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Common eigenstates of two particles' center-of-mass coordinates and mass-weighted relative momentum

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We give the explicit form of the common eigenstate $|\xi\rangle$ of the center-of-mass coordinate $\mu_1 Q_1+\mu_2 Q_2$ and the mass-weighted relative momentum $\mu_2 P_1-\mu_1 P_2$ of two particles, which is more complicated than the common eigenstate of the other pair of commutative operators Q_2-Q_1 and P_1+P_2. The orthonormal and completeness relation of $|\xi\rangle$ are investigated easily by virtue of the technique of integration within an ordered product of operators. The normally ordered squeezing operator for $\mu_1 Q_1+\mu_2 Q_2$ and $\mu_2 P_1-\mu_1 P_2$ is also derived by using the $|\xi\rangle$ representation.

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I. INTRODUCTION

In Ref. [1] the explicit form of the common eigenvectors $|\eta\rangle$ of the relative position Q_1-Q_2 and the total momentum P_1+P_2 of two particles are constructed. The fact that $[Q_1-Q_2,P_1+P_2]=0$ was considered by Einstein, Podolsky, and Rosen [2] in their argument that the quantum-mechanical state vector is not complete. As shown in Ref. [1] the state $|\eta\rangle$, expressed in two-model Fock space, is given by

$$|\eta\rangle=\exp[-\tfrac{1}{2}|\eta|^2+\eta a^\dagger-\eta^* b^\dagger+a^\dagger b^\dagger]|00\rangle\ , \qquad (1)$$

where $[a,a^\dagger]=[b,b^\dagger]=1$, $|00\rangle$ is the ground state. Having noticed another commutator $[Q_1+Q_2,P_1-P_2]=0$, the simultaneous eigenstate of Q_1+Q_2 and P_1-P_2 is also derived in Ref. [1]. In the present work we go a step further to consider the common eigenstate of the center-of-mass coordinate $Q_{\text{c.m.}}=\mu_1 Q_1+\mu_2 Q_2$ and the mass-weighted relative momentum $\mu_2 P_1-\mu_1 P_2\equiv P_r$, where

$$\mu_1=\frac{m_1}{m_1+m_2},\quad \mu_2=\frac{m_2}{m_1+m_2},\quad \mu_1+\mu_2=1 \qquad (2)$$

are the reduced masses, and m_1 (m_2) is the mass of the first (second) particle. In solving dynamical problems it is frequently useful to convert from individual particle coordinates Q_1 and Q_2 to center-of-mass and relative coordinates (correspondingly, the individual particle momenta transform to total momentum and mass-weighted relative momentum) [3]

$$Q_{\text{c.m.}}=\mu_1 Q_1+\mu_2 Q_2,\quad Q_r=Q_2-Q_1\ , \qquad (3)$$

$$P_{\text{c.m.}}=P_1+P_2,\quad P_r=\mu_2 P_1-\mu_1 P_2\ . \qquad (4)$$

Since $[Q_{\text{c.m.}},P_r]=0$, it is meaningful to construct the simultaneous eigenstates of $Q_{\text{c.m.}}$ and P_r in terms of conventional creation and annihilation operators. We shall make full use of the technique of integration within an ordered product (IWOP) [4] of operators to examine the orthogonality and completeness relation of such eigenvectors. As one can see from Sec. II, these eigenvectors take more complicated forms than Eq. (1), as they must be μ_1 or μ_2 dependent. In Sec. III we study orthonormal and completeness properties of $|\xi\rangle$. In Sec. IV we derive the squeezing operator for both $Q_{\text{c.m.}}$ and P_r.

II. THE COMMON EIGENSTATES OF $Q_{\text{c.m.}}$ AND P_r

We show that in the two-mode Fock space spanned by

$$|nm\rangle=\frac{a^{\dagger n}b^{\dagger m}}{\sqrt{n!m!}}|00\rangle \qquad (5)$$

the common eigenstates of $Q_{\text{c.m.}}$ and P_r are all given by

$$|\xi\rangle=\exp\left\{-\frac{|\xi|^2}{2}+\frac{1}{\sqrt{\lambda}}[(\mu_1+\mu_2)\xi+(\mu_1-\mu_2)\xi^*]a^\dagger+\frac{1}{\sqrt{\lambda}}[(\mu_1+\mu_2)\xi^*-(\mu_1-\mu_2)\xi]b^\dagger+\frac{1}{\lambda}[(\mu_2^2-\mu_1^2)(a^{\dagger 2}-b^{\dagger 2})-4\mu_1\mu_2 a^\dagger b^\dagger]\right\}|00\rangle\ , \qquad (6)$$

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where $\xi=\xi_1+i\xi_2$ is an arbitrary complex number, and

$$\lambda=2(\mu_1^2+\mu_2^2) . \tag{7}$$

We want to prove that $|\xi\rangle$ is the common eigenstate of the pair operators

$$Q_{\text{c.m.}}=\frac{1}{\sqrt{2}}[\mu_1(a+a^\dagger)+\mu_2(b+b^\dagger)] ,$$
$$P_r=\frac{-1}{\sqrt{2}i}[\mu_1(b-b^\dagger)-\mu_2(a-a^\dagger)] . \tag{8}$$

For this purpose we act with a and b on $|\xi\rangle$, respectively, to derive

$$a|\xi\rangle=\left[\frac{2}{\sqrt{\lambda}}(\mu_1\xi_1+i\mu_2\xi_2)-4\frac{\mu_1\mu_2}{\lambda}b^\dagger+\frac{2}{\lambda}(\mu_2^2-\mu_1^2)a^\dagger\right]|\xi\rangle , \tag{9}$$

$$b|\xi\rangle=\left[\frac{2}{\sqrt{\lambda}}(\mu_2\xi_1-i\mu_1\xi_2)-4\frac{\mu_1\mu_2}{\lambda}a^\dagger-\frac{2}{\lambda}(\mu_2^2-\mu_1^2)b^\dagger\right]|\xi\rangle . \tag{10}$$

It then follows from Eqs. (9), (10), and (7) that

$$(\mu_1a+\mu_2b)|\xi\rangle=[\sqrt{\lambda}\xi_1-(\mu_1a^\dagger+\mu_2b^\dagger)]|\xi\rangle , \tag{11}$$

$$(\mu_1b-\mu_2a)|\xi\rangle=(-i\sqrt{\lambda}\xi_2-\mu_2a^\dagger+\mu_1b^\dagger)|\xi\rangle . \tag{12}$$

Combining the results of Eqs. (11), (12), and (6)–(8), we obtain the eigenvalue equations

$$Q_{\text{c.m.}}|\xi\rangle=\left[\frac{\lambda}{2}\right]^{1/2}\xi_1|\xi\rangle=\sqrt{\mu_1^2+\mu_2^2}\xi_1|\xi\rangle , \tag{13}$$

$$P_r|\xi\rangle=\left[\frac{\lambda}{2}\right]^{1/2}\xi_2|\xi\rangle=\sqrt{\mu_1^2+\mu_2^2}\xi_2|\xi\rangle . \tag{14}$$

In particular, when $\mu_1=\mu_2$, Eq. (6) reduces to Eq. (14) of Ref. [1]. Comparing Eq. (6) with Eq. (14) of Ref. [1], we see that the former is more complicated than the latter, so the generalization in this section is nontrivial.

III. SOME PROPERTIES OF $|\xi\rangle$

We now examine if $|\xi\rangle$ spans a completeness relation. By virtue of the IWOP and the normal ordering form of the two-mode vacuum projector

$$|00\rangle\langle 00|=:\exp[-a^\dagger a-b^\dagger b]: , \tag{15}$$

we can easily perform the following integration:

$$\int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=\int\frac{d^2\xi}{\pi}:\exp\left\{-|\xi|^2+\frac{\xi}{\sqrt{\lambda}}[(\mu_1+\mu_2)(a^\dagger+b)+(\mu_1-\mu_2)(a-b^\dagger)]-a^\dagger a-b^\dagger b+\frac{\xi^*}{\sqrt{\lambda}}[(\mu_1-\mu_2)(a^\dagger-b)+(\mu_1+\mu_2)(b^\dagger+a)]+\frac{1}{\lambda}(\mu_2^2-\mu_1^2)(a^{\dagger 2}-b^{\dagger 2}+a^2-b^2)-\frac{4}{\lambda}\mu_1\mu_2(a^\dagger b^\dagger+ab)\right\}:=1 . \tag{16}$$

Next we calculate the overlap $\langle\xi'|\xi\rangle$. In terms of (13)–(14) we see that

$$\left[\frac{2}{\lambda}\right]^{1/2}\langle\xi'|(Q_{\text{c.m.}}+iP_r)|\xi\rangle=(\xi_1+i\xi_2)\langle\xi'|\xi\rangle=(\xi_1'+i\xi_2')\langle\xi'|\xi\rangle . \tag{17}$$

Thus

$$(\xi-\xi')\langle\xi'|\xi\rangle=0 , \tag{18}$$

which leads to

$$\langle\xi'|\xi\rangle=\pi\delta^{(2)}(\xi'-\xi) . \tag{19}$$

Hence $|\xi\rangle$ is an orthonormal eigenstate of $Q_{\text{c.m.}}$ and P_r.

IV. SIMULTANEOUSLY SQUEEZING UNITARY TRANSFORMATION FOR $Q_{\text{c.m.}}$ and P_r

Since $[Q_{\text{c.m.}},P_r]=0$, it is possible to derive the squeezing operator for both $Q_{\text{c.m.}}$ and P_r. After using $\xi=\xi_1+i\xi_2$ and reexpressing $|\xi\rangle$ as

$$|\xi\rangle \equiv |\xi_1,\xi_2\rangle = \exp\left\{-\frac{|\xi|^2}{2}+\frac{2}{\sqrt{\lambda}}[(\mu_1\xi_1+i\mu_2\xi_2)a^\dagger+(\mu_2\xi_1-i\mu_1\xi_2)b^\dagger] -\frac{4}{\lambda}\mu_1\mu_2 a^\dagger b^\dagger+\frac{\mu_2^2-\mu_1^2}{\lambda}(a^{\dagger 2}-b^{\dagger 2})\right\}|00\rangle , \tag{20}$$

we construct the following integral form unitary operator:

$$U=\int\frac{d^2\xi}{\pi}\sqrt{\sigma\nu}|\sigma\xi_1,\nu\xi_2\rangle\langle\xi_1,\xi_2|, \quad d^2\xi\equiv d\xi_1 d\xi_2 , \tag{21}$$

where σ and μ are two independent positive numbers. The unitarity of U can be seen from (19) by calculating

$$UU^\dagger=\sigma\nu\int\frac{d^2\xi}{\pi}|\sigma\xi_1,\nu\xi_2\rangle\langle\sigma\xi_1,\nu\xi_1|=1 .$$

Using (15), (20), and (7) as well as the IWOP technique, we can perform the integration in (21) and obtain

$$\begin{aligned}U=\sqrt{\sigma\nu}\int\frac{d^2\xi}{\pi}:\exp\Bigg\{&-\frac{\xi_1^2}{2}(1+\sigma^2)-\frac{\xi_2^2}{2}(1+\nu^2)+\frac{2}{\sqrt{\lambda}}\xi_1[(\mu_1 a^\dagger+\mu_2 b^\dagger)\sigma+\mu_1 a+\mu_2 b]\\&+\frac{2i}{\sqrt{\lambda}}\xi_2[(\mu_2 a^\dagger-\mu_1 b^\dagger)\nu-\mu_2 a+\mu_1 b]-\frac{4\mu_1\mu_2}{\lambda}(a^\dagger b^\dagger+ab)\\&+\frac{\mu_2-\mu_1}{\lambda}(a^{\dagger 2}-b^{\dagger 2}+a^2-b^2)-a^\dagger a-b^\dagger b\Bigg\}:\\
=\frac{2\sqrt{\sigma\nu}}{\sqrt{L}}\exp\Bigg\{&\frac{1}{\lambda L}\{(\sigma^2-\nu^2)(\mu_1^2+\mu_2^2)(a^{\dagger 2}+b^{\dagger 2})\\&+[(\mu_2^2-\mu_1^2)(a^{\dagger 2}-b^{\dagger 2})-4\mu_1\mu_2 a^\dagger b^\dagger](1-\sigma^2\nu^2)\}\Bigg\}\\
\times:\exp\left[(a^\dagger b^\dagger)(g-\underline{1})\begin{bmatrix}a\\b\end{bmatrix}\right]:\exp\Bigg\{&\frac{1}{\lambda L}\{(\nu^2-\sigma^2)(\mu_1^2+\mu_2^2)(a^2+b^2)\\&+[(\mu_2^2-\mu_1^2)(a^2-b^2)-4\mu_1\mu_2 ab](\sigma^2\nu^2-1)\}\Bigg\} ,\end{aligned} \tag{22}$$

where $\underline{1}$ is a 2×2 unit matrix, and

$$L=(1+\sigma^2)(1+\nu^2),\quad g=\frac{4}{\lambda L}\begin{bmatrix}\sigma\mu_1^2(1+\nu^2)+\nu\mu_2^2(1+\sigma^2) & -\mu_1\mu_2[(1+\sigma^2)\nu-\sigma(1+\nu^2)]\\ -\mu_1\mu_2[(1+\sigma^2)\nu-\sigma(1+\nu^2)] & \sigma\mu_2^2(1+\nu^2)+\nu\mu_1^2(1+\sigma^2)\end{bmatrix} . \tag{23}$$

Note that

$$\det g=\frac{4\nu\sigma}{L},\quad g^{-1}=\frac{1}{\sigma\nu\lambda}\begin{bmatrix}\sigma\mu_2^2(1+\nu^2)+\nu\mu_1^2(1+\sigma^2) & \mu_1\mu_2[(1+\sigma^2)\nu-\sigma(1+\nu^2)]\\ \mu_1\mu_2[(1+\sigma^2)\nu-\sigma(1+\nu^2)] & \sigma\mu_1^2(1+\nu^2)+\nu\mu_2^2(1+\sigma^2)\end{bmatrix} . \tag{24}$$

Let

$$S\equiv:\exp\left[(a^\dagger b^\dagger)(g-\underline{1})\begin{bmatrix}a\\b\end{bmatrix}\right]: ; \tag{25}$$

we can derive

$$\begin{aligned}sas^{-1}&=(g^{-1})_{11}a+(g^{-1})_{12}b ,\\ sbs^{-1}&=(g^{-1})_{21}a+(g^{-1})_{22}b .\end{aligned} \tag{26}$$

It then follows from Eqs. (25), (26), and (22) that

$$\begin{aligned}UaU^{-1}=&(g^{-1})_{11}a+(g^{-1})_{12}b\\&-[2k_1(g^{-1})_{11}+(g^{-1})_{12}k_3]a^\dagger\\&-[(g^{-1})_{11}k_3+2(g^{-1})_{12}k_2]b^\dagger ,\end{aligned} \tag{27}$$

where

$$\begin{aligned}k_1&=\frac{1}{\lambda L}[(1-\sigma^2\nu^2)(\mu_2^2-\mu_1^2)+(\sigma^2-\nu^2)(\mu_1^2+\mu_2^2)] ,\\ k_2&=\frac{1}{\lambda L}[(\sigma^2\nu^2-1)(\mu_2^2-\mu_1^2)+(\sigma^2-\nu^2)(\mu_1^2+\mu_2^2)] ,\\ k_3&=\frac{4\mu_1\mu_2}{\lambda L}(\sigma^2\nu^2-1) .\end{aligned} \tag{28}$$

Similarly, we have

$$UbU^{-1}=(g^{-1})_{21}a+(g^{-1})_{22}b-[2k_1(g^{-1})_{21}+(g^{-1})_{22}k_3]a^\dagger-[(g^{-1})_{21}k_3+2(g^{-1})_{22}k_2]b^\dagger \ . \tag{29}$$

As a result of Eqs. (27)–(29) and (8), we obtain

$$UQ_{\text{c.m.}}U^{-1}=\sigma^{-1}Q_{\text{c.m.}} \ , \tag{30}$$

$$UP_rU^{-1}=\nu^{-1}P_r \ . \tag{31}$$

Comparing Eqs. (30) and (31), we see that U is indeed an operator simultaneously squeezing $Q_{\text{c.m.}}$ and P_r. We can also employ the normally ordered form of U to calculate

$$UQ_rU^{-1}=\frac{2}{\sigma\lambda}[(\mu_2-\mu_1)(\mu_1Q_1+\mu_2Q_2)+\sigma\nu(\mu_1+\mu_2)(\mu_1Q_2-\mu_2Q_1)] \ , \tag{32}$$

$$UP_{\text{c.m.}}U^{-1}=\frac{2}{\nu\lambda}[(\mu_2-\mu_1)(\mu_2P_1-\mu_1P_2)+\sigma\nu(\mu_1+\mu_2)(\mu_1P_1+\mu_2P_2)] \ . \tag{33}$$

Especially, when $\mu_1=\mu_2$, Eqs. (32) and (33) reduce to

$$U(\mu_1=\mu_2)Q_rU^{-1}(\mu_1=\mu_2)=\nu Q_r \ ,$$

$$U(\mu_1=\mu_2)P_{\text{c.m.}}U^{-1}(\mu_1=\mu_2)=\sigma P_{\text{c.m.}} \ .$$

In summary, we have found a representation $|\xi\rangle$ in two-mode Fock space, which possesses orthonormal and completeness relation. The IWOP technique plays an essential role in our derivations.

APPENDIX

Here we give a rigorous proof for Eq. (15). Without loss of generality we prove $|0\rangle\langle 0|=:e^{-a^\dagger a}:$ for the single-mode case. Actually, supposing $|0\rangle\langle 0|=:W:$, W is to be determined. From the completeness relation of the Fock state

$$\sum_{n=0}^{\infty}|n\rangle\langle n|=1, \quad |n\rangle=\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \ ,$$

we have

$$1=\sum_{n,n'=0}^{\infty}|n\rangle\langle n'|\frac{1}{\sqrt{n\ n'!}}\left[\frac{d}{dZ^*}\right]^n(z^*)^{n'}\Bigg|_{Z^*=0}$$

$$=e^{a^\dagger d/dZ^*}|0\rangle\langle 0|e^{Z^*a}\Bigg|_{Z^*=0}=:e^{a^\dagger a}W: \ . \tag{A1}$$

Then we use the property of the normal ordering symbol, that is, the symbol : : which is within another symbol that can be deleted, to compare (A1) with $1=:e^{a^\dagger a}e^{-a^\dagger a}:$ and obtain $:W:=:e^{-a^\dagger a}:$. Further, using the mathematical formula

$$\int\frac{d^2\xi}{\pi}e^{-|\xi|^2+\lambda\xi+\sigma\xi^*}=e^{\lambda\sigma}$$

and the property that the order of Bose operators a,b and $a^\dagger,b^\dagger$ within : : can be permuted, we perform the integration of Eq. (16) to reach the result of completeness of $|\xi\rangle$.

[1] Fan Hong-yi and J. R. Klauder, Phys. Rev. A **49**, 704 (1994).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] Fan Hong-yi and J. VanderLinda, Phys. Rev. A **43**, 1604 (1991).
[4] Fan Hong-yi, H. R. Zaidi, and J. R. Klauder, Phys. Rev. D **35**, 1831 (1987); Fan Hong-yi, Phys. Rev. A **47**, 2379 (1993).