Even and odd coherent states for multimode parametric systems

V. V. Dodonov
Moscow Institute of Physics and Technology, Zhukovsky 140160, Russia
and Laboratorio de Cuernavaca, Instituto de Física, Universidad Nacional Autónoma de México, 62191 Cuernavaca, Morelos, Mexico

V. I. Man'ko
Lebedev Physics Institute, Moscow, Russia
and Departimento di Scienze Fisiche, Universita di Napoli "Federico II," Mostra d'Oltremare, Padia 20-80125 Napoli, Italy

D. E. Nikonov
Department of Physics, Texas A&M University, College Station, Texas 77843-4242
and Moscow Institute of Physics and Technology, Zhukovsky 140160, Russia
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The multimode even and odd coherent states (multimode Schrödinger cat states) are constructed for polymode parametric oscillators of the electromagnetic field. The evolution of the photon distribution function is evaluated explicitly. The distribution function is expressed in terms of multivariable Hermite polynomials; the means and dispersions of the function are calculated. The conditions for the existence of squeezing are formulated. The correlations between the different modes of the Schrödinger cat states are studied. The transformation of the initial Schrödinger cat states under the action of a resonant external force is investigated.

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I. INTRODUCTION

Even and odd coherent states for one-mode systems were introduced in [1]. Their properties were studied, e.g., in [2–5], and recently they have been shown to represent a special set of a more general class of nonclassical states considered by Nieto and Truax [6]. The multimode even and odd coherent states and the photon statistics for these states have been considered in Ref. [7]. The even coherent states are similar in a sense to the squeezed vacuum states [8] since they are the superpositions of the photon number states with even numbers of quanta. In particular, the photon statistics of the one-mode even and odd coherent states exhibits properties which are typical for the other nonclassical states of light [5]. For this reason the even coherent light may be used in the interferometric gravitational wave detectors to give the same effect of increasing the sensitivity of these devices, which could be produced by the replacement of the vacuum state by the squeezed vacuum light at the unused port of the interferometer [9].

The even and odd coherent states (Schrödinger cat states [10,11]) may be generated in different processes [12–19]. Gea-Banacloche [12] showed a possibility of the appearance of Schrödinger cat states in the resonant Jaynes-Cummings model. Gerry and Hach [19] demonstrated a possibility to generate even and odd coherent states for the long-time evolution of the competition between two-photon absorption and two-photon parametric processes for a special initial field state. Recently, a possibility to generate the quantum superposition of macroscopically distinguishable states in two cavities was shown [20]. In Ref. [21] methods of engineering different kinds of field states (including Schrödinger cat states) were suggested. Agarwal *et al.* have studied [22] the possibility of generating two-mode cat states by a continuous measurement.

In the present paper we consider the evolution of the one-mode and multimode even and odd coherent states due to the parametric excitation of a quantum system, since this process is known to produce squeezing in the quadratures of initially coherent light (see, for example, [23,24]). It may correspond, e.g., to the evolution of the Schrödinger cat states in a resonator filled with a medium which parameters vary in time [25,26]. We shall demonstrate that the multimode Schrödinger cat states of Ref. [7] turn into multimode squeezed Schrödinger cat states, since each coherent component determining the even and odd coherent states transforms into a multimode squeezed correlated state.

Another goal of our work is to study the photon statistics of multimode even and odd light with the account of the squeezing in the multimode coherent components through which the Schrödinger cat states are expressed. We construct the multimode photon distribution function (PDF) and calculate the means and dispersions related to this distribution. The remarkable characteristic feature of the photon distribution functions of the nonclassical states of light is their oscillating and sometimes almost irregular behavior for certain combinations of the parameters. There exist at least two mechanisms producing strong oscillations of the PDF. The first one is related to large squeezing in the quadrature components [27–29]. Strong statistical correlations between the quadrature components also produce the oscillations of the PDF [30]. But these types of oscillations are small in comparison with the oscillations of the PDF, which are observed in the even or odd coherent states due to the absence in these states of the contributions of the odd or even number states, respectively [1]. The PDF of the multimode Schrödinger cat states turns out to be a deformed product of the Poisson distributions related to different modes. However, this function is not factorized into the product of the unidimensional Poisson distributions due to the statistical correlations between the modes [7].

The amplitude of oscillations in real experimental situations must be smaller than that in ideal cases due to various parasite effects. It is evident, for instance, that the thermal noise smooths the oscillations in any state. Nonetheless, a number of papers were devoted to the quantitative estimations of the temperature corrections to the PDFs of the Schrödinger cat states and squeezed states [31–36]. Here we analyze another mechanism destroying the oscillations in the Schrödinger cat states due to the possible presence of linear terms in the Hamiltonian, which describe the influence of some external classical force. Such a force "spoils" the even or odd states by adding to them terms with the opposite parity. Therefore the amplitudes of the oscillations of the corresponding PDFs decrease. However, this mechanism has not been investigated in detail until now.

The material of the paper is organized in the following manner. In Sec. II we review the properties of the multimode Schrödinger cat states discussed in Ref. [7]. The properties of polymode parametric quadratic systems and the multimode squeezed Schrödinger cat states resulting from the time evolution of the Schrödinger cat states of a parametric electromagnetic field oscillator are presented in Sec. III. The photon distribution functions in the parametrically excited Schrödinger cat states in the absence of an external force are studied in Sec. IV, while the transformations of these functions under the action of the time-dependent external driving force are considered in Sec. V.

II. MULTIMODE EVEN AND ODD COHERENT STATES

For each mode of the electromagnetic field, the eigenvectors of its annihilation operator a_j (coherent states) with the eigenvalue α_j are generated from the vacuum state $|0\rangle$ by the displacement operator [37], namely,

$$|\alpha_j\rangle = D(\alpha_j)|0\rangle \equiv \exp(\alpha_j a_j^\dagger - \alpha_j^* a_j)|0\rangle. \tag{1}$$

In [7] N-mode even and odd coherent states defined as

$$|\alpha_\pm\rangle = N_\pm(|\alpha\rangle \pm |-\alpha\rangle) \tag{2}$$

were considered, where $|\alpha\rangle$ is a direct product of coherent states in each mode:

$$|\alpha\rangle = |\alpha_1, \alpha_2, \ldots, \alpha_N\rangle. \tag{3}$$

The normalization constants

$$N_+ = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\cosh|\alpha|^2}}\ , \tag{4}$$

$$N_- = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\sinh|\alpha|^2}} \tag{5}$$

contain the square of the parameter complex vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. The presence of this nonfactorizable factor expresses statistical correlations between the modes. The action rules for annihilation operators on these states

$$a_j|\alpha_+\rangle = \alpha_j\sqrt{\tanh|\alpha|^2}|\alpha_-\rangle, \tag{6}$$

$$a_j|\alpha_-\rangle = \alpha_j\sqrt{\coth|\alpha|^2}|\alpha_+\rangle \tag{7}$$

allow us to calculate the means and the dispersions of quadratures and photon numbers. The states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ are orthonormal. Thus using Eqs. (6) and (7) we get

$$\langle\alpha_\pm|a_j|\alpha_\pm\rangle = \langle\alpha_\pm|a_j^\dagger|\alpha_\pm\rangle = 0. \tag{8}$$

So the multimode dispersions coincide with the second moments for the operators

$$\langle\alpha_\pm|a_j a_k|\alpha_\pm\rangle = \alpha_j\alpha_k = \langle\alpha_\pm|a_j^\dagger a_k^\dagger|\alpha_\pm\rangle^*. \tag{9}$$

Further, the expressions for other elements of the dispersion matrix are

$$\sigma_+(a_j^\dagger, a_k) \equiv \langle\alpha_+|\tfrac{1}{2}(a_j^\dagger a_k + a_k a_j^\dagger)|\alpha_+\rangle = \alpha_j^*\alpha_k\tanh|\alpha|^2 + \tfrac{1}{2}\delta_{jk}, \tag{10}$$

$$\sigma_-(a_j^\dagger, a_k) \equiv \langle\alpha_-|\tfrac{1}{2}(a_j^\dagger a_k + a_k a_j^\dagger)|\alpha_-\rangle = \alpha_j^*\alpha_k\coth|\alpha|^2 + \tfrac{1}{2}\delta_{jk}. \tag{11}$$

These expressions allow us to calculate the dispersions of canonical coordinate q and momentum p,

$$\sigma(q,q) = \sigma(a,a^+) + \mathrm{Re}\sigma(a,a)\ , \tag{12}$$

$$\sigma(p,p) = \sigma(a,a^+) - \mathrm{Re}\sigma(a,a)\ , \tag{13}$$

$$\sigma(p,q) = \mathrm{Im}\sigma(a,a). \tag{14}$$

Specifically, this yields, for a special case of one mode in general, nonzero correlation and squeezing

$$\sigma(p,q)_\pm = \mathrm{Im}(\alpha^2)\ , \tag{15}$$

$$\sigma(q,q)_+ = \frac{1}{2} + |\alpha|^2\tanh|\alpha|^2 + \mathrm{Re}\alpha^2, \tag{16}$$

$$\sigma(q,q)_- = \frac{1}{2} + |\alpha|^2\coth|\alpha|^2 + \mathrm{Re}\alpha^2, \tag{17}$$

$$\sigma(p,p)_+ = \frac{1}{2} + |\alpha|^2\tanh|\alpha|^2 - \mathrm{Re}\alpha^2, \tag{18}$$

$$\sigma(p,p)_- = \frac{1}{2} + |\alpha|^2\coth|\alpha|^2 - \mathrm{Re}\alpha^2. \tag{19}$$

The case of zero correlation corresponds to real α^2. If this expression is positive, it yields a variance less than that in the vacuum state for $\sigma(p,p)_+$ only.

The mean photon numbers $n_j = a_j^\dagger a_j$ prove to be

$$\langle\alpha_+|n_j|\alpha_+\rangle = |\alpha_j|^2 \tanh|\alpha|^2, \tag{20}$$

$$\langle\alpha_-|n_j|\alpha_-\rangle = |\alpha_j|^2 \coth|\alpha|^2. \tag{21}$$

Dispersions for the photon numbers

$$\sigma_{jk\pm} = \langle\alpha_\pm|n_j n_k|\alpha_\pm\rangle - \langle\alpha_\pm|n_j|\alpha_\pm\rangle\langle\alpha_\pm|n_k|\alpha_\pm\rangle \tag{22}$$

can be expressed as

$$\sigma_{jk+} = |\alpha_j|^2|\alpha_k|^2 \mathrm{sech}^2|\alpha|^2 + |\alpha_j|^2 \tanh|\alpha|^2 \delta_{jk}, \tag{23}$$

$$\sigma_{jk-} = -|\alpha_j|^2|\alpha_k|^2 \mathrm{cosech}^2|\alpha|^2 + |\alpha_j|^2 \coth|\alpha|^2 \delta_{jk}. \tag{24}$$

Comparing the dispersion of the photon number σ_{jj} at some fixed j with its mean value in the same mode n_j, we see that for the even cat state the photon statistics is always super-Poissonian and for the odd cat states it is always sub-Poissonian. This result has been discussed in Ref. [5] for one-mode Schrödinger cat states.

III. PARAMETRIC CAT STATES FOR A MULTIMODE OSCILLATOR

Our first goal is to consider the time evolution of the Schrödinger cat states under the action of a generic quadratic multimode Hermitian Hamiltonian [38,39]

$$H = \frac{1}{2}\mathbf{q}\mathbf{B}(t)\mathbf{q} + \mathbf{C}(t)\mathbf{q}. \tag{25}$$

Here $\mathbf{q}$ is a Schrödinger vector operator $(\mathbf{p},\mathbf{x}) = (p_1, p_2, \ldots, p_N, x_1, x_2, \ldots, x_N)$ composed of the momentum and coordinate operators (quadrature components) for each mode. Coefficients $B_{\mu\nu} = B_{\nu\mu}$ of the $2N \times 2N$ matrix $\mathbf{B}(t) = \|B_{\mu\nu}\|$ and $2N$ vector $\mathbf{C}(t)$ may be arbitrary functions of time. Canonical coordinates are expressed through the annihilation and creation operators as (we assume $\hbar = 1$)

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix} = \mathbf{u}\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}, \tag{26}$$

where the $2N \times 2N$ unitary block matrix $\mathbf{u}$ reads

$$\mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\mathbf{1} & i\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \tag{27}$$

($\mathbf{1}$ is the $N \times N$ unity matrix). The Hamiltonian can be reexpressed in terms of creation and annihilation operators via (26) ($\mathbf{u}^t$ means the transposed matrix)

$$\mathcal{B}(t) = \mathbf{u}^t\mathbf{B}(t)\mathbf{u}, \tag{28}$$

$$H = \frac{1}{2}\begin{pmatrix} \mathbf{a} & \mathbf{a}^\dagger \end{pmatrix}\mathcal{B}(t)\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \begin{pmatrix} \mathbf{f} & \mathbf{f}^* \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \tag{29}$$

The initial wave function of the multimode coherent state reads (see, e.g., [38])

$$\langle\mathbf{x}|\alpha\rangle = \pi^{-N/4}\exp\left(-\frac{\mathbf{x}^2}{2} + \sqrt{2}\mathbf{x}\alpha - \frac{|\alpha|^2}{2} - \frac{\alpha\alpha}{2}\right). \tag{30}$$

In this section we will confine ourselves to the case when the linear terms in the Hamiltonian (25) or (29) are absent. Taking into account the expression for its evolution under the action of Hamiltonian (25) found in Ref. [38] and Eq. (2), we obtain the wave functions of the squeezed cat states $|\alpha_\pm, t\rangle$, which coincide at the initial time with the states $|\alpha_\pm\rangle$ described in Sec. I:

$$\begin{aligned}\langle\mathbf{x}|\alpha_\pm, t\rangle = {} & \pi^{-N/4}[\det\lambda_p]^{-1/2} \\ & \times\exp\left[-\frac{1}{2}\mathbf{x}\lambda_p^{-1}\lambda_q\mathbf{x} - \frac{1}{2}\alpha\lambda_p^*\lambda_p^{-1}\alpha - \frac{|\alpha|^2}{2}\right] \\ & \times 2N_\pm\left\{\begin{array}{l}\cosh\left(\sqrt{2}\mathbf{x}\lambda_p^{-1}\alpha\right) \\ \sinh\left(\sqrt{2}\mathbf{x}\lambda_p^{-1}\alpha\right)\end{array}\right., \end{aligned} \tag{31}$$

where cosh corresponds to α_+ and sinh to α_-. Here λ_p and λ_q are $N \times N$ matrices $\lambda_p = \lambda_1 - i\lambda_3$ and $\lambda_q = \lambda_4 + i\lambda_2$, where matrices λ_j, $j = 1, 2, 3, 4$, form a symplectic block matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \tag{32}$$

satisfying the first-order linear differential equation

$$\dot{\Lambda} = \Lambda\Sigma\mathbf{B} \tag{33}$$

with the initial condition $\Lambda(0)$ being the unity matrix. The $2N \times 2N$ commutator matrix Σ is defined as

$$[q_\mu, q_\nu] = -i(\Sigma)_{\mu\nu}, \tag{34}$$

$$\Sigma = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{35}$$

The explicit formula (31) enables us to calculate the average values and variances of the quadrature components as well as the creation and annihilation operators at moment t. However, it is more convenient to use for this purpose the time-dependent Schrödinger operators — integrals of motion $\mathbf{b}(t)$ and $\mathbf{b}^\dagger(t)$ which coincide at the initial moment with time-independent creation and annihilation operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ [38,39]. The former are expressed in terms of the latter via the matrix $M = \mathbf{u}^{-1}\Lambda\mathbf{u}$:

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{b}^\dagger \end{pmatrix} = \begin{pmatrix} \xi & \eta \\ \eta^* & \xi^* \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} \equiv M\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \tag{36}$$

Matrix M satisfies the differential equation

$$\dot{M} = iM\Sigma\mathcal{B} \tag{37}$$

with the initial condition that M is a unity matrix at zero time. Due to this equation M is a symplectic matrix $M\Sigma M^t = \Sigma$. Consequently, the inverse transformation to (36) reads

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} = \begin{pmatrix} \xi^* & -\eta \\ -\eta^* & \xi \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{b}^\dagger \end{pmatrix}. \tag{38}$$

The $N \times N$ matrices ξ and η are related to matrices λ_p and λ_q as

$$\xi = \frac{1}{2}(\lambda_p + \lambda_q), \qquad \eta = \frac{1}{2}(\lambda_q - \lambda_p). \tag{39}$$

Using Eq. (38) we can calculate the evolution of the means and the dispersions of the creation and annihilation operators (8)–(24). The average values of single operators remain zero. The second-order averages are

$$\begin{aligned}\langle\alpha_+|a_j a_l|\alpha_+\rangle = {}& (\xi^*\alpha)_j(\xi^*\alpha)_l + (\eta\alpha^*)_j(\eta\alpha^*)_l - (\xi^*\eta^t)_{jl} \\ & - \tanh|\alpha|^2[(\eta\alpha^*)_j(\xi^*\alpha)_l \\ & + (\xi^*\alpha)_j(\eta\alpha^*)_l],\end{aligned} \tag{40}$$

$$\begin{aligned}\langle\alpha_-|a_j a_l|\alpha_-\rangle = {}& (\xi^*\alpha)_j(\xi^*\alpha)_l + (\eta\alpha^*)_j(\eta\alpha^*)_l - (\xi^*\eta^t)_{jl} \\ & - \coth|\alpha|^2[(\eta\alpha^*)_j(\xi^*\alpha)_l \\ & + (\xi^*\alpha)_j(\eta\alpha^*)_l],\end{aligned} \tag{41}$$

$$\begin{aligned}\langle\alpha_+|a_j^\dagger a_l|\alpha_+\rangle = {}& -(\eta^*\alpha)_j(\xi^*\alpha)_l - (\xi\alpha^*)_j(\eta\alpha^*)_l + (\eta^*\eta^t)_{jl} \\ & + \tanh|\alpha|^2[(\eta^*\alpha)_j(\eta\alpha^*)_l \\ & + (\xi\alpha^*)_j(\xi^*\alpha)_l],\end{aligned} \tag{42}$$

$$\begin{aligned}\langle\alpha_-|a_j^\dagger a_l|\alpha_-\rangle = {}& -(\eta^*\alpha)_j(\xi^*\alpha)_l - (\xi\alpha^*)_j(\eta\alpha^*)_l + (\eta^*\eta^t)_{jl} \\ & + \coth|\alpha|^2[(\eta^*\alpha)_j(\eta\alpha^*)_l \\ & + (\xi\alpha^*)_j(\xi^*\alpha)_l] .\end{aligned} \tag{43}$$

The mean photon numbers are obtained from formulas (42) and (43) by setting $j = l$. The dispersions of the photon numbers turn out to be

$$\begin{aligned}\sigma_{jj+} = {}& (\xi^*\xi^t)(\eta^*\eta^t) + (\xi^*\eta^t)(\eta^*\xi^t) + \mathrm{sech}^2|\alpha|^2[(\xi\alpha^*)^2(\xi^*\alpha)^2 + (\eta\alpha^*)^2(\eta^*\alpha)^2 + 2(\xi\alpha^*)(\xi^*\alpha)(\eta^*\alpha)(\eta\alpha^*)] \\ & -[(\xi^*\xi^t) + (\eta^*\eta^t)][(\xi\alpha^*)(\eta\alpha^*) + (\xi^*\alpha)(\eta^*\alpha)] - (\xi^*\eta^t)[(\xi\alpha^*)(\xi\alpha^*) + (\eta^*\alpha)(\eta^*\alpha)] \\ & -(\eta^*\xi^t)[(\xi^*\alpha)(\xi\alpha^*) + (\eta\alpha^*)(\eta\alpha^*)] + \tanh|\alpha|^2[2(\eta^*\eta^t)(\eta^*\alpha)(\eta\alpha^*) + 2(\eta^*\eta^t)(\xi^*\alpha)(\xi\alpha^*) \\ & +2(\xi^*\eta^t)(\eta^*\alpha)(\xi\alpha^*) + 2(\eta^*\xi^t)(\xi^*\alpha)(\eta\alpha^*) + (\xi^*\xi^t)(\xi^*\alpha)(\xi\alpha^*) + (\xi^*\xi^t)(\eta^*\alpha)(\eta\alpha^*) \\ & -(\xi^*\alpha)^2(\eta^*\alpha)(\xi\alpha^*) - (\eta^*\alpha)^2(\xi^*\alpha)(\eta\alpha^*) - (\eta\alpha^*)^2(\eta^*\alpha)(\xi\alpha^*) - (\xi\alpha^*)^2(\xi^*\alpha)(\eta\alpha^*)] ,\end{aligned} \tag{44}$$

$$\begin{aligned}\sigma_{jj-} = {}& (\xi^*\xi^t)(\eta^*\eta^t) + (\xi^*\eta^t)(\eta^*\xi^t) - \mathrm{cosech}^2|\alpha|^2[(\xi\alpha^*)^2(\xi^*\alpha)^2 + (\eta\alpha^*)^2(\eta^*\alpha)^2 + 2(\xi\alpha^*)(\xi^*\alpha)(\eta^*\alpha)(\eta\alpha^*)] \\ & -[(\xi^*\xi^t) + (\eta^*\eta^t)][(\xi\alpha^*)(\eta\alpha^*) + (\xi^*\alpha)(\eta^*\alpha)] - (\xi^*\eta^t)[(\xi\alpha^*)(\xi\alpha^*) + (\eta^*\alpha)(\eta^*\alpha)] \\ & -(\eta^*\xi^t)[(\xi^*\alpha)(\xi\alpha^*) + (\eta\alpha^*)(\eta\alpha^*)] + \coth|\alpha|^2[2(\eta^*\eta^t)(\eta^*\alpha)(\eta\alpha^*) + 2(\eta^*\eta^t)(\xi^*\alpha)(\xi\alpha^*) \\ & +2(\xi^*\eta^t)(\eta^*\alpha)(\xi\alpha^*) + 2(\eta^*\xi^t)(\xi^*\alpha)(\eta\alpha^*) + (\xi^*\xi^t)(\xi^*\alpha)(\xi\alpha^*) + (\xi^*\xi^t)(\eta^*\alpha)(\eta\alpha^*) \\ & -(\xi^*\alpha)^2(\eta^*\alpha)(\xi\alpha^*) - (\eta^*\alpha)^2(\xi^*\alpha)(\eta\alpha^*) - (\eta\alpha^*)^2(\eta^*\alpha)(\xi\alpha^*) - (\xi\alpha^*)^2(\xi^*\alpha)(\eta\alpha^*)] .\end{aligned} \tag{45}$$

The parentheses in these formulas mean the jth element of a vector or the jjth element of a matrix.

To illustrate the results obtained let us consider an example of a one-mode parametric oscillator described with the Hamiltonian (we use dimensionless variables)

$$H = \frac{p^2}{2} + \Omega^2(t)\frac{x^2}{2}, \qquad \Omega(0) = 1. \tag{46}$$

In this case the elements of the 2×2 transformation matrix (32) read

$$\lambda_1 = \varepsilon_r \equiv \mathrm{Re}\varepsilon, \tag{47}$$

$$\lambda_2 = -\dot{\varepsilon}_r, \tag{48}$$

$$\lambda_3 = -\varepsilon_i \equiv \mathrm{Im}\varepsilon, \tag{49}$$

$$\lambda_4 = \dot{\varepsilon}_i, \tag{50}$$

where ε is the solution of the classical oscillator equation

$$\ddot{\varepsilon}(t) + \Omega^2(t)\varepsilon(t) = 0 \tag{51}$$

satisfying the normalization condition

$$\dot{\varepsilon}\varepsilon^* - \varepsilon\dot{\varepsilon}^* = 2i. \tag{52}$$

The wave functions of the time-dependent cat states read

$$\langle x|\alpha_+, t\rangle = \langle x|0,t\rangle 2N_+ \exp\left(-\frac{|\alpha|^2}{2} - \frac{\alpha^2}{2}\frac{\varepsilon^*}{\varepsilon}\right) \times \cosh\left(\frac{\sqrt{2}\alpha x}{\varepsilon}\right), \tag{53}$$

$$\langle x|\alpha_-, t\rangle = \langle x|0,t\rangle 2N_- \exp\left(-\frac{|\alpha|^2}{2} - \frac{\alpha^2}{2}\frac{\varepsilon^*}{\varepsilon}\right) \times \sinh\left(\frac{\sqrt{2}\alpha x}{\varepsilon}\right), \tag{54}$$

where the evolution of the vacuum state is

$$\langle x|0,t\rangle = (\pi)^{-1/4}\varepsilon^{-1/2}\exp\left(\frac{i\dot{\varepsilon}x^2}{2\varepsilon}\right). \tag{55}$$

The parameters of transformation (38) expressing the evolution of the annihilation and creation operators in the one-mode case are

$$\xi = \frac{\varepsilon - i\dot{\varepsilon}}{2}, \qquad \eta = \frac{-\varepsilon - i\dot{\varepsilon}}{2}. \tag{56}$$

Expressions (40)–(43) allow us to obtain the variances of the coordinate and the momentum (12)–(14)

$$\sigma(q,q)_+ = \frac{1}{2} + |\eta|^2 - \mathrm{Re}(\xi^*\eta) + |\xi-\eta|^2|\alpha|^2 \tanh|\alpha|^2 + \mathrm{Re}[(\xi\alpha^* - \eta\alpha^*)^2], \tag{57}$$

$$\sigma(q,q)_- = \frac{1}{2} + |\eta|^2 - \mathrm{Re}(\xi^*\eta) + |\xi-\eta|^2|\alpha|^2 \coth|\alpha|^2 + \mathrm{Re}[(\xi\alpha^* - \eta\alpha^*)^2], \tag{58}$$

$$\sigma(p,p)_+ = \frac{1}{2} + |\eta|^2 + \mathrm{Re}(\xi^*\eta) + |\xi+\eta|^2|\alpha|^2 \tanh|\alpha|^2 - \mathrm{Re}[(\xi\alpha^* + \eta\alpha^*)^2], \tag{59}$$

$$\sigma(p,p)_- = \frac{1}{2} + |\eta|^2 + \mathrm{Re}(\xi^*\eta) + |\xi+\eta|^2|\alpha|^2 \coth|\alpha|^2 - \mathrm{Re}[(\xi\alpha^* + \eta\alpha^*)^2]. \tag{60}$$

As in Sec. II, one of the cases of pure squeezing is real $\alpha^2 > 0$ and real $\xi^*\eta$. Then the largest squeezing will be achieved for the variance $\sigma(p,p)_+$ in case of real ξ and η:

$$\sigma(p,p)_+ = (\xi+\eta)^2 \left\{1/2 + \alpha^2 \left[\tanh(\alpha^2) - 1\right]\right\}, \tag{61}$$

where we have taken into account the identity

$$|\xi|^2 - |\eta|^2 = 1, \tag{62}$$

which is a consequence of Eqs. (52) and (56). This variance is less when ξ and η have opposite signs. It can be made infinitely small for $\xi^2 \gg 1$, namely,

$$\sigma(p,p)_+ \approx \left\{1/2 + \alpha^2 \left[\tanh(\alpha^2) - 1\right]\right\} \frac{1}{4\xi^2}. \tag{63}$$

This expression in turn reaches its minimum with respect to α for $\alpha \approx 0.8$, so that

$$\sigma(p,p)_+^{\min} \approx 0.055/(\xi^2). \tag{64}$$

IV. PHOTON DISTRIBUTION FUNCTIONS IN SQUEEZED CAT STATES

Our next goal is to obtain the photon distribution function for the multimode *squeezed cat* state. For this purpose we first calculate the matrix elements between the squeezed Schrödinger cat states $|\alpha_\pm, t\rangle$ and a number (Fock) state $|\mathbf{n}\rangle$, where $\mathbf{n} = (n_1, n_2, ..., n_N)$ is the vector of photon numbers in all modes. For the squeezed coherent states (of which the squeezed Schrödinger cat states consist) this matrix element was found in [39] [the subscript 0 means that we consider the case of zero linear terms in Hamiltonians (25) or (29)]:

$$\langle \mathbf{m}|\alpha, t\rangle_0 = \frac{\exp\left(\frac{1}{2}\alpha\eta^*\xi^{-1}\alpha - \frac{1}{2}|\alpha|^2\right)}{\mathbf{m}!^{1/2}(\det\xi)^{1/2}} H_{\mathbf{m}}^{\{\xi^{-1}\eta\}}(\eta^{-1}\alpha), \tag{65}$$

where $H_{\mathbf{m}}^{\{\mathbf{R}\}}(\mathbf{y})$ is the N-dimensional Hermite polynomial defined via the generating function (here $\mathbf{a}$ and $\mathbf{y}$ are N-dimensional vectors and $\mathbf{R}$ is a symmetric $N \times N$ matrix)

$$\exp\left(-\frac{1}{2}\mathbf{aRa} + \mathbf{aRy}\right) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \frac{\mathbf{a}^{\mathbf{m}}}{\mathbf{m}!} H_{\mathbf{m}}^{\{\mathbf{R}\}}(\mathbf{y}), \tag{66}$$

$$\mathbf{m}! = m_1! m_2! \cdots m_N!,$$

$$\mathbf{a}^{\mathbf{n}} = a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N},$$

$$\sum_{\mathbf{m}=\mathbf{0}}^{\infty} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty}.$$

Using the representation of a cat state in terms of the coherent states (2), we obtain the photon number distribution

$$(P_{\mathbf{m}}^{\pm})_0 = |\langle \mathbf{m}|\alpha_\pm, t\rangle_0|^2 = \left[1 \pm (-1)^{\sum_i m_i}\right]^2 \times N_\pm^2 \frac{\exp\left[\mathrm{Re}(\alpha\eta^*\xi^{-1}\alpha) - |\alpha|^2\right]}{\mathbf{m}!|\det\xi|} \times \left|H_{\mathbf{m}}^{\{\xi^{-1}\eta\}}(\eta^{-1}\alpha)\right|^2. \tag{67}$$

In deriving this formula we use the property of the multimode Hermite polynomial to be either an even or an odd function if the sum of the indices of the polynomial is respectively an even or an odd number. This property may be easily proved from the definition of the Hermite polynomial through the generating function.

Formula (67) demonstrates an example of the interference in the discrete configuration space of photon numbers: it yields the factor 4 in the case of the constructive interference of two amplitudes and the factor zero in the destructive case. So in the even states we can observe only an even number of photons, while the probability of having an odd number of photons equals zero. Conversely, in the odd states the probability of having an even number of photons equals zero. When the probability $(P_{\mathbf{m}}^{\pm})_0$ is not equal to zero, we have the equality

$$(P_{\mathbf{m}}^{\pm})_0 = 4N_\pm^2 \mathcal{P}_{\mathbf{m}}, \tag{68}$$

where $\mathcal{P}_{\mathbf{m}}$ is the photon distribution function for polymode squeezed and correlated light found in Ref. [40]. Consequently, the envelope of the distribution function $(P_{\mathbf{m}}^{\pm})_0$ has the same shape as that of the photon distribution function for polymode squeezed and correlated light (if one neglects the fast oscillations connected with zeros at even or odd numbers).

In the one-mode case we use the property of the Hermite polynomials

$$H_n^{\{r\}}(y) = \left(\frac{r}{2}\right)^{n/2} H_n\left(y\sqrt{\frac{r}{2}}\right) \quad (69)$$

to rewrite (67) as

$$(P_m^{\pm})_0 = [1 \pm (-1)^m]^2 N_{\pm}^2 \frac{\exp\left[\mathrm{Re}(\alpha\eta^*\xi^{-1}\alpha) - |\alpha|^2\right]}{m!|\xi|} \times \left|\left(\frac{\eta}{2\xi}\right)^m H_m^2\left(\frac{\alpha}{\sqrt{2\eta\xi}}\right)\right|. \quad (70)$$

In the unsqueezed case (when $\eta = 0$ and $|\xi| = 1$) the PDF reduces to the quasi-Poissonian form

$$(P_m^{\pm})_0 = [1 \pm (-1)^m]^2 N_{\pm}^2 \frac{|\alpha|^{2m}}{m!} \exp\left(-|\alpha|^2\right). \quad (71)$$

Let us consider the special case of the parametric resonance when the eigenfrequency of the oscillator changes harmonically at twice the frequency (we assume $\kappa \ll 1$)

$$\Omega^2 = \frac{1 + \kappa\cos(2t)}{1 + \kappa}. \quad (72)$$

The approximate solution to Eq. (51) in this case was found in [41] (see also [25]):

$$\varepsilon = \cosh(\kappa t/4)\exp(it) - i\sinh(\kappa t/4)\exp(-it), \quad (73)$$

$$\xi = \cosh(\kappa t/4)\exp(it), \quad (74)$$

$$\eta = i\sinh(\kappa t/4)\exp(-it). \quad (75)$$

Note that this solution satisfies exactly the initial conditions and the normalization constraint (52). The PDF in this case is obtained trivially by substituting the above values into (70):

$$(P_m^{\pm})_0 = [1 \pm (-1)^m]^2 \times N_{\pm}^2 \frac{\exp\left[\tanh(\kappa t/4)\mathrm{Im}(\alpha^2) - |\alpha|^2\right]}{2^m m! \cosh(\kappa t/4)} \times [\tanh(\kappa t/4)]^m \left|H_m\left(\frac{\alpha}{\sqrt{i\sinh(\kappa t/2)}}\right)\right|^2. \quad (76)$$

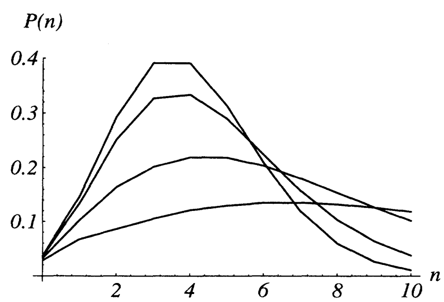

FIG. 1. Envelope of the photon distribution for the cat state $\alpha = 2$ evolving under the parametric excitation (69) without the external force, at times $\kappa t = 0, 1, 2, 3$ (from the highest to the lowest maximum).

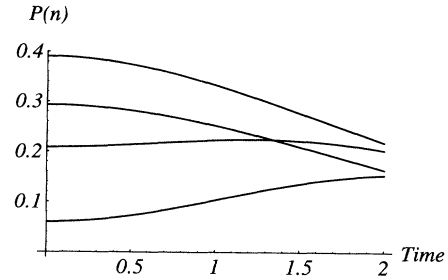

FIG. 2. Probability of finding four, two, six and eight photons (from the upper to the lower curve) in the cat state $\alpha = 2$ evolving under the parametric excitation (69) without the external force. Time is in units of κt.

This formula is simplified in the long-time limit $\kappa t \gg 1$, when the argument of the Hermite polynomial tends to zero. Then the nonzero probabilities can be expressed as

$$P_{2l}^{+} = \frac{(2l)!\exp\left[\mathrm{Im}(\alpha^2)\right]}{2^{2l}(l!)^2\cosh(\kappa t/4)\cosh(|\alpha|^2)}, \quad (77)$$

$$P_{2l+1}^{-} = \frac{(2l+1)!|\alpha|^2\exp\left[\mathrm{Im}(\alpha^2)\right]}{2^{2l-1}(l!)^2\cosh(\kappa t/4)\sinh(\kappa t/2)\sinh(|\alpha|^2)}. \quad (78)$$

These relations hold for not very large values of the integer l, when $2l \ll \exp(\kappa t/4)$. If $1 \ll 2l \ll \exp(\kappa t/2)$, then we may use the known asymptotics of the Hermite polynomials [42] and the Stirling formula for the factorials to rewrite Eq. (76) as

$$P_{2l}^{+} = \frac{2\exp\left[\mathrm{Im}(\alpha^2) - 4le^{-\kappa t/2} - \kappa t/4\right]}{\sqrt{\pi l}\cosh(|\alpha|^2)} \times \left\{\cos^2\left[2\sqrt{2l}e^{-\kappa t/4}\mathrm{Re}\left(\alpha e^{-i\pi/4}\right)\right] + \sinh^2\left[2\sqrt{2l}e^{-\kappa t/4}\mathrm{Im}\left(\alpha e^{-i\pi/4}\right)\right]\right\}, \quad (79)$$

$$P_{2l+1}^{-} = \frac{2\exp\left[\mathrm{Im}(\alpha^2) - 4le^{-\kappa t/2} - \kappa t/4\right]}{\sqrt{\pi l}\sinh(|\alpha|^2)} \times \left\{\sin^2\left[2\sqrt{2l}e^{-\kappa t/4}\mathrm{Re}\left(\alpha e^{-i\pi/4}\right)\right] + \sinh^2\left[2\sqrt{2l}e^{-\kappa t/4}\mathrm{Im}\left(\alpha e^{-i\pi/4}\right)\right]\right\}. \quad (80)$$

The envelope of the PDF (76) is depicted in Fig. 1. We see that as times increases, the distribution becomes more flat and the probabilities of higher photon numbers increase. The temporal dependence of these probabilities is shown in Fig. 2.

V. DESTRUCTION OF THE SCHRÖDINGER CAT STATES BY EXTERNAL FORCES

Let us consider now the evolution of the Schrödinger cat state distribution function due to the influence of linear quadratures terms in the Hamiltonian (29). In this case the transition amplitude from a coherent state to a number state (in the notation of Sec. III) is given by Eq. (5.58) from Ref. [39]:

$$\langle \mathbf{m}|\alpha,t\rangle = A_f \frac{\exp\left[\frac{1}{2}\alpha\eta^*\xi^{-1}\alpha + \alpha(\gamma^* - \eta^*\xi^{-1}\gamma) - \frac{1}{2}|\alpha|^2\right]}{\mathbf{m}!^{1/2}(\det\xi)^{1/2}} H_{\mathbf{m}}^{\{\xi^{-1}\eta\}}(\eta^{-1}(\alpha-\gamma)), \quad (81)$$

where A_f is given by the formula

$$A_f = \exp\left(\frac{1}{2}\gamma\eta^*\xi^{-1}\gamma - \frac{|\gamma|^2}{2}\right). \quad (82)$$

The N vector γ is the solution to the equation

$$-i\dot{\gamma} = \xi\mathbf{f}^* - \eta\mathbf{f}, \qquad \gamma(0) = 0. \quad (83)$$

The quanta distribution function is given by the formula

$$P_{\mathbf{m}}^{\pm} = N_{\pm}^2 |\langle \mathbf{m}|\alpha,t\rangle \pm \langle \mathbf{m}|-\alpha,t\rangle|^2. \quad (84)$$

In the previous case the amplitudes had simple parity properties and because of this, the expression for the distribution function had no cross terms. In the presence of linear terms in the interaction Hamiltonian, the amplitudes have no properties of either even or odd functions of α. For this reason we will have four terms after factoring (84), that are no longer different by merely a number factor. Combining all four terms we could write the expression for the photon distribution function in terms of the products of the multivariable Hermite polynomials. The main difference from the previous distribution is that now the probabilities do not turn into zero for either even or odd photon numbers. So the property of even (or odd) Schrödinger cat states not to contain states with an odd (or even) number of photons is destroyed. Note that just this property leads to strong oscillations of the photon distribution function for the Schrödinger cat states. The additional oscillations are produced by squeezing in the coherent components of Schrödinger cat states. For squeezed and correlated states, the oscillating character of the photon distribution function was studied, e.g., in [27,30] for the one-mode case and in [28,29] for the two-mode squeezed vacuum state. It is obvious that in the polymode squeezed even and odd coherent states the photon distribution function preserves the property of having strong oscillations. The linear terms in the interaction Hamiltonian reduce the oscillations related to the interference of the two squeezed components of the Schrödinger cat states, thus producing a nonzero probability of having both an even and an odd number of photons.

To analyze the process of the destruction of Schrödinger cat states by the external force we confine ourselves to the simplest case of an oscillator with a constant frequency $\Omega = 1$. Then $\eta = 0$ and $\xi = e^{it}$, so that

$$\gamma(t) = i\int_0^t e^{i\tau}\mathbf{f}^*(\tau)\,d\tau. \quad (85)$$

Instead of proceeding to the limit $\eta \to 0$ in the amplitude (81) it is more convenient to use the generating function for this amplitude [see, e.g., Eq. (5.13) of Ref. [39]]

$$\langle \beta|\alpha,t\rangle = (\det\xi)^{-1/2}\exp\left[-\frac{1}{2}\beta\xi^{-1}\eta\beta + \beta\xi^{-1}(\alpha-\gamma) - \frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2} + \frac{1}{2}\alpha\eta^*\xi^{-1}\alpha + \alpha(\gamma^* - \eta^*\xi^{-1}\gamma) + \frac{1}{2}\gamma\eta^*\xi^{-1}\gamma - \frac{|\gamma|^2}{2}\right]. \quad (86)$$

Equations (84) and (86) in the case of $\eta = 0$ lead to the following expression for the photon distribution function:

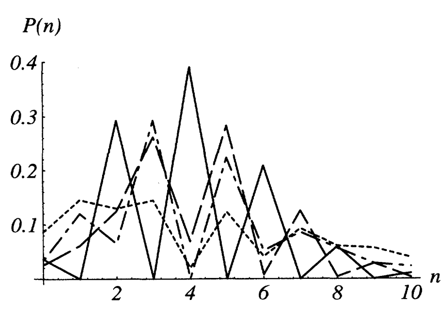

FIG. 3. Photon distribution for a cat state with $\alpha = 2$ in an oscillator with constant frequency $\Omega = 1$ driven by the permanent force $f = 0.4$ at the instants of time $t = 0$ (solid curve), $\pi/4$ (dashed curve), $\pi/2$ (dot-dashed curve), and $3\pi/4$ (dotted curve).

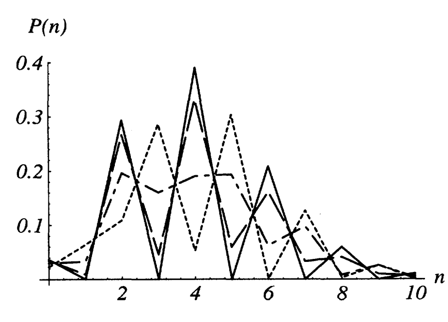

FIG. 4. Same as in Fig. 3, but for the resonant force $f = 0.5\exp(it)$ at times $t = 0$ (solid curve), 0.2 (dashed curve), 0.4 (dot-dashed curve), and 0.6 (dotted curve).

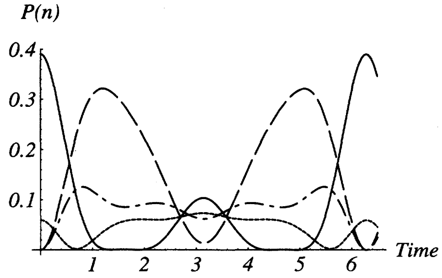

FIG. 5. Probability of finding three (dashed), four (solid), seven (dot-dashed), and eight (dotted curve) photons in the case of Fig. 3 as a function of time under the action of the permanent force $f = 0.4$.

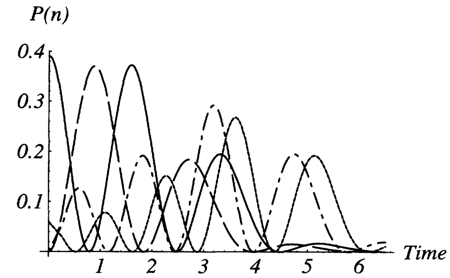

FIG. 6. Same as in Fig. 5, but for the resonant force $f = 0.5\exp(it)$.

$$P_m^{\pm} = \frac{N_{\pm}^2}{m!}\exp\left(-|\alpha|^2 - |\gamma|^2\right)\left(|\alpha-\gamma|^{2m}\exp\left[2\mathrm{Re}(\alpha\gamma^*)\right] + |\alpha+\gamma|^{2m}\exp\left[-2\mathrm{Re}\{\alpha\gamma^*)\right] \pm 2(-1)^m \mathrm{Re}\{[(\alpha-\gamma)(\alpha^*+\gamma^*)]^m \times \exp\left[2i\mathrm{Im}(\alpha\gamma^*)\right]\}\right) . \quad (87)$$

For small values of parameter γ this formula yields the following nonzero probabilities of observing the photon numbers with the "incorrect parity:"

$$P_m^{\pm}(\text{opposite parity}) = 4|\gamma|^2 N_{\pm}^2 \frac{|\alpha|^{2m-2}}{m!} e^{-|\alpha|^2}\{m^2 + |\alpha|^4 - 2m|\alpha|^2 \times \cos\left[2\left(\phi_\alpha - \phi_\gamma\right)\right]\}, \quad (88)$$

where ϕ_α and ϕ_γ are the phases of the complex numbers α and γ.

Note that function $\gamma(t)$ may be large even for small linear terms in the Hamiltonian (29); for instance, in the resonance case $f(t) = f_0 e^{it}$, this quantity increases proportionally to time $\gamma = if_0 t$ [see Eq. (85)]. Therefore it is interesting to consider the limit case of Eq. (87) when $|\gamma| \gg |\alpha|$:

$$P_m^{\pm} = 2N_{\pm}^2 \frac{|\gamma|^{2m}}{m!}\exp\left(-|\alpha|^2 - |\gamma|^2\right) \times \{\cosh\left[2\mathrm{Re}(\alpha\gamma^*)\right] \mp (-1)^m \cos\left[2\mathrm{Im}(\alpha\gamma^*)\right]\} . \quad (89)$$

We see that the linear terms in the interaction Hamiltonian tend to remove strong oscillations of the distribution function exhibited in the initial Schrödinger cat states. For instance, if $2\mathrm{Im}(\alpha\gamma^*) = (\pi/2) + k\pi$, then we have no oscillations at all; moreover, the distributions in the even and the odd states have an identical dependence on the number m, differing only by the normalization constants. Even more striking is the behavior of the distribution function when $\mathrm{Re}(\alpha\gamma^*) = 0$. Then the situation where the probability to observe an *even* (*odd*) number of photons in the initially *even* (*odd*) state is equal to zero (up to small corrections of the order of $|\alpha/\gamma|^2$) is possible. Various dependences of the distribution function (87) on the parameters m and γ are illustrated in the following figures.

In Fig. 3 we see the PDF of a cat state initially with $\alpha = 2$, evolving under the action of a permanent driving force. The originally even cat state loses its property: the probability of the even number decreases while the probability of the odd ones increases. The same feature is observed for the resonant force (see Fig. 4). However, looking at the temporal evolution of the PDF under the action of the permanent force (Fig. 5), we see that, first, the probabilities of some photon numbers are close to zero for a significant length of time. In other words, the state is converted to almost the opposite one: even to odd and vice versa. Second, we notice a remarkable property that for a permanent force in a constant-frequency oscillator, the state returns to exactly its original value after each period of oscillation. In our case it corresponds to time $T = 2\pi$. Thus the cat state is never completely destroyed. More surprisingly, similar behavior occurs for a resonant force. There is an oscillating term in the curly brackets of (89); thus under the action of a small force the state will return close to the original one after a period $T = \pi/(\alpha f_0)$, which is equal to π for the case in Fig. 6. We see, however, that after each period this revival of the cat state becomes weaker.

VI. CONCLUSION

We have shown explicitly that the photon distribution function for the multimode parametrically excited squeezed Schrödinger cat states differs essentially from the product of Poisson distributions describing the multimode coherent state. Moreover, we have demonstrated how the linear terms in the interaction Hamiltonian destroy the property of Schrödinger cat states to have only even or only odd numbers of photons. This kind of interaction is nontrivial in the sense that in the resonance case, an arbitrary, small, time-dependent external force can either completely destroy the oscillations or transform even states to almost odd ones and vice versa. For a permanent force the PDF returns exactly to its initial value after one period. For a resonant force the states returns close to the original one after a period determined by the force and the amplitude of the cat state.

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