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ARTICLES

Bell's-inequality experiment employing four harmonic oscillators

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We show that Bell's-inequality experiments can be carried out using four harmonic oscillators, pairs of which interact with each other at suitable instants of time to allow for the exchange of energy. Between interactions, the oscillators are shifted in phase relative to each other in a controllable manner. Energy becomes distributed among the four harmonic oscillators in a manner that is incompatible with a local realism picture in which information is exchanged only when the oscillators interact with each other. Entanglement is dynamically generated from an initial product state.

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In the usual Bell-inequality experiments [1–4] a pair of particles is initially prepared in an entangled state. Each particle is then delivered to a detector in which one of several observables is measured, the choice of observable being left to the experimenter. The statistics of the measurements reported by the detectors exhibit correlations that are stronger than allowed by local realism. A number of such experiments have been proposed [5–8] and even performed [9–15] in which the particles are photons. Since photons are excitations of modes of the electromagnetic field, which are harmonic oscillators, it is not surprising that Bell's-inequality experiments can be performed using harmonic oscillators. In fact, the gedanken experiment, employing four distinguishable harmonic oscillators, described here is a direct analog of an optical Bell-inequality experiment we have described previously [16]. It is worthwhile, however, to consider Bell's-inequality experiments in a manner that emphasizes harmonic oscillator wave functions because, as is clear from our development here, Bell's-inequality experiments can *actually* be carried out with harmonic oscillators. We note here that the initial state of the system is an unentangled direct product state. Entanglement arises dynamically as the oscillators are allowed to exchange energy with each other. This illustrates that the mechanism for producing entanglement may be an explicit part of the actual apparatus rather than be implicit in the preparation of the initial state. The analysis provided in the text is carried out entirely in the language of interacting harmonic oscillators. In the Appendix the connection between the present formalism and the second quantized formalism we employed in the analysis of the optical experiment [16] is given. Although the focus here is on a Bell's-inequality experiment, the tools presented can be applied to other Einstein-Podolsky-Rosen [17] experiments that have optical realizations to demonstrate that these experiments can be performed using harmonic oscillators. In particular, the techniques can be readily applied to an optical version of the Greenberger-Horne-Zeilinger experiment [18–20] recently proposed by us [21] and Reid and Munro [22] and an optical version [23] of a local realism violating experiment proposed by Hardy [24] in order to demonstrate that these experiments can be carried out with, respectively, six or four harmonic oscillators.

The four harmonic oscillators used in the present gedanken experiment can each be materially different. For example, oscillator O1 could be a mass attached to a spring, oscillator O2 could be a torsion oscillator, oscillator O3 could be an electrical oscillator consisting of an inductor and a capacitor, and oscillator O4 could be of some other construction. Two of the oscillators, say oscillator O1 and oscillator O3, are prepared in their first excited state. Oscillators O2 and O4 would then be prepared in their ground state. These oscillators are transported in the manner shown schematically in Fig. 1 in which it is shown that at certain times, pairs of oscillators are brought into contact with each other through couplings that allow the interchange of energy. First, oscillators O1 and O2 are brought into contact with each other and oscillators O3 and O4 are brought into contact with each other. After the oscillators have been separated again, each oscillator is subjected to a controllable phase shift. Finally, oscillators O1 and O4 are brought into contact and oscillators O2 and O3 are brought into contact. After the oscillators are separated for the final time, one measures the energy of each oscillator to determine the energy eigenstate that the oscillator is in. The probability distribution of the occupation of these energy eigenstates, conditioned on the settings of the phase shifters, is not consistent with a local realism picture in which the oscillators interchange information only while they are in contact with each other.

We now describe the interaction that is used to couple a pair of harmonic oscillators. For simplicity, it will be assumed that the oscillators all have the same resonant frequency. We will use units in which $\hbar$ is 1. In addition, the scale used to measure the displacement of a given oscillator will be chosen so that the energy eigenstates of each oscillator have the standard form

$$\begin{aligned}\psi_0(x) &= \pi^{-1/4}e^{-x^2/2},\\ \psi_1(x) &= 2^{1/2}\pi^{-1/4}xe^{-x^2/2},\\ \psi_2(x) &= 2^{-1/2}\pi^{-1/4}(2x^2-1)e^{-x^2/2},\\ &\ \vdots\end{aligned} \tag{1}$$

Since the interactions between oscillators in each case last for a finite duration, it will be convenient to work in an interaction picture. The interaction Hamiltonian $H_I(r,s)$ between a pair of oscillators Or and Os, where r and s are labels that take on the values 1–4, is taken to be of the form

$$H_I(r,s) = \kappa(x_r x_s + p_r p_s), \tag{2}$$

where x_r and p_r are the position and the momentum operators for the oscillator Or and x_s and p_s are the position and the momentum operators for oscillator Os. In the position representation, the momentum operators p_r are given by

$$p_r = -i\frac{\partial}{\partial x_r}\ . \tag{3}$$

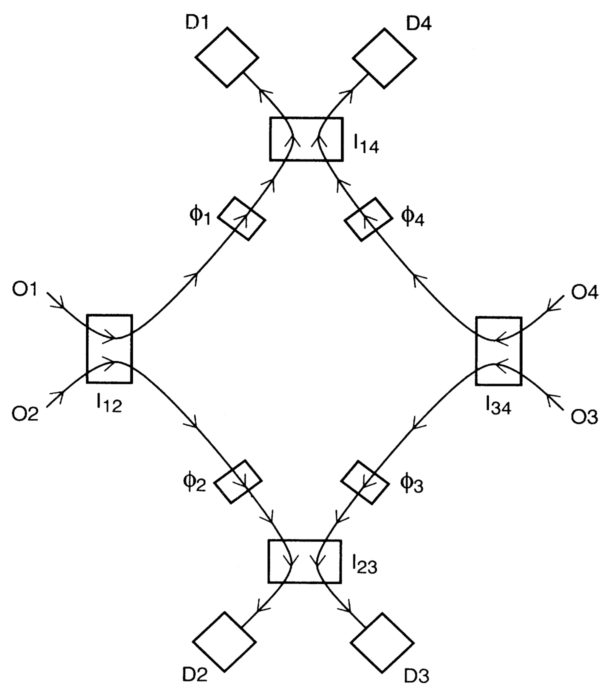

FIG. 1. Schematic of how oscillators are transported to and from interaction regions in order to perform a Bell's-inequality experiment. The oscillators are labeled O1, O2, O3, and O4 and their paths of transport are indicated. The regions where pairs of oscillators interact to exchange energy are denoted I_{12}, I_{34}, I_{23}, and I_{14}. The phase shifters, which advance or retard the phase of the oscillators, are denoted ϕ_1, ϕ_2, ϕ_3, and ϕ_4. The detectors, which determine which energy eigenstate a given oscillator is in, are denoted by D1, D2, D3, and D4.

Before continuing with the analysis, a discussion is in order regarding the possibility of physically realizing an interaction Hamiltonian of the form shown in Eq. (2). A coupling of the form $\kappa x_r x_s$ is quite familiar in a mechanical context. In particular, if one connects a spring between two mechanical oscillators, the energy of the spring is given by $k(x_r - x_s)^2/2$, where k is the spring constant. This energy thus gives rise to the coupling term $-kx_r x_s$ in the Hamiltonian. A coupling term of the form $\kappa p_r p_s$ involving the momenta of the two oscillators is unusual in a mechanical context. It is, however, quite familiar in an electrical context where charge plays the role of position and current plays the role of velocity. In an electrical context the coupling $\kappa p_r p_s$ can be recognized as a mutual inductance. To be more explicit, consider the two electrical harmonic oscillators depicted in Fig. 2. The oscillator E1 consists of the capacitor C_1 and the inductor L_1 and the oscillator E2 consists of the capacitor C_2 and the inductor L_2. The oscillators have been coupled together through a common current path into which a mutual capacitor C_M has been inserted. In addition, the inductors are coupled to each other through fringing magnetic fields, giving rise to a mutual inductance M. The Hamiltonian for the system can be written in the form

$$H = \frac{L_1}{2}\left(\frac{dQ_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{1}{C_1}+\frac{1}{C_M}\right)Q_1^2 \tag{4}$$

$$+\frac{L_2}{2}\left(\frac{dQ_2}{dt}\right)^2 + \frac{1}{2}\left(\frac{1}{C_2}+\frac{1}{C_M}\right)Q_2^2 \tag{5}$$

$$+M\frac{dQ_1}{dt}\frac{dQ_2}{dt} + \frac{Q_1Q_2}{C_M}, \tag{6}$$

where Q_1 and Q_2 are the charges associated with oscillators E1 and E2, respectively. Provided the values of the mutual inductance M and the mutual capacitance C_M are chosen properly, the last line Eq. (6) is an interaction

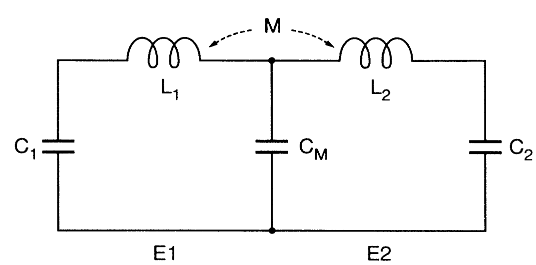

FIG. 2. Two coupled electrical harmonic oscillators. Oscillator E1 consists of the inductor L_1 and capacitor C_1. Similarly, oscillator E2 consists of the inductor L_2 and capacitor C_2. Two means of interaction between the two oscillators are provided. One coupling interaction is provided by the mutual inductance M between $L1$ and $L2$, which results from the fringing magnetic fields of one inductor penetrating the other inductor. The other coupling is provided by the capacitor C_M, which has been inserted into a current path that both oscillators share in common. This capacitor gives rise to a mutual capacitance.

of the form Eq. (2). More generally, since linear transducers exist that convert displacement or velocity into charge or current and vice versa, electromechanical couplers can be devised that will give rise to effective interactions of the form Eq. (2) between mechanical oscillators as well. As shown in the Appendix, the interaction, Eq. (2), performs the same function that a beam splitter performs in an optical experiment.

The function of the interaction $H_I(r,s)$ is to allow harmonic oscillators Or and Os to exchange energy. Thus, although we have specialized to the interaction given in Eq. (2), other interactions could be used. Effects produced by turning the interaction on or off will be neglected. Such effects can be made negligibly small in a variety of ways, such as by turning the interaction on and off adiabatically or by making the coupling strength κ small and correspondingly increasing the interaction time.

We now consider the evolution of wave functions under the influence of the interaction Hamiltonian $H_I(r,s)$. Let $\psi(x_r,x_s)$ be a wave function in the interaction picture. Under the influence of the interaction Hamiltonian, the wave function evolves according to

$$i\frac{\partial\psi}{\partial t}=H_I\psi \ , \tag{7}$$

where we have suppressed the r and s indices. As shown in the Appendix, this differential equation can be expressed as the integral equation

$$\psi_{\rm out}(x_r,x_s)=\int dx_r'dx_s'G(x_r,x_s;x_r',x_s')\psi_{\rm in}(x_r',x_s') \ , \tag{8}$$

where $\psi_{\rm in}$ is the wave function before interaction and $\psi_{\rm out}$ is the wave function after interaction. The kernel is given by

$$G(x_r,x_s;x_r',x_s')=\frac{\csc(\kappa t)}{2\pi}\exp\{i[\cot(\kappa t)(x_rx_s+x_r'x_s') - \csc(\kappa t)(x_rx_s'+x_sx_r')]\} \ , \tag{9}$$

where t is the interaction time. For the Bell's-inequality experiment, the duration of the interaction between oscillators will be chosen such that

$$\kappa t=\pi/4 \ . \tag{10}$$

For this choice of interaction time, the interaction performs the following transformations listed here for later reference:

$$\psi_{\rm out}(x_r,x_s)=\begin{cases}\psi_0(x_r)\psi_0(x_s) & \text{if } \psi_{\rm in}=\psi_0(x_r)\psi_0(x_s)\\ \frac{1}{\sqrt2}[\psi_0(x_r)\psi_1(x_s)-i\psi_1(x_r)\psi_0(x_s)] & \text{if } \psi_{\rm in}=\psi_0(x_r)\psi_1(x_s)\\ \frac{1}{\sqrt2}[-i\psi_0(x_r)\psi_1(x_s)+\psi_1(x_r)\psi_0(x_s)] & \text{if } \psi_{\rm in}=\psi_1(x_r)\psi_0(x_s)\\ -\frac{i}{\sqrt2}[\psi_0(x_r)\psi_2(x_s)+\psi_2(x_r)\psi_0(x_s)] & \text{if } \psi_{\rm in}=\psi_1(x_r)\psi_1(x_s).\end{cases} \tag{11}$$

We note that, except for the ground state $\psi_0(x_r)\psi_0(x_s)$, the other product states $\psi_{\rm in}$ are converted into entangled states by the interaction.

The phase of an oscillator can be advanced or retarded through a change of its spring constant or its mass. Here we will employ both effects to shift the oscillator's phase. The interaction Hamiltonian for a phase shifter is taken to be

$$H_\Omega=\frac{\Omega}{2}(p^2+x^2) \ . \tag{12}$$

Again, in the interaction picture, the wave function $\psi(x)$ is governed by the equation

$$i\frac{\partial\psi}{\partial t}=H_\Omega\psi \ . \tag{13}$$

As shown in the Appendix, this differential equation can be expressed as the integral equation

$$\psi_{\rm out}(x)=\int dx'K(x,x')\psi_{\rm in}(x') \ , \tag{14}$$

where $\psi_{\rm in}$ is the wave function before the interaction and $\psi_{\rm out}$ is the wave function after the interaction. The kernel is given by

$$K(x,x')=\frac{1}{\sqrt{2\pi i\sin(\Omega t)}}\exp[i\cot(\Omega t)(x^2+x'^2)/2 - i\csc(\Omega t)xx'] \ . \tag{15}$$

For the analysis of the Bell's-inequality experiment we will need to know the following results for how wave functions are transformed under Eqs. (14) and (15):

$$\psi_{\rm out}(x)=\begin{cases}e^{-i\phi/2}\psi_0(x) & \text{if } \psi_{\rm in}(x)=\psi_0(x)\\ e^{-i3\phi/2}\psi_1(x) & \text{if } \psi_{\rm in}(x)=\psi_1(x),\end{cases} \tag{16}$$

where the phase angle ϕ is given by

$$\phi=\Omega t \ . \tag{17}$$

We now have all the tools necessary to determine the probability distribution describing how the quanta of energy are distributed among the four oscillators at the end of the Bell's-inequality experiment. The initial wave function Ψ_a is the state for which oscillators O1 and O3 have been prepared in their first excited state while oscillators O2 and O4 have been prepared in their ground state. The initial state vector is thus

$$\Psi_a(x_1,x_2,x_3,x_4)=\psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4) \ . \tag{18}$$

Oscillators O1 and O2 are brought near each other and allowed to interact and oscillators O3 and O4 are brought near each other and allowed to interact. Each pair involves one oscillator in its first excited state and one oscillator in its ground state. One sees from Eq. (11) that the interactions, Eq. (2), will result in a superposition in which one or the other oscillator of a pair is excited. Let Ψ_b denote the state vector after this first set of interactions. From Eq. (11) it follows that this state vector has the form

$$\Psi_b(x_1,x_2,x_3,x_4) = \Psi_{b,\mathrm{Bell}}(x_1,x_2,x_3,x_4) + \Psi_{b,\mathrm{back}}(x_1,x_2,x_3,x_4), \tag{19}$$

where the part of the wave function that will give rise to Bell's-inequality violations is given by

$$\Psi_{b,\mathrm{Bell}}(x_1,x_2,x_3,x_4) = \frac{1}{2}[\psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4) - \psi_0(x_1)\psi_1(x_2)\psi_0(x_3)\psi_1(x_4)] \tag{20}$$

and the part of the wave function that will give rise to background events is given by

$$\Psi_{b,\mathrm{back}}(x_1,x_2,x_3,x_4) = -\frac{i}{2}[\psi_1(x_1)\psi_0(x_2)\psi_0(x_3)\psi_1(x_4) + \psi_0(x_1)\psi_1(x_2)\psi_1(x_3)\psi_0(x_4)]\ . \tag{21}$$

Let Ψ_c denote the state vector after the harmonic oscillators have been phase shifted and let ϕ_r, where $r \in \{1,2,3,4\}$, denote the phase by which the rth oscillator has been shifted. From Eq. (16) it follows that the state vector Ψ_c is given by

$$\Psi_c(x_1,x_2,x_3,x_4) = \Psi_{c,\mathrm{Bell}}(x_1,x_2,x_3,x_4) + \Psi_{c,\mathrm{back}}(x_1,x_2,x_3,x_4), \tag{22}$$

where the part of the wave function that will give rise to Bell's-inequality violations is given by

$$\Psi_{c,\mathrm{Bell}}(x_1,x_2,x_3,x_4) = \frac{e^{i\phi_s}}{2}[e^{-i\theta}\psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4) - e^{i\theta}\psi_0(x_1)\psi_1(x_2)\psi_0(x_3)\psi_1(x_4)] \tag{23}$$

and the part of the wave function that will give rise to background events that do not contribute to local realism violations is given by

$$\Psi_{c,\mathrm{back}}(x_1,x_2,x_3,x_4) = -\frac{ie^{i\phi_s}}{2}[e^{-i\theta_d}\psi_1(x_1)\psi_0(x_2)\psi_0(x_3)\psi_1(x_4) \quad e^{i\theta_d}\psi_0(x_1)\psi_1(x_2)\psi_1(x_3)\psi_0(x_4)]. \tag{24}$$

The phases are given by

$$\phi_s = \phi_1 + \phi_2 + \phi_3 + \phi_4, \tag{25}$$

$$\theta = \frac{1}{2}[\phi_1 - \phi_2 + \phi_3 - \phi_4], \tag{26}$$

$$\theta_d = \frac{1}{2}[\phi_1 - \phi_2 - \phi_3 + \phi_4]\ . \tag{27}$$

After being phase shifted the oscillators interact one last time via an interaction of the form Eq. (2). In this case oscillators O1 and O4 interact and oscillators O2 and O3 interact. Let Ψ_d denote the wave function after the interaction. Using Eq. (11) one finds that the wave function has the form

$$\Psi_d(x_1,x_2,x_3,x_4) = \Psi_{d,\mathrm{Bell}}(x_1,x_2,x_3,x_4) + \Psi_{d,\mathrm{back}}(x_1,x_2,x_3,x_4)\ , \tag{28}$$

where the part of the wave function responsible for Bell's inequalities is given by

$$\begin{aligned}\Psi_{d,\mathrm{Bell}}(x_1,x_2,x_3,x_4) = {}& \frac{e^{i\phi_s}}{2}\cos(\theta)[\psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4) - \psi_0(x_1)\psi_1(x_2)\psi_0(x_3)\psi_1(x_4)] \\ & -\frac{e^{i\phi_s}}{2}\sin(\theta)[\psi_1(x_1)\psi_1(x_2)\psi_0(x_3)\psi_0(x_4) + \psi_0(x_1)\psi_0(x_2)\psi_1(x_3)\psi_1(x_4)]\end{aligned} \tag{29}$$

and the part of the wave function that gives rise to background events is given by

$$\begin{aligned}\Psi_{d,\mathrm{back}}(x_1,x_2,x_3,x_4) = {}& -\frac{e^{i(\phi_s-\theta_d)}}{2\sqrt{2}}[\psi_0(x_1)\psi_0(x_2)\psi_0(x_3)\psi_2(x_4) + \psi_2(x_1)\psi_0(x_2)\psi_0(x_3)\psi_0(x_4)] \\ & -\frac{e^{i(\phi_s+\theta_d)}}{2\sqrt{2}}[\psi_0(x_1)\psi_0(x_2)\psi_2(x_3)\psi_0(x_4) + \psi_0(x_1)\psi_2(x_2)\psi_0(x_3)\psi_0(x_4)]\ .\end{aligned} \tag{30}$$

The probability distribution describing how the energy is distributed among the four oscillators is now simply determined by squaring the probability amplitudes for particular configurations of energy eigenstates given by Eqs. (29) and (30). Since the initial state vector consists of one in which two quanta of energy are present, the final state also contains two quanta of energy. We can denote the outcome of a single run of the Bell's-inequality experiment by a pair of integers (r,s) specifying which harmonic oscillators were found in their excited state at the end of the run. In particular, 13 denotes the event where oscillator O1 and oscillator O3 are in their excited state. Similarly, 24 denotes the event for which oscillator O2 and O4 are found to be in their excited state. We introduce the set A defined by

$$A = \{13, 24\} . \tag{31}$$

Similarly, let B denote the set of events for which oscillator O1 and oscillator O2 are found to be excited or oscillator O3 and oscillator O4 are found to be excited

$$B = \{12, 34\} . \tag{32}$$

Finally, let D denote the events where a single oscillator is doubly excited. For example, by 11 we denote the event where oscillator O1 is found to be in its second excited state, i.e., it has two quanta of vibration. Hence D is given by

$$D = \{11, 22, 33, 44\} . \tag{33}$$

Let $P(\alpha)$ denote the probability that the event α will occur where $\alpha \in A \bigcup B \bigcup D$. The probabilities are then given by

$$P(\alpha) = \begin{cases} \frac{1}{4}\cos^2(\theta) & \text{if } \alpha \in A \\ \frac{1}{4}\sin^2(\theta) & \text{if } \alpha \in B \\ \frac{1}{8} & \text{if } \alpha \in D \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

This probability distribution is identical to one obtained by us earlier for an optical Bell's-inequality experiment [16]. By performing an ensemble of experiments in which suitable choices of phases ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are made, one can show that the correlations in the distribution of energies among the four oscillators violate local realism. In particular, let the phase shifters ϕ_1 and ϕ_4 be controlled by experimenter 1 and let the phase shifters ϕ_2 and ϕ_3 be controlled by experimenter 2. As depicted in Fig. 1, harmonic oscillator O1 is only allowed to interact with harmonic oscillator O4 after the phase shifts and harmonic oscillator O2 is only allowed to interact with harmonic oscillator O3 after the phase shifts. Therefore, one argues, from a local realism point of view, that the phase settings experimenter 1 chooses should not influence what detectors D2 and D3 report. Similarly, the phase settings experimenter 2 chooses should not influence what detector 1 reports. By performing a suitable topological distortion of the configuration depicted in Fig. 1, it is apparent that phase shifters ϕ_1 and ϕ_4 and the interaction region I_{14} can be moved very far from the phase shifters ϕ_2 and ϕ_3 and the interaction region I_{23}. By moving these two sets of components sufficiently far from each other, experimenter 1 can delay his choice of phase settings so that when the choice is made there is insufficient time to causally communicate this information to detectors D2 and D3 and, similarly, experimenter 2 can delay his choice of phase settings so that when the choice is made there is insufficient time to causally communicate this information to detectors D1 and D4. The probability distribution Eq. (34), however, has correlations that are sufficiently strong that the system acts, from a local realism point of view, as if the settings chosen by the experimenters are being communicated to the detectors acausally. The derivation of Bell's inequalities violated by the probability distribution Eq. (34) have been presented by us elsewhere [16] and will not be repeated here, particularly since these derivations follow standard arguments given by Clauser and Horne [25] and by Wigner [26] and Belinfante [27]. It suffices to say that the quantity $\theta_1 = (\phi_1 - \phi_4)/2$ controlled by experimenter 1 and the quantity $\theta_2 = -(\phi_2 - \phi_3)/2$ controlled by experimenter 2 act like the polarization analyzer angles in the usual optical Bell's-inequality experiments [5, 9–12].

With the derivation of Eq. (34) we have thus shown that Bell's-inequality experiments can be performed using four harmonic oscillators that are initially prepared in the product state $\psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4)$, where ψ_0 and ψ_1 are the ground state and the first excited state of an harmonic oscillator. In the optical equivalent, this state vector, written in second quantized notation, is $a_1^\dagger a_3^\dagger|0\rangle$, where $a_1^\dagger$ and $a_3^\dagger$ are the creation operators for modes 1 and 3. In terms of photon wave packets this state vector has the symmetrized form $\frac{1}{\sqrt{2}}[f_1(\mathbf{r}_1,t)f_3(\mathbf{r}_2,t) + f_1(\mathbf{r}_2,t)f_3(\mathbf{r}_1,t)]$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position coordinates of the two photons and f_1 and f_3 are the envelope functions for the wave packets in which the photons reside. This last state is entangled.

For the oscillators we manifestly *produce* entangled states dynamically through Hamiltonian evolution by "turning on" interactions between pairs of oscillators. The oscillators could be different and are distinguishable, so one need not symmetrize the initial wave function. If one views the harmonic oscillators simply as boxes that carry quanta of vibration, say phonons or photons, these quanta are indistinguishable (even if one is a phonon and the other is a photon) and the initial wave function written in terms of these "particles" must be symmetrized.

APPENDIX

Here a derivation of the kernels Eq. (9) and Eq. (15) is provided. The derivation of Eq. (9) proceeds by first obtaining a related kernel for a beam splitter. This kernel is then used to guess the solution of Eq. (7). This process establishes the equivalence between a beam splitter mode transformation and the mode transformation induced by the interaction Hamiltonian Eq. (2). The derivation of Eq. (15) proceeds the same way, except that we can cite other literature [28] for much of the derivation. By establishing that the mode transformations performed by the interaction Hamiltonians H_I and H_Ω are equivalent to those performed by optical beam splitters and phase shifters, we have shown that the Bell's-inequality experiment employing distinguishable mechanical harmonic oscillators is equivalent to an optical experiment recently proposed by us [16].

Consider a beam splitter performing the mode transformation

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \tag{A1}$$

where a_1 and a_2 are the annihilation operators for the input modes and b_1 and b_2 are the annihilation operators for the output modes. These annihilation operators satisfy the usual boson commutation relations. We now introduce the following position and momentum operators for the input modes

$$X_1 = \frac{1}{\sqrt{2}}(a_1 + a_1^\dagger), \tag{A2}$$

$$P_1 = -\frac{i}{\sqrt{2}}(a_1 - a_1^\dagger), \tag{A3}$$

$$X_2 = \frac{1}{\sqrt{2}}(a_2 + a_2^\dagger), \tag{A4}$$

$$P_2 = -\frac{i}{\sqrt{2}}(a_2 - a_2^\dagger) . \tag{A5}$$

We also introduce a similar set of position and momentum operators for the output modes of the beam splitter

$$\bar{X}_1 = \frac{1}{\sqrt{2}}(b_1 + b_1^\dagger), \tag{A6}$$

$$\bar{P}_1 = -\frac{i}{\sqrt{2}}(b_1 - b_1^\dagger), \tag{A7}$$

$$\bar{X}_2 = \frac{1}{\sqrt{2}}(b_2 + b_2^\dagger), \tag{A8}$$

$$\bar{P}_2 = -\frac{i}{\sqrt{2}}(b_2 - b_2^\dagger) . \tag{A9}$$

These position and momentum operators correspond to the actual position and momentum operators for mechanical oscillators. For the optical system these position and momentum operators are the operators for the two amplitude components that are in quadrature for an electromagnetic field mode. These observables of the electromagnetic field can be measured using homodyne detection techniques [29–32]. In this section we will use the convention that barred coordinates or operators are those associated with the beam splitter output ports, while unbarred coordinates and operators are those associated with the input ports of the beam splitter. The mode transformation Eq. (A1) leads to the following transformation among the position and momentum operators at the input and the output of the beam splitter

$$\bar{X}_1 = \cos(\theta)X_1 + \sin(\theta)P_2, \tag{A10}$$

$$\bar{P}_1 = \cos(\theta)P_1 - \sin(\theta)X_2, \tag{A11}$$

$$\bar{X}_2 = \sin(\theta)P_1 + \cos(\theta)X_2, \tag{A12}$$

$$\bar{P}_2 = -\sin(\theta)X_1 + \cos(\theta)P_2 . \tag{A13}$$

After a bit of algebra these equations can be rearranged into the convenient form

$$P_2 + \cot(\theta)X_1 - \csc(\theta)\bar{X}_1 = 0, \tag{A14}$$

$$P_1 + \cot(\theta)X_2 - \csc(\theta)\bar{X}_2 = 0, \tag{A15}$$

$$\bar{P}_1 - \cot(\theta)\bar{X}_2 + \csc(\theta)X_2 = 0, \tag{A16}$$

$$\bar{P}_2 - \cot(\theta)\bar{X}_1 + \csc(\theta)X_1 = 0 . \tag{A17}$$

These equations can now be used to generate differential equations that the transformation kernel must satisfy. To show how this is done [33], consider Eq. (A14). Let $|x_1, x_2\rangle$ denote the position eigenstate having the eigenvalues x_1 and x_2 for the unbarred position operators X_1 and X_2, respectively and let $|\bar{x}_1, \bar{x}_2\rangle$ denote the position eigenstate having the eigenvalues $\bar{x}_1$ and $\bar{x}_2$ for the barred position operators $\bar{X}_1$ and $\bar{X}_2$, respectively. Then, applying the position eigenstate $\langle \bar{x}_1, \bar{x}_2|$ to the left-hand side of Eq. (A14) and $|x_1, x_2\rangle$ to the right-hand side, one obtains

$$\langle \bar{x}_1, \bar{x}_2|P_2|x_1, x_2\rangle + \cot(\theta)x_1\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle - \csc(\theta)\bar{x}_1\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = 0 . \tag{A18}$$

Using the relationship

$$\langle \bar{x}_1, \bar{x}_2|P_2|x_1, x_2\rangle = i\frac{\partial}{\partial x_2}\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle , \tag{A19}$$

Eq. (A18) yields

$$\left[i\frac{\partial}{\partial x_2} + \cot(\theta)x_1 - \csc(\theta)\bar{x}_1\right]\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = 0 . \tag{A20}$$

Using similar techniques, one obtains from Eqs. (A15)–(A17) the following additional differential equations that the kernel $\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle$ must satisfy:

$$\left[i\frac{\partial}{\partial x_1} + \cot(\theta)x_2 - \csc(\theta)\bar{x}_2\right]\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = 0 , \tag{A21}$$

$$\left[i\frac{\partial}{\partial \bar{x}_1} + \cot(\theta)\bar{x}_2 - \csc(\theta)x_2\right]\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = 0 , \tag{A22}$$

$$\left[i\frac{\partial}{\partial \bar{x}_2} + \cot(\theta)\bar{x}_1 - \csc(\theta)x_1\right]\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = 0 . \tag{A23}$$

Integrating these equations, one finds that the kernel has the form

$$\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = c(\theta)e^{i\{\cot(\theta)[x_1x_2+\bar{x}_1\bar{x}_2]-\csc(\theta)[\bar{x}_1x_2+\bar{x}_2x_1]\}} , \tag{A24}$$

where $c(\theta)$ is an integration constant that remains to be determined. The normalization of the position eigenstates fixes the constant $c(\theta)$, up to a phase, through the relation

$$\int d\bar{x}_1 d\bar{x}_2 \langle x_1', x_2'|\bar{x}_1, \bar{x}_2\rangle\langle \bar{x}_1, \bar{x}_2|x_1, x_2\rangle = \langle x_1', x_2'|x_1, x_2\rangle = \delta(x_1' - x_1)\delta(x_2' - x_2) . \tag{A25}$$

One finds that the norm of c is given by

$$|c(\theta)| = \frac{|\csc\theta|}{2\pi} . \tag{A26}$$

Having obtained the transformation kernel, in the position representation, for the transformation performed by the beam splitter Eq. (A24), we show that this kernel also provides the solution to Eq. (7) if θ is taken to be time dependent. In order for $\langle\bar{x}_1,\bar{x}_2|\psi\rangle$, given by

$$\langle\bar{x}_1,\bar{x}_2|\psi\rangle=\int dx_1dx_2\langle\bar{x}_1,\bar{x}_2|x_1,x_2\rangle\langle x_1,x_2|\psi\rangle\ , \tag{A27}$$

to be a solution of the Schrödinger equation

$$i\frac{\partial\langle\bar{x}_1,\bar{x}_2|\psi\rangle}{\partial t}=\langle\bar{x}_1,\bar{x}_2|H_I|\psi\rangle\ , \tag{A28}$$

where

$$H_I=\kappa(\bar{x}_1\bar{x}_2+\bar{p}_1\bar{p}_2)\ , \tag{A29}$$

the kernel $\langle\bar{x}_1,\bar{x}_2|x_1,x_2\rangle$ must satisfy the differential equation

$$i\frac{\partial\langle\bar{x}_1,\bar{x}_2|x_1,x_2\rangle}{\partial t}=\kappa\left(\bar{x}_1\bar{x}_2-\frac{\partial}{\partial\bar{x}_1}\frac{\partial}{\partial\bar{x}_2}\right)\langle\bar{x}_1,\bar{x}_2|x_1,x_2\rangle\ . \tag{A30}$$

Substituting Eq. (A24) into this equation, one finds that θ must satisfy the differential equation

$$\frac{\partial\theta}{\partial t}=-\kappa \tag{A31}$$

and $c(\theta)$ must satisfy the differential equation

$$\frac{1}{c(\theta)}\frac{\partial\theta}{\partial t}=\kappa\cot(\theta)\ . \tag{A32}$$

These equations can be satisfied by the choice

$$\theta=-\kappa t \tag{A33}$$

and

$$c(\theta)=\frac{\csc(\kappa t)}{2\pi}\ . \tag{A34}$$

With the choice of integration constants implied by Eqs. (A33) and (A34), one has

$$\lim_{t\to 0}\langle\bar{x}_1,\bar{x}_2|\psi\rangle=\langle x_1,x_2|\psi\rangle\ , \tag{A35}$$

that is, the solution Eq. (A24) satisfies the proper initial conditions. Gathering the results Eqs. (A24), (A33), and (A34), one sees that the kernel $\langle\bar{x}_1,\bar{x}_2|x_1,x_2\rangle$ is identical to the kernel $G(\bar{x}_1,\bar{x}_2;x_1,x_2)$ of Eq. (9). We have thus shown that the mode transformation Eq. (A1) performed by a beam splitter is identical to the mode transformation performed by the interaction Hamiltonian Eq. (2).

We now show that the mode transformation performed by an optical phase shifter is equivalent to the mode transformation performed by the interaction Hamiltonian H_Ω given in Eq. (12). The mode transformation performed by a phase shifter has the form

$$b=e^{-i\phi}a\ , \tag{A36}$$

where a is the annihilation operator for the mode entering the phase shifter and b is the annihilation operator for the mode leaving the phase shifter. Again, one can introduce the position and momentum operators

$$X=\frac{1}{\sqrt{2}}(a+a^\dagger), \tag{A37}$$

$$P=-\frac{i}{\sqrt{2}}(a-a^\dagger), \tag{A38}$$

$$\bar{X}=\frac{1}{\sqrt{2}}(b+b^\dagger), \tag{A39}$$

$$\bar{P}=-\frac{i}{\sqrt{2}}(b-b^\dagger)\ . \tag{A40}$$

The barred position and momentum operators are related to the unbarred position and momentum operators via the transformation

$$\bar{X}=\cos(\phi)X+\sin(\phi)P, \tag{A41}$$

$$\bar{P}=-\sin(\phi)X+\cos(\phi)X\ . \tag{A42}$$

Using techniques similar to those used previously [28], one can show that the transition matrix $\langle\bar{x}|x\rangle$ has the form

$$\langle\bar{x}|x\rangle=N(\phi)\exp[i\cot(\phi)(x^2+\bar{x}^2)/2-i\csc(\phi)\bar{x}x]\ , \tag{A43}$$

where the square of the norm of the normalization constant $N(\phi)$ is given by

$$|N(\phi)|^2=\frac{1}{2\pi|\sin\phi|}\ . \tag{A44}$$

We use this as our guess for the functional form of the propagator for the Hamiltonian H_Ω where now ϕ becomes time dependent. In particular, if $\langle\bar{x}|\psi\rangle$, where

$$\langle\bar{x}|\psi\rangle=\int dx\langle\bar{x}|x\rangle\langle x|\psi\rangle\ , \tag{A45}$$

is the solution to the Schrödinger equation

$$i\frac{\partial\langle\bar{x}|\psi\rangle}{\partial t}=H_\Omega\langle\bar{x}|\psi\rangle \tag{A46}$$

with H_Ω given by

$$H_\Omega=\frac{\Omega}{2}\left[-\frac{\partial^2}{\partial\bar{x}^2}+\bar{x}^2\right]\ , \tag{A47}$$

then the propagator $\langle\bar{x}|x\rangle$ must satisfy the differential equation

$$i\frac{\partial\langle\bar{x}|x\rangle}{\partial t}=\frac{\Omega}{2}\left[-\frac{\partial^2}{\partial\bar{x}^2}+\bar{x}^2\right]\langle\bar{x}|x\rangle\ . \tag{A48}$$

Substituting Eq. (A43) into this differential equation, one finds that ϕ must satisfy

$$\frac{\partial\phi}{\partial t}=\Omega \tag{A49}$$

and the normalization constant must satisfy the differential equation

$$\frac{1}{N(\phi)}\frac{\partial N(\phi)}{\partial t} = -\frac{\Omega}{2}\cot\phi\ . \tag{A50}$$

The solution to these differential equations that satisfies the initial condition

$$\lim_{t\to 0}\langle\bar{x}|\psi\rangle = \langle x|\psi\rangle \tag{A51}$$

is

$$\phi = \Omega t \tag{A52}$$

and

$$N(\phi) = \frac{1}{\sqrt{2\pi i \sin(\Omega t)}}\ . \tag{A53}$$

Substituting Eqs. (A52) and (A53) into Eq. (A43), one sees that the propagator $\langle\bar{x}|x\rangle$ is equal to the kernel $K(\bar{x},x)$ given in Eq. (15). We have thus provided a derivation of Eq. (15) and demonstrated that the Hamiltonian H_Ω, acting over the time interval t given in Eq. (A52), performs the same function as an optical beam splitter. We have thus established the equivalence between the mechanical or electromechanical oscillator Bell inequality experiment presented here and the optical Bell inequality experiment we described previously [16].

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