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Nonclassical effects in phase space

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We introduce a quantitative measure of nonclassical behavior based on negative regions of quasiprobability distributions. Zeros of the antinormal ordered phase-space representation characterize a state to be maximally nonclassical with respect to this measure. All pure states of a particle or of a single field mode, except those with Gaussian quasiprobability distributions, have one or more zeros in this representation.

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It is a fundamental principle of quantum mechanics that one cannot, with absolute precision, specify both the position and momentum of a particle. Consequently, the classical notion of phase space with the state of particle described by a single point has to be modified when dealing with quantum systems. In its place, we have a family of phase-space quasiprobability distributions, depending on a real parameter s, corresponding to different operator orderings [1,2]. These quasiprobability distributions have been widely employed, especially in quantum optics where they have been used to calculate and illustrate the properties of nonclassical states of light [3]. The quasiprobability distributions are not simply calculational tools; two of the family of distributions have been successfully measured for optical fields [4]. Nonclassical behavior is connected with properties of the quasiprobability that are incompatible with their interpretation as true probability distributions. In particular, nonclassical effects are associated with negative values of the quasiprobability for some values of the ordering parameter s. This is the motivation for our work in which we present a measure for the degree of nonclassical behavior for a state. In place of the often employed division of states into "classical" and "nonclassical," we provide a quantitative measure indicating how "classical" or "nonclassical" a given state is. This measure is based on the existence of negative regions of the s-ordered quasiprobability distributions [1].

The normalized quasiprobability distributions $W(\alpha,s)$ for a state with density matrix ρ are defined to be [1,5]

$$W(\alpha,s)=\frac{1}{\pi^2}\int \exp(\alpha\eta^*-\alpha^*\eta)C(\eta,s)d^2\eta \,, \tag{1}$$

where $C(\eta,s)$ is the characteristic function

$$C(\eta,s)=\mathrm{Tr}\{\rho\exp(\eta\hat{a}^{+}-\eta^*\hat{a})\}\exp\left[\frac{s}{2}|\eta|^2\right] . \tag{2}$$

Here $\hat{a}$ is the annihilation operator for a single field mode or is a non-Hermitian combination of the position and momentum operators for a particle [1]. The quasiprobability distributions (1) are a generalization of the well-known Glauber-Sudarshan (P), Wigner (W), and Husimi (Q) quasiprobability distributions corresponding to the values $s=1$, 0, and -1, respectively. The P, W, and Q quasiprobability distributions give rise to normally, symmetrically, and antinormally ordered moments of the creation and annihilation operators, respectively. In general, the quasiprobability distributions are only well behaved for certain values of the ordering parameter s. However, for $s<-1$ the distribution will be positive definite and for $s<0$ it will always be regular in the sense that it can be expressed in terms of functions. For $s>0$ the quasiprobability distributions will, in general, be expressed in terms of generalized functions such as δ functions and their derivatives [6]. These properties are illustrated in Fig. 1.

Two characteristic functions for the same state but different values of s differ by a factor depending on the difference in the values of s:

$$C(\eta,\bar{s})=C(\eta,\tilde{s})\exp\left\{-\frac{(\bar{s}-\tilde{s})}{2}|\eta|^2\right\} . \tag{3}$$

For $\bar{s}>\tilde{s}$ it follows that the two quasiprobability distributions are related by a convolution of the form

$$W(\alpha,\tilde{s})=\int W(\beta,\bar{s})\left[\frac{2}{\pi(\bar{s}-\tilde{s})}\exp\left\{-\frac{2|\alpha-\beta|^2}{\bar{s}-\tilde{s}}\right\}\right]d^2\alpha \,. \tag{4}$$

The convolution means that $W(\alpha_0,\tilde{s})$ may be calculated from the function $W(\alpha,\bar{s})$ by averaging with a Gaussian at $\alpha=\alpha_0$. The relation (4) can be understood by observing that $W(\alpha,s)$ satisfies a sourceless diffusion equation with s taking the role of reverse time [1]. The diffusion

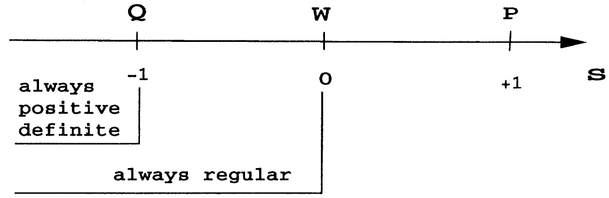

FIG. 1. The principal properties of the s-quasiprobability distributions.

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smooths $W(\alpha,s)$ in the direction of decreasing s. This allows us to prove the following theorem: *If* $W(\alpha,\bar{s})\geq 0$ *for all* α *and* $W(\alpha_0,\bar{s})=0$ *for at least one* α_0, *then there is no* $\bar{\bar{s}}$ *with* $\bar{\bar{s}}>\bar{s}$ *and* $W(\alpha,\bar{\bar{s}})\geq 0$ *for all* α. The proof follows directly from the convolution (4). The Gaussian in the integrand is strictly positive and the function $W(\beta,\bar{s})$ is, by assumption, positive semidefinite; it follows that the integral, and hence $W(\alpha,\bar{\bar{s}})$, cannot take a negative value.

A second consequence of Eq. (4) is that the parameter space of s is divided into two intervals belonging to well-behaved and not so well-behaved quasiprobability distributions. The division between these parts, illustrated in Fig. 2, occurs at a particular value of s, which we denote s_c. This *critical parameter* may be regarded as a measure of the degree of nonclassical behavior associated with a given state [7]. We can readily regain a simpler, qualitative distinction between classical and nonclassical states by defining a particular value of the order parameter S as a threshold and referring to states as classical if $s_c\geq S$ and as nonclassical states otherwise. if we set the threshold to be $+1$ then we recover the existence of negative regions of the P function as the criterion for nonclassical behavior [8]. For $S=0$ the existence of negative parts of the Wigner function form the criterion. It is worth noting that $s_c\geq -1$ as the Q function is always non-negative. Moreover, for all pure states, s_c is less than or equal to 1 and equals 1 only for the coherent states [9]. We seek a general expression for the value of the critical parameter valid for all pure states.

We begin by specializing the first result to the value $\bar{s}=-1$. If $W(\alpha_0,-1)=Q(\alpha_0)=0$ for at least one value α_0, then the theorem implies that $W(\alpha,s)$ has negative regions for all $s>-1$. As we have already noted, the Q function is non-negative and hence states represented by Q functions with one or more zeros have the minimum possible value of the critical parameter. It only remains to determine which pure states have Q functions with one or more zeros.

A pure state represented by the ket $|\Psi\rangle$ has a Q function that may be expressed in terms of the overlap between $|\Psi\rangle$ and the coherent states $|\alpha\rangle$ [1],

$$Q(\alpha)=|\langle\Psi|\alpha\rangle|^2/\pi \ , \tag{5}$$

which is clearly positive-semidefinite and will have zeros if and only if $\langle\Psi|\alpha_0\rangle=0$ for at least one coherent state $|\alpha_0\rangle$. We will now show that such zeros exist for all states except the coherent states and the ideal squeezed states [10]. The proof of this result follows a proof

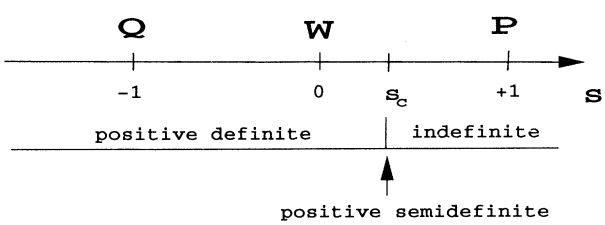

FIG. 2. The critical parameter s_c separates all the positive definite s-quasiprobability distributions from those that are negative for some parts of phase space.

developed by Hudson to study the properties of the Wigner function [11].

A general state of a single field mode or harmonic oscillator may be represented as a superposition of the number states $|n\rangle$:

$$|\Psi\rangle=\sum_{n=0}^{\infty}c_n|n\rangle \ . \tag{6}$$

That all pure states can be represented in this form follows from the completeness of the number states. We define the complex, analytic function $f(\alpha)$ to be

$$f(\alpha)=\exp\left[\frac{|\alpha|^2}{2}\right]\langle\Psi|\alpha\rangle=\sum_{n=0}^{\infty}c_n^*\frac{\alpha^n}{\sqrt{n!}} \ . \tag{7}$$

It is clear from this definition that the Q function will have zeros if and only if $f(\alpha)$ has zeros. We can readily bound the magnitude of $f(\alpha)$ by the inequality

$$|f(\alpha)|^2\leq\exp(|\alpha|^2) \ . \tag{8}$$

This characterizes the sum to be of growth order less than or equal to 2 [12]. This, together with the analyticity of $f(\alpha)$, allows us to apply Hadamard's theorem [12], which states that any function that is analytic on the whole complex plane, has no zeros, and is restricted in growth to be of order 2 or less, *must* be of the form

$$f(\alpha)=\exp(A\alpha^2+B\alpha+C) \ , \tag{9}$$

where A, B, and C are complex constants. This form corresponds to a Gaussian Q function. It is relatively straightforward to show that the only states for which $f(\alpha)$ has this form are the squeezed coherent states [10] $|\zeta,\beta\rangle$, which are related to the ground state $|0\rangle$ by a unitary transformation

$$|\zeta,\beta\rangle=\exp(\beta a^+-\beta^*a)\exp(\tfrac{1}{2}\{\zeta a^{+2}-\zeta^*a^2\})|0\rangle \ , \tag{10}$$

where ζ and β are arbitrary complex numbers. All other pure states will have Q functions with at least one zero, and hence all quasiprobability distributions $W(\alpha,s)$ with $x>-1$ will be negative for some values of α.

The squeezed states (10) will not lead to well-behaved quasiprobability distributions for all values of s. The regions corresponding to well-behaved and not so well-behaved quasiprobability distributions will be separated by a critical parameter related to the squeezing parameter ζ:

$$s_c=\exp(-2|\zeta|) \ . \tag{11}$$

The results derived here are strictly valid only for single modes. This is because Hadamard's theorem is proven in general for functions of only a single complex variable. However, Soto and Claverie [13] have proven a restricted version of the theorem and this is all that is needed to extend the results of this paper to multimode states.

We conclude that the critical value of s dividing well-behaved from not so well-behaved quasiprobability distributions is -1 for all pure states except those with Gaussian Q functions. These states are the familiar coherent squeezed states. The behavior of the critical parameter suggests that of all states the Gaussian states are the most nearly classical. This conclusion is in agreement with that reached by studying a number of other methods to distinguish between classical and nonclassical behavior [14].

The division of the quasiprobability distributions into well-behaved and not so well-behaved regions of s space is not restricted to pure states. Our measure of the degree of nonclassical behavior is still well-defined for mixed states. For the mixed states the critical value can take on any value greater than -1 with values near to -1 occurring for mixed states that are close to the pure states other than those with Gaussian Wigner functions. Zeros of the Q function for a mixed state lead to $s_c = -1$ as stated above. We will apply the ideas developed in this paper to mixed states and to the study of mode dynamics elsewhere.

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