrum is determined from the overall distribution and not from its components.

The energy states of the harmonic oscillator are degenerate in the quantum numbers n_r and l; the degeneracy is $2n_r + l$. Therefore, the probability of finding a particle with a certain energy is the sum of all the transition probabilities into the states with this energy. This quantity can be calculated directly from the "exact" wave function in Cartesian coordinates, and the result is the Poisson's distribution

$$P_n = \frac{a^n}{n!} e^{-a} \tag{68}$$

in the variable $n = 2n_r + l$. The test was done by summing the transition probabilities in Fig. 3 and compared with Poisson's distribution. In Fig. 4 we show the result of the least square fit of the numerical probabilities (circles) to Poisson's distribution (solid line). The fit is nearly exact, which only confirms that our numerical procedure leading to the transition probabilities is reliable.

In the analysis of the example we did not take into account the energy shift of the states, discussed in the preceding section. This is an additional complication which needs separate discussion. However, on the level of the qualitative arguments, based on the coupling term (64), we can predict that the simplicity of the dynamics will be lost due to the mixing of all the three space directions. One of the consequences, for example, will be that the Poisson type distribution of the states of the harmonic oscillator will not hold true. Also it is not *a priori* clear that classical and quantum results will be exactly equal, but there is no reason to expect that they will be far apart. The prediction is based on the experience from the previous analysis of the quantum-classical correspondence [15–18], however, only the detailed analysis should confirm this.

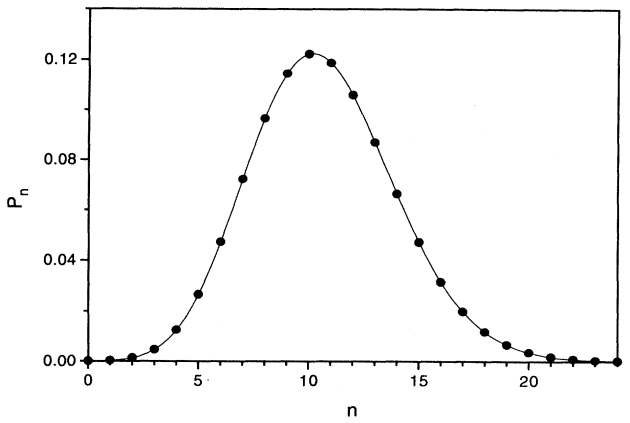


FIG. 4. The total transition probabilities (circles) into the energy states of the harmonic oscillator fitted to the Poisson's distribution (line), at $t = 4000$.

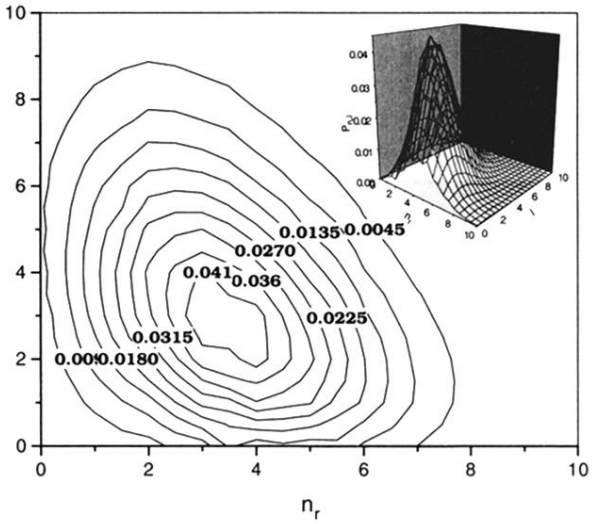
Strictly speaking, there are two kinds of radiative frequency shifts: one coming from the effect of the energy shift of the bound states and the other coming from the effect of the spontaneous emission. The first, as we mentioned, was not discussed in detail, but the second was taken into account in the solution of Eq. (30). The effect of the latter is negligible because it is too small to be noted in calculations, as can be deduced from (33). This frequency shift is exactly equal to the one which is obtained in the treatment of the same problem by the Abraham-Lorentz theory [19], the expected result because the equation which one needs to solve in the Abraham-Lorentz theory is the same as (32). The important thing to note, however, is that this frequency shift is the same in both the classical and the semiclassical treatment which we developed here.

The efforts to formulate the quantum (semiclassical) harmonic oscillator problem with the "damping" term are very old. There are at least two distinct directions in this pursuit: one which is based on treating the damping term empirically [14,20–23] in analogy with the friction term in classical theory, or through formulation of the so-called "stochastic electrodynamics" [24,25]. In the first line of formulation the basis is the Abraham-Lorentz theory of the RRI, and from the classical-dynamics equation the effort was concentrated in finding the analogous quantum equation. One should mention though that the effort had a much wider aim, formulating a general quantum equation which includes dissipative processes, where the RRI is one of them. However, here we refer only to the part which bears direct relevance to the RRI. The difference between our approach and the one which we referred to is great. First, in our approach we can describe the additional (in fact the most important) frequency shift, while in that treatment it is difficult to imagine how this shift can be included in the dynamics equations. However, the most revealing difference between the two formulations is in the attitude towards the properties of the probability distribution in phase space. It was argued be Greenberger [20] that "friction" causes all the

velocity part of the phase space density to be reduced to zero, which is indeed the case for any particle which loses energy. However, in our treatment of the RRI, zero "friction" is a result of the stationary probability distribution and zero (or stationary) current, which does not imply that the momentum part of the phase space distribution is reduced to zero.

The "stochastic electrodynamics" is closer to our treatment because the concept of the probability distributions is introduced, and a number of the relationships which we defined in the beginning are formulated. However, there are a number of steps in this theory which set it apart from our approach. For example, in that treatment one deals with the ensemble of harmonic oscillators rather than a single oscillator whose distribution of coordinates and momenta obey the uncertainty principle. As an ensemble the oscillators produce an EM field (random, hence the name "stochastic electrodynamics") which interacts back with the oscillators. In all the treatment there is no use of the retardation effects, nor the treatment of the Coulomb term. Furthermore, the basic dynamics equation is of the Abraham-Lorentz type, where the connection between the EM field and the trajectories is not very clearly defined. Based on the previous comparisons it is, therefore, believed that our formulation of the RRI is the alternative which has advantages in many respects. However, more detailed studies are necessary to confirm this.

[1] O. Heaviside, Nature **67**, 6 (1902).
[2] M. Abraham, Ann. Phys. **10**, 156 (1903).
[3] M. O. Scully and M. Sargent III, Phys. Today **25** (3), 38 (1972).
[4] P. W. Milonni, in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A. O. Barut (Plenum, New York, 1980), p. 1.
[5] P. A. M. Dirac, Nature **137**, 298 (1936).
[6] For a collection of views on the QED there is a good historical account in A. Pais, *Inward Bound* (Clarendon, New York, 1988), Chap. 16.
[7] J. R. Ackerhalt, P. L. Knight, and J. H. Eberly, Phys. Rev. Lett. **30**, 456 (1973).
[8] E. Fermi, Rendiconti Lincei. **5**, 795 (1927).
[9] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
[10] E. T. Jaynes, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 35.
[11] S. D. Bosanac, J. Phys. A **27**, 1723 (1994).
[12] S. D. Bosanac, Fiz. **B3**, 21 (1994).
[13] S. D. Bosanac, Phys. Rev. A **50**, 2899 (1994).
[14] H. Dekker, Phys. Rep. **80**, 1 (1981).
[15] S. D. Bosanac, J. Chem. Phys. **95**, 5732 (1991).
[16] S. D. Bosanac, Z. Phys. D **28**, 195 (1993).
[17] S. D. Bosanac, Z. Phys. D **24**, 325 (1992).
[18] N. Doslic and S. D. Bosanac, Z. Phys. D **32**, 261 (1995).
[19] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Chap. 17.7.
[20] D. M. Greenberger, J. Math. Phys. **20**, 762 (1979).
[21] D. M. Greenberger, J. Math. Phys. **20**, 771 (1979).
[22] E. G. Harris, Phys. Rev. A **42**, 3685 (1990).
[23] H. G. Oh, H. R. Lee, T. F. George, and C. I. Um, Phys. Rev. A **39**, 5515 (1989).
[24] T. W. Marshall, Proc. R. Soc. London Ser. A **276**, 475 (1963).
[25] T. H. Boyer, in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A. O. Barut (Plenum, New York, 1980), p. 49.



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