Phase operators on Hilbert space

John A. Vaccaro
Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft an der Humboldt–Universität zu Berlin, Rudower Chaussee 5, 12484 Berlin, Germany
(Received 15 September 1994; revised manuscript received 7 December 1994)

A simple formalism is introduced for describing the quantum phase of a single-mode field (or, equivalently, a harmonic oscillator) using a particular infinite-dimensional Hilbert space H_{sym}. This space is spanned by vectors which represent states of the single field mode containing non-negative numbers of photons. It has a symmetry in the sense that it contains vectors in the neighborhood of both the vacuum state and the infinite photon number state. A striking property of H_{sym} is that it supports Hermitian and unitary phase operators. Indeed the Pegg-Barnett unitary phase operators are shown to converge *strongly* on the Hilbert space H_{sym} and so the corresponding limit points are ordinary operators on H_{sym}. This can be compared with the situation for the infinite-dimensional Hilbert space H conventionally used in quantum optics on which the Pegg-Barnett unitary phase operators converge weakly, in general.

PACS number(s): 42.50.−p, 03.65.Ca

I. INTRODUCTION

The quantum phase of a single-mode field has received increasing attention during recent years [1]. In this paper we consider only the basic problem of defining Hermitian and unitary phase operators. This problem was first investigated by Dirac over 60 years ago [2]. The problem itself can be defined in different ways depending on the level of generality. For example, previous authors have required the phase operators to operate on a vector space whose elements either represent the states of the field mode alone [2–13] or the product of field mode states with the states of some apparatus or other device [14,15] where the vector space is either an infinite-dimensional Hilbert space [2–12,14,15], or a more-general space [13,16], etc. Before we begin, therefore, we must frame the problem we are considering in precise terms. We define it as follows. Find the Hermitian phase operator that is canonically conjugate to the photon number operator and is defined on an infinite-dimensional Hilbert space which is spanned by the (non-negative) photon number states of the single mode field. In this paper we give the solution to this problem. We acknowledge, however, that by defining the quantum phase problem in different terms, it is possible to arrive at different solutions (see, e.g., [3–19]).

The work of many in this field (e.g., Refs. [4,6,10–13]) has as its focus the construction of raising and lowering operators for the photon number states. These operators are required to be unitary in order to represent exponential phase operators consistently. However, it is well known that the infinite-dimensional Hilbert space H, which is conventionally used in quantum optics, presents an impasse for this task. The difficulty is that the lowering operation destroys the vacuum component of every vector. Thus such operators cannot be unitary on H. Two important methods have been introduced recently for overcoming this problem and obtaining unitary phase operators [20]. The first is due to Newton [11] and independently Barnett and Pegg [12] (see also [21,22]), who extended the photon number spectrum to negative infinity so that the vacuum state is mapped to the -1 state when acted on by the lowering operator. We shall refer to this method as the NBP formalism in this paper. The second method was introduced by Pegg and Barnett [13] and involves a special procedure for taking the infinite-dimensional limit of the vector space. We shall refer to this latter method as the Pegg-Barnett formalism. Their limiting procedure allows the use of vectors that belong to a space that is larger than H [16]. In this case the vacuum state vector is mapped to a vector which lies outside H and which has a divergent mean photon number. In fact, the Pegg-Barnett formalism solves the most general statement of the problem of phase in which there is no restriction on the state space other than that it is spanned by photon number states of non-negative number. However, we are interested here in a solution to a more restricted problem as stated above and neither of these two approaches provides the solution.

There is, fortunately, yet another approach that has not been explored previously [23]. Consider for the moment adjoining to H the vector representing the state of "infinite photon number" [24] as the vector to which the vacuum state vector will be mapped by the lowering operation. This would allow the definition of operators of the form $\exp(\mathrm{i}\phi)$ and $\exp(-\mathrm{i}\phi)$, which do not destroy the vacuum state vector. However, the problem then is that the infinite photon number state vector is destroyed by the lowering operation. To make the raising and lowering operators unitary we need to adjoin a collection of vectors in the neighborhood of the infinite photon number vector. The resulting Hilbert space then contains a symmetry in the sense that vectors representing states in the neighborhood of both zero and infinite photon number are included and this allows it to support the unitary operators required. We label the larger space as H_{sym}.

In this paper we introduce a simple formalism based on

H_{sym} for describing the phase of the single-mode field. It gives the solution to the problem stated above. We give the details of the construction of H_{sym} in Sec. II. In Sec. III we introduce various operators on H_{sym} as the strong limits of Pegg-Barnett operator sequences. Following this is a comparison in Sec. IV between the formalism introduced here and the NBP and Pegg-Barnett formalisms. A discussion is given in Sec. V.

II. THE INFINITE-DIMENSIONAL HILBERT SPACE H_{sym}

We begin by following earlier approaches [8,13] in that we consider a sequence of finite-dimensional spaces but quickly depart from previous treatments in two important ways. The first is in the relationship between spaces of different dimensions: we specify a method of stepping from one space to a space of higher dimension that differs from the previous treatments. The second is that we construct an infinite-dimensional Hilbert space from the finite-dimensional spaces and then consider infinite-rank operators on the infinite-dimensional space: this differs from the Pegg-Barnett formalism where the infinite-dimensional limit is only taken of expectation values and not directly of the operators or the spaces.

Let Λ be an infinite set of orthonormal vectors $|n\rangle_\Lambda$ labeled by the positive integers, i.e., ${}_\Lambda\langle n|m\rangle_\Lambda = \delta_{n,m}$ for $n, m = 1, 2, \ldots$. We wish to stress that the vector $|n\rangle_\Lambda$ does not necessarily represent the state of n photons. The set Λ is any countable infinite orthonormal set labeled in any order. The identification of any vector with a particular photon number state depends on the definition of the number operator which we give later. For example, defining $\hat{N} \equiv 0|\gamma\rangle_{\Lambda\Lambda}\langle\gamma| + 1|\epsilon\rangle_{\Lambda\Lambda}\langle\epsilon| + 2|\chi\rangle_{\Lambda\Lambda}\langle\chi| + \cdots$, where $\gamma, \epsilon, \chi, \ldots$ are unique positive integers, implies that the vectors $|\gamma\rangle_\Lambda, |\epsilon\rangle_\Lambda, |\chi\rangle_\Lambda, \ldots$ represent the 0,1,2,... photon number states, respectively.

We construct finite-dimensional spaces from Λ as follows. First we introduce a new set of symbols which are related to the old by

$$|n\rangle_s \equiv |\xi(n,s)\rangle_\Lambda. \tag{1}$$

Here $\xi(n,s)$ is a positive integer function of n and s, where $n = 0, 1, 2, \ldots, s$ and $s = 0, 1, 2, \ldots$, which we use to label the elements of Λ. We will give specific examples of $\xi(n,s)$ later. For the moment we note that different choices of $\xi(n,s)$ lead to different infinite-dimensional Hilbert spaces. Now let

$$\{|n\rangle_s\}_{n=0,1,2,\ldots,s} \tag{2}$$

be a set of $(s+1)$ linearly independent vectors. This constrains the function $\xi(n,s)$ to the extent that for each given value of $s \geq 0$ it takes on $(s+1)$ *different* values for $n = 0, 1, 2, \ldots, s$. The set in (2) spans a $(s+1)$-dimensional Hilbert space Ψ_s, which is equivalent to that used by Pegg and Barnett. Accordingly we can define on Ψ_s all the Pegg-Barnett operators, including the phase operators. For example, consider the photon number operator $\hat{N}_s$ [13], which is given here by

$$\hat{N}_s = \sum_{n=0}^{s} n|n\rangle_{ss}\langle n| \,. \tag{3}$$

Clearly $\hat{N}_s|n\rangle_s = n|n\rangle_s$ and so in the Pegg-Barnett formalism $|n\rangle_s$ is the eigenvector of $\hat{N}_s$ with eigenvalue n.

So far we have not specified the relationship between Ψ_s and $\Psi_{s'}$ for $s' > s$. If we were to strictly follow the relationship used by Popov and Yarunin [8] and Pegg and Barnett [13] we would define

$$\xi(n,s) = \zeta(n), \tag{4}$$

where $\zeta(n)$ is a positive integer function of n only [e.g., the function $\xi(n,s) = n+1$ satisfies this]. We would then find that the completion of the infinite union $\Theta \equiv \bigcup_{s=0}^{\infty} \Psi_s$, for which the inner product is given by

$$\langle g, s|f, s'\rangle = \sum_{n=0}^{k} g_n^* f_n,$$

where

$$|f, s\rangle = \sum_{n=0}^{s} f_n |n\rangle_s \in \Theta, \tag{5}$$

$$|g, s\rangle = \sum_{n=0}^{s} g_n |n\rangle_s \in \Theta, \tag{6}$$

and k is the smaller of s and s', yields the Hilbert space H'. The general vector $|h\rangle = \sum_{n=0}^{\infty} h_n |\zeta(n)\rangle_\Lambda$ belonging to H' has the property that $\sum_{n=0}^{\infty} |h_n|^2 < \infty$. This restriction [25] implies, in particular, that H' does not contain a vector corresponding to the vector $|s\rangle_s$ in the Pegg-Barnett formalism in the infinite-s limit and so H' cannot support unitary phase operators corresponding to those in the Pegg-Barnett formalism. Indeed, defining operators on H' as the (weak) limit points of the Pegg-Barnett phase operators yields the Susskind-Glogower and the Popov-Yarunin operators [4,8], which attribute the vacuum state with nonrandom phase properties. The reason this situation arises here is that an asymmetry is produced by adding new vectors as s increases only at the end of the sequence of vectors $|0\rangle_s, |1\rangle_s, |2\rangle_s, \ldots, |s\rangle_s$ as given by Eq. (4). This corresponds essentially to adding new vectors at the upper end of the photon number spectrum, which restricts the resulting Hilbert space so that it contains only elements whose number state coefficients h_n vanish as $n \to \infty$. Alternatively, in order to produce a symmetry between both ends of the photon number spectrum, we add new states, in the following, to the *middle* of the sequence of vectors. To simplify the analysis we consider only odd s from now on; the extension to all values of s is straightforward.

We define the relationship between $\xi(n,s)$ and $\xi(m,s+2)$ for odd s by

$$\xi(n,s+2)=\begin{cases}\xi(n,s) & \text{for } 0\le n\le \frac{1}{2}(s-1)\\ \xi(n-2,s) & \text{for } \frac{1}{2}(s+5)\le n\le s+2\end{cases} \tag{7}$$

and where the new values $\xi(n,s+2)$ for $n=\frac{1}{2}(s+1),\frac{1}{2}(s+3)$ are different from the values of $\xi(n,s)$ for $n=0,1,2,\ldots,s$. This relationship is depicted graphically in Fig. 1. There are many functions $\xi(n,s)$ that satisfy this; a specific example is given by

$$\xi(n,s)=\begin{cases}2n+1 & \text{for } \; 0\le n\le \frac{1}{2}(s-1)\\ 2(s-n)+2 & \text{for } \; \frac{1}{2}(s+1)\le n\le s.\end{cases}$$

From the relationship between $\xi(n,s)$ for different s given by Eq. (7) we find that the inner product of two vectors in the infinite union $\Theta\equiv\bigcup_{s=0}^{\infty}\Psi_s$ is now given by

$$\langle g,s|f,s'\rangle=\sum_{n=0}^{(k-1)/2} g_n^* f_n+\sum_{n=(k+1)/2}^{k} g_{s-k+n}^* f_{s'-k+n},$$

where, as before, k is the smaller of s and s' and $|g,s\rangle$ and $|f,s\rangle$ are defined in Eqs. (5) and (6). The completion of Θ with the new function $\xi(n,s)$ defined by Eq. (7) is the infinite-dimensional Hilbert space H_{sym} discussed above. We highlight some of the properties of H_{sym} in the remainder of this section.

It is helpful to introduce a new notation so that we can eliminate reference to the parameter s. Consider the vector given by Eq. (5) with $\xi(n,s)$ now given by Eq. (7) and set

$$\bar{f}_n=\begin{cases}f_n & \text{for } \; n\ge 0\\ f_{s+1+n} & \text{for } \; n<0\end{cases}$$

so that

$$|f,s\rangle=\sum_{n=0}^{(s-1)/2}\bar{f}_n|n\rangle_s+\sum_{n=-1}^{-(s+1)/2}\bar{f}_n|s+1+n\rangle_s\,. \tag{8}$$

We note that in the Pegg-Barnett formalism the vector $|s\rangle_s$ is an eigenstate of $\hat{N}_s$ with eigenvalue s. Since we are ultimately interested only in the infinite-s limit, we see that $|s\rangle_s$ will eventually correspond to a state of *infinite photon number*. Also we find from Eqs. (1) and (7) that $|s-n\rangle_s$ and $|n\rangle_s$ are independent of s for $s>2n\ge 0$ and so we define

$$|n\rangle\equiv|n\rangle_s, \tag{9}$$

$$|\widetilde{\infty}-n\rangle\equiv|s-n\rangle_s \tag{10}$$

for $0\le n\le(s-1)/2$. We use the symbol $\widetilde{\infty}$ in expression (10) as a reminder of the divergent nature of the photon number eigenvalues; the properties of the vector $|\widetilde{\infty}-n\rangle$ can be inferred from that of the right-hand side of Eq. (10). Figure 2 illustrates the use of the new notation. From Eqs. (8)–(10) we see that the general vector $|h\rangle$ belonging to H_{sym} can now be written in the form

$$|h\rangle=\sum_{n=0}^{\infty}h_n|n\rangle+\sum_{n=-1}^{-\infty}h_n|\widetilde{\infty}+1+n\rangle, \tag{11}$$

where h_n are complex numbers for $n=0,\pm1,\pm2,\ldots$. From the orthogonality of the vectors in Eqs. (9) and (10) we find that the square of the norm of $|h\rangle$ is

$$\langle h|h\rangle=\sum_{n=-\infty}^{\infty}|h_n|^2<\infty \tag{12}$$

and the innerproduct between two vectors $|f\rangle,|g\rangle\in H_{sym}$ of the form given by Eq. (11) is

$$\langle f|g\rangle=\sum_{n=-\infty}^{\infty}f_n^* g_n\,.$$

The subspace of H_{sym} containing vectors in the form of Eq. (11) but whose coefficients h_n are zero for $n<0$ is clearly the Hilbert space H that is conventionally used in quantum optics. Thus, by simply choosing the relationship *between* each of the spaces Ψ_s according to Eq. (7) we are able to construct the Hilbert space H_{sym} which contains H as a subspace. In particular, H_{sym} also contains the vector $|\widetilde{\infty}\rangle$, which does not belong to H. The importance of this vector is that it corresponds to $|s\rangle_s$ in the Pegg-Barnett formalism in the infinite-s limit. Put more precisely, the sequence $|0\rangle_0,|1\rangle_1,|2\rangle_2,\ldots,|s\rangle_s,\ldots$ *converges* to $|\widetilde{\infty}\rangle$ in H_{sym} because, in fact, $|s\rangle_s=|\widetilde{\infty}\rangle$ for all s. The presence of the vectors $|\widetilde{\infty}-n\rangle$ for $n=0,1,2,\ldots$ allows H_{sym} to support the unitary phase operators treated in the next section.

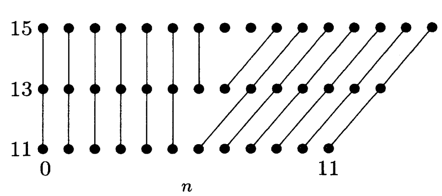

FIG. 1. Relationship between $\xi(n,s),\xi(n,s+2)$, and $\xi(n,s+4)$ for $s=11$. The solid disks represent values of ξ and the straight lines connect equal values. In particular, $\xi(n,s+2)$ has $s+3$ unique values for $n=0,1,2,\ldots,s+2$ of which $s+1$ values are inherited from $\xi(n,s)$.

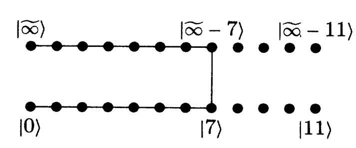

FIG. 2. Diagram representing a portion of the number state basis of H_{sym} using the new notation of Eqs. (9) and (10). Each disk represents a vector in the orthonormal set $\{|n\rangle,|\widetilde{\infty}-n\rangle\}_{n=0,1,2,\ldots}$. The line connects the vectors spanning Ψ_s for $s=15$. The whole basis of H_{sym} can be represented by extending the diagram indefinitely to the right. The bottom row of disks in the extended diagram represents the basis of the subspace H.

III. OPERATORS ON H_{sym}

We now introduce infinite-rank operators on H_{sym}. In particular we wish to define the photon number and phase operators as a canonically conjugate pair. We adopt the Pegg-Barnett approach [13] for defining these operators via sequences. But whereas in the Pegg-Barnett formalism it is only the limit points of expectation values that are considered, here we find the limit points on H_{sym} of the operator sequences themselves.

The Pegg-Barnett number operator $\hat{N}_s$ given by Eq. (3) takes the form

$$\hat{N}_s = \sum_{n=0}^{(s-1)/2} n|n\rangle\langle n| + \sum_{n=-1}^{-(s+1)/2} (s+1+n)|\widetilde{\infty}+1+n\rangle\langle\widetilde{\infty}+1+n|$$

in the new notation. It is not difficult to show that the sequence $\hat{N}_1, \hat{N}_3, \hat{N}_5, \ldots$ has the strong limit

$$\hat{N} \equiv \sum_{n=0}^{\infty} n|n\rangle\langle n| \tag{13}$$

on the domain $D(\hat{N})$, which contains vectors of the form Eq. (11), but whose coefficients satisfy

$$\sum_{n=0}^{\infty} |nh_n|^2 < \infty$$

and $h_n = 0$ for $n < 0$. Thus we identify the vector $|n\rangle$ with the state of n photons. However, one may ask whether we are justified in identifying $|\widetilde{\infty}\rangle$ as a state of infinite photon number when this vector lies outside the domain of $\hat{N}$. To see that the answer is "yes" consider the sequence of inverse number operators $(1+\hat{N}_s)^{-1}$ for $s = 1, 3, 5, \ldots$, which is easily found to converge strongly on the whole space H_{sym} to

$$\sum_{n=0}^{\infty}(1+n)^{-1}|n\rangle\langle n| + 0\left(\sum_{n=0}^{-\infty} |\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n|\right) .$$

The vectors $|\widetilde{\infty}-n\rangle$ for $n = 0, 1, 2, \ldots$ are degenerate eigenvectors of this operator with eigenvalue zero. In this respect all of these vectors do, in fact, correspond to states of infinite photon number. It is therefore quite natural that such vectors lie outside the domain of $\hat{N}$. We note also from Eq. (10) that these vectors are orthonormal: $\langle\widetilde{\infty}-n|\widetilde{\infty}-m\rangle = \delta_{n,m}$ for $n, m = 0, 1, 2, \ldots$, which gives further support for our notation. Care should be taken, however, not to confuse the label $\widetilde{\infty}-n$ as representing "infinity minus n." Rather the vectors $|\widetilde{\infty}-n\rangle$ are a collection of orthonormal vectors representing states of infinite photon number.

The Pegg-Barnett unitary phase operators [13] are given here by

$$\begin{aligned}\exp(i\hat{\phi}_{\theta,s}) &= \sum_{n=0}^{s-1} |n\rangle_{ss}\langle n+1| + |s\rangle_{ss}\langle 0| \\ &= \sum_{n=0}^{(s-1)/2} |n\rangle\langle n+1| \\ &\quad + \sum_{n=-1}^{-(s-1)/2} |\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n+1| + |\widetilde{\infty}\rangle\langle 0| \\ &= [\exp(-i\hat{\phi}_{\theta,s})]^\dagger\end{aligned}$$

for $s = 1, 3, 5, \ldots$. We have set the "phase window" [13] to $[0, 2\pi)$ and used Eqs. (9) and (10) in the second line. The same expression results from other choices of the phase window provided an appropriate sequence of values of s is chosen accordingly [26]. The sequence of operators $\exp(i\hat{\phi}_{\theta,s})$ and $\exp(-i\hat{\phi}_{\theta,s})$ are found to converge strongly on H_{sym} to $\exp(i\hat{\phi})$ and $\exp(-i\hat{\phi})$, respectively, where

$$\begin{aligned}\exp(i\hat{\phi}) &= \sum_{n=0}^{\infty} |n\rangle\langle n+1| \\ &\quad + \sum_{n=-1}^{-\infty} |\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n+1| + |\widetilde{\infty}\rangle\langle 0| \qquad (14) \\ &= [\exp(-i\hat{\phi})]^\dagger . \qquad (15)\end{aligned}$$

From the orthogonality of the vectors $|n\rangle$ and $|\widetilde{\infty}-n\rangle$ for $n = 0, 1, 2, \ldots$ we find that

$$\exp(i\hat{\phi})\exp(-i\hat{\phi}) = \exp(-i\hat{\phi})\exp(i\hat{\phi}) = \hat{1},$$

where

$$\hat{1} \equiv \sum_{n=0}^{\infty} |n\rangle\langle n| + \sum_{n=0}^{-\infty} |\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n|$$

is the unit operator for H_{sym}. Thus the *limit points are unitary on* H_{sym}. This contrasts markedly with the limits of the same sequences on H where the limit points are the Susskind-Glogower operators which are not unitary [27]. The unitarity of $\exp(i\hat{\phi})$ guarantees that there is a corresponding Hermitian phase angle operator on H_{sym}. We introduce this operator via the phase eigenstate basis.

The eigenstates of $\exp(i\hat{\phi})$ do not belong to H_{sym}, but can be accommodated in a rigged Hilbert space H^R_{sym} [28–30]. We describe H^R_{sym} briefly here. Let Ξ be the space of all linear combinations of the vectors $|n\rangle$ and $|\widetilde{\infty}-n\rangle$, for $n = 0, 1, 2, \ldots$, and let Ω_{sym} be the nuclear space consisting of all vectors $|h\rangle$ [given by Eq. (11)] in H_{sym} which satisfy

$$\sum_{n=-\infty}^{\infty} |h_n|^2|n|^m < \infty$$

for $m = 0, 1, 2, \ldots$. The space Ω^x_{sym}, which is conjugate to Ω_{sym}, contains all those vectors $|f\rangle \in \Xi$ satisfying $\langle f|\omega\rangle < \infty$ for all $|\omega\rangle \in \Omega_{sym}$ and for which $\langle f|\omega\rangle$ is a continuous linear functional of $|\omega\rangle$ on Ω_{sym}. The triplet $\Omega_{sym} \subset H_{sym} \subset \Omega^x_{sym}$ is our rigged Hilbert space H^R_{sym}. It is not difficult to show that the vector

$$|\theta\rangle = (2\pi)^{-\frac{1}{2}}\left[\sum_{n=0}^{\infty}\exp(in\theta)|n\rangle + \sum_{n=-1}^{-\infty}\exp(in\theta)|\widetilde{\infty}+1+n\rangle\right],$$

where θ is real, belongs to Ω^{x}_{sym} and also that it has the property

$$\exp(i\hat{\phi})|\theta\rangle = \exp(i\theta)|\theta\rangle .$$

Thus $|\theta\rangle$, which we call a phase state, is clearly an eigenvector of $\exp(i\hat{\phi})$. The phase states $|\theta\rangle$ for θ in a 2π interval form a complete orthonormal set, e.g.,

$$\hat{1} = \int_{2\pi}|\theta\rangle\langle\theta|d\theta,$$

$$\langle\theta'|\theta\rangle = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\theta-\theta')] \equiv \delta(\theta-\theta')$$

where $\delta(\theta-\theta')$ is the 2π-periodic Dirac delta distribution. The completeness property allows an alternate, equivalent definition of the unitary phase operator:

$$\exp(i\hat{\phi}) = \int_{2\pi}\exp(i\theta)|\theta\rangle\langle\theta|d\theta .$$

It follows from the orthonormal property of the phase states that $\exp(i\hat{\phi})$ is the exponential of the phase angle operator given by

$$\hat{\phi}_\theta = \int_{\theta_0}^{\theta_0+2\pi}\theta|\theta\rangle\langle\theta|d\theta$$

for arbitrary θ_0. The value of θ_0 defines a 2π phase window in the usual way [13].

Strictly speaking, the commutator of $\hat{N}$ with $\hat{\phi}_\theta$ does not exist because the range of $\hat{\phi}_\theta$ lies outside the domain $D(\hat{N})$ of $\hat{N}$ and so the product $\hat{N}\hat{\phi}_\theta$ is not an operator on H_{sym}. However, it is still possible to evaluate the expectation value $\langle f|\hat{N}\hat{\phi}_\theta|f\rangle$ for $|f\rangle \in D(\hat{N})$ by operating $\hat{N}$ first on $\langle f|$. Thus we find

$$\langle f|[\hat{N},\hat{\phi}_\theta]|f\rangle = i(1-2\pi|\langle f|\theta_0\rangle|^2)$$

for all $|f\rangle \in D(\hat{N})$. It can then be shown by a trivial extension of the usual proof [31] that all $|g\rangle \in D(\hat{N}^2)$ satisfy the Heisenberg uncertainty relation

$$\langle\Delta\hat{N}^2\rangle\langle\Delta\hat{\phi}_\theta{}^2\rangle \geq \frac{1}{4}(1-2\pi|\langle g|\theta_0\rangle|^2)^2 .$$

These last two expressions agree with the corresponding expressions found in the Pegg-Barnett formalism.

It remains for us to show that $\hat{\phi}_\theta$ is canonically conjugate to the photon number operator $\hat{N}$. We note that $|\theta\rangle$ contains the broadest spread of number states possible. Moreover, $\hat{N}$ is the generator of phase shifts as given by

$$\langle\theta|\exp(i\hat{N}\delta)|h\rangle = \langle\theta-\delta|h\rangle$$

for all $|h\rangle \in H \cap \Omega_{sym}$, where δ is real, and $\hat{\phi}_\theta$ is the generator of number shifts as can be seen from the matrix elements of $\exp(i\hat{\phi})$ in Eq. (14). These results show that $\hat{\phi}_\theta$ is canonically conjugate to $\hat{N}$ in the sense defined by Pegg *et al.* [13,32]. We also note that

$$\langle h|\exp(i\hat{N})\exp(i\hat{\phi}_\theta)|h\rangle = \exp(-i)\langle h|\exp(i\hat{\phi}_\theta)\exp(i\hat{N})|h\rangle ,$$

where $|h\rangle \in H$, and so $\hat{\phi}_\theta$ is also canonically conjugate to $\hat{N}$ in the Weyl sense [33].

Hence, to summarize, we have constructed a Hilbert space H_{sym} which is spanned by non-negative number states and found on it the Hermitian phase operator canonically conjugate to the photon number operator. Thus *we have solved the phase problem defined in the Introduction.*

We now pursue the nature of the operators on H_{sym} a little further and, in particular, examine the polar decomposition of the annihilation operator sought by Dirac [2]. The annihilation and creation operators are given in the Pegg-Barnett formalism [13] by

$$\begin{aligned}\hat{a}_s &= \sum_{n=0}^{s-1}\sqrt{n+1}|n\rangle_{ss}\langle n+1| \\ &= \sum_{n=0}^{(s-1)/2}\sqrt{n+1}|n\rangle\langle n+1| \\ &\quad + \sum_{n=-1}^{-(s-1)/2}\sqrt{s+n+1}|\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n+1| \\ &= (\hat{a}_s^\dagger)^\dagger .\end{aligned}$$

The strong limits of the sequences of $\hat{a}_s$ and $\hat{a}_s^\dagger$ are found to be $\hat{a}$ and $\hat{a}^\dagger$, respectively, where

$$\hat{a} = \sum_{n=0}^{\infty}\sqrt{n+1}|n\rangle\langle n+1| = (\hat{a}^\dagger)^\dagger$$

on the domain given by vectors $|h\rangle$ of the form Eq. (11) whose coefficients satisfy

$$\sum_{n=0}^{\infty}|h_n|^2 n < \infty$$

and $h_n = 0$ for $n < 0$. Clearly $\hat{a}^\dagger\hat{a} = \hat{N}$ and

$$[\hat{a},\hat{a}^\dagger] = \sum_{n=0}^{\infty}|n\rangle\langle n| . \tag{16}$$

Indeed, $\hat{N}$, $\hat{a}$, and $\hat{a}^\dagger$ are, in fact, *identical to the corresponding operators in the conventional approach which uses H as the state space.* The right-hand side of Eq. (16) is the unit operator for H. We note that H contains the domain of $\hat{a}$ and $\hat{a}^\dagger$ and so the above commutator cannot operate on a space larger than H. However, since all physical states belong to H, the commutator is also physically indistinguishable from the unit operator $\hat{1}$ on H_{sym}.

The operator $\exp(i\hat{\phi})$ generates a state of infinite photon number from the vacuum state $\exp(i\hat{\phi})|0\rangle = |\widetilde{\infty}\rangle$. Can such an operation be produced through a physical interaction? The answer has been pointed out previously by Barnett and Pegg [12,13]. All physical interactions between the field mode and other systems involve interaction terms which are functions of the annihilation and creation operators. Since the annihilation operator destroys the vacuum state we see that it is not possible for this operation to occur in physical situations. It follows that a field mode that is initially in a state with a finite mean photon number and that interacts with a system that has a finite energy will always have a finite mean photon number. Thus the state of a field mode in physical situations is always represented by a vector that belongs to H.

Finally, we note that the annihilation operator $\hat{a}$, which operates on a dense subset of H, can be decomposed into unitary phase and photon number operator parts

$$\hat{a} = \exp(i\hat{\phi})\sqrt{\hat{N}}\ . \tag{17}$$

This is *the polar decomposition of the very operator* $\hat{a}$ *sought by Dirac* [35]. An interesting feature of this decomposition is that $\hat{a}$ is a product of two operators, one of which ($\exp i\hat{\phi}$) operates on a larger space than H while the other ($\sqrt{\hat{N}}$) operates on a dense subset of H. We now see the reason why the previous attempts to find the decomposition of $\hat{a}$ using only H were unsuccessful: the unitary phase operator could not be constructed because it operates on a larger space.

IV. COMPARISON WITH RELATED FORMALISMS

Unitary phase operators are possible on H_{sym} because the vacuum state can be mapped to the state $|\widetilde{\infty}\rangle$, which is orthogonal to all the states $|n\rangle$ for $n = 0, 1, 2, \ldots$. In the NBP formalism [11,12] the vacuum component is also mapped to a state that is orthogonal to the same set of photon number states. This leads us to examine the relationship between the two formalisms.

Newton [11] introduced a fictitious spin-$\frac{1}{2}$ property to the field mode to enable its mathematical description to include a photon number operator $\hat{L}$ that has both positive *and negative* integer eigenvalues. Barnett and Pegg also considered extending the photon number spectrum to negative infinity, but without the spin property [12]. We now construct an operator on H_{sym} with a spectrum equivalent to that of $\hat{L}$ for comparison. Consider the sequence of operators

$$\begin{aligned}\hat{\mathcal{L}}_s &\equiv \sum_{n=0}^{(s-1)/2} n|n\rangle_{ss}\langle n| + \sum_{n=(s+1)/2}^{s} (n-s-1)|n\rangle_{ss}\langle n| \\ &= \hat{N}_s - (s+1)\sum_{n=-1}^{-(s+1)/2} |\widetilde{\infty}+n+1\rangle\langle\widetilde{\infty}+n+1|\end{aligned}$$

for $s = 1, 3, 5, \ldots$. Thus $\hat{\mathcal{L}}_s$ is diagonal in the photon number basis, but with different eigenvalues to $\hat{N}_s$. The sequence converges strongly to

$$\hat{\mathcal{L}} \equiv \sum_{n=0}^{\infty} n|n\rangle\langle n| + \sum_{n=-1}^{-\infty} n|\widetilde{\infty}+n+1\rangle\langle\widetilde{\infty}+n+1| \tag{18}$$

on a dense subspace of H_{sym}. The eigenvalue equation for $\hat{\mathcal{L}}$ is

$$\begin{aligned}\hat{\mathcal{L}}|n\rangle &= n|n\rangle, \\ \hat{\mathcal{L}}|\widetilde{\infty}-n\rangle &= -(n+1)|\widetilde{\infty}-n\rangle\end{aligned}$$

for $n = 0, 1, 2, \ldots$. If we relabel the vectors by their $\hat{\mathcal{L}}$ eigenvalues

$$\begin{aligned}|n\rangle_{\mathcal{L}} &\equiv |n\rangle, \\ |-(n+1)\rangle_{\mathcal{L}} &\equiv |\widetilde{\infty}-n\rangle\end{aligned}$$

for $n = 0, 1, 2, \ldots$, we then have

$$\hat{\mathcal{L}} = \sum_{n=-\infty}^{\infty} n|n\rangle_{\mathcal{L}\mathcal{L}}\langle n|\ .$$

This shows that $\hat{\mathcal{L}}$ and H_{sym} are mathematically equivalent to $\hat{L}$ and the corresponding space used by Newton and also Barnett and Pegg. Does this mean that here we are simply reinterpreting the negative photon number states in the NBP formalism as states of infinite photon number? The answer is quite clearly "no" for the following reasons. Imagine that we map the Hilbert space used in the NBP formalism onto H_{sym} by identifying a state of n photons in the NBP formalism with $|n\rangle$ for $n \geq 0$ and $|\widetilde{\infty}+n+1\rangle$ for $n < 0$. The NBP photon number operator would then be given by $\hat{\mathcal{L}}$ in Eq. (18) which operates on states of infinite photon number whereas our operator $\hat{N}$, given by Eq. (13), is the conventional photon number operator which operates on a dense subset of H only. The time evolution of a state in the NBP formalism [36] would be given by a Hamiltonian that is proportional to $\hat{\mathcal{L}}$ in contrast to that here, where the Hamiltonian is proportional to $\hat{N}$. The NBP annihilation operator would be given by

$$\sum_{n=0}^{\infty}\sqrt{n+1}|n\rangle\langle n+1| + \sum_{n=-1}^{-\infty}\sqrt{|n|}|\widetilde{\infty}+n\rangle\langle\widetilde{\infty}+n+1|, \tag{19}$$

which acts on states of infinite photon number whereas our operator $\hat{a}$ is the conventional annihilation operator acting on a dense subset of H. Clearly the two formalisms differ by more than a simple reinterpretation of the photon number states. Indeed, it is only when the corresponding interpretation of negative or infinite photon numbers is used in each formalism that the respective photon number and annihilation operators have the forms expected of them. For example, vectors that are interpreted as negative photon number states must belong to the domain of the photon number operator, whereas vectors that are interpreted as infinite photon number states cannot belong to the domain of this operator. Thus the interpretations of negative and infinite photon number states are necessarily associated with the NBP and the new formalisms, respectively.

Even so, one may ask how the concept of a state of a negative number of photons differs from that of a state of an infinite number of photons when neither can be occupied physically. We note that both negative and infinite photon number states are orthogonal to states in H and so the distance [based on the norm in Eq. (12)] between any state in H and a negative or infinite photon number state is $\sqrt{2}$. Thus neither negative nor infinite photon number states are approached closely by states in H. Despite this we can describe a procedure, which is based on the limiting procedure of the Pegg-Barnett formalism, that provides a physical interpretation of the states of infinite photon number. Imagine that a field is prepared in the photon number state $|n\rangle$, where n is some given positive integer, and that the physical properties of the field are calculated accordingly. Imagine further that this is repeated for an infinite sequence of increasing values of n. We note that this procedure involves a sequence of states that does not converge strongly to any vector in H; in fact, it can be shown [16] that the sequence is represented by a vector in a space that is *larger* (i.e., more general) than H. The results of some of the calculations of the physical properties will converge as $n \to \infty$, e.g., $\langle(\hat{N}+1)^{-1}\rangle \to 0$ and $\langle\Delta\hat{\phi}_\theta{}^2\rangle = \pi^2/3$ independent of n. The *limit points* of these results represent the properties of a *state of infinite photon number*. This procedure therefore provides a concrete interpretation of states of infinite photon number. In contrast, there is no analogous interpretation of negative photon number states.

Moreover, unlike states of negative photon number, states that have a divergent mean photon number already appear in the conventional approach, which uses the Hilbert space H to represent the state of the field mode. For example, the state $|B\rangle = \sum(n+1)^{-1}|n\rangle$ which belongs to H has a divergent mean photon number. Thus, in this respect, our use of the state $|\widetilde{\infty}\rangle$ is not exceptional [34].

This brings us to an important point regarding the interpretation of quantum phase. We note that if the initial state of the field mode is represented by a vector in H, then in both the NBP and the present formalisms it remains in H for all times under physical interactions. Despite the differences mentioned above, the calculations in both formalisms will yield the same physical results because the corresponding operators are identical for vectors in H. The approach taken in this paper, however, gives a new interpretation of these physical results in terms of Hilbert-space operators which are defined on non-negative photon number states only. Thus it gives *a more conventional interpretation of the representation of quantum phase on an infinite-dimensional Hilbert space.*

We have used the operator sequences from the Pegg-Barnett formalism [13] as a basis for building phase operators on the space H_{sym}. This gives a close relationship between the formalism presented here and the Pegg-Barnett formalism. In particular, the calculations in both give the same physical results. The feature that distinguishes the two is that we have taken *the limit of operator sequences on an infinite-dimensional space*, whereas the corresponding limit in the Pegg-Barnett formalism is taken only of *expectation values calculated on finite-dimensional spaces.* Let us examine this point more closely. Our use of an infinite-dimensional Hilbert space places a restriction, which is absent from the Pegg-Barnett formalism, on the vectors we can treat, as shown by Eq. (12). For example, the vector $|(s+1)/2\rangle_s$, which is easily handled in the Pegg-Barnett formalism in the infinite-s limit, does not have a corresponding representation in H_{sym}. This indicates that the Pegg-Barnett formalism is a more *general* formalism in that it allows more-general vectors [16]. On the other hand, by relaxing the generality a little we are able to retain the physically important features of the Pegg-Barnett formalism and omit fine mathematical features of no physical value in the problem at hand.

More importantly, the existence of the strong limits of the Pegg-Barnett phase operators on H_{sym} means that infinite-rank operators on a Hilbert space can be used directly to represent observables of the field. Since the $s \to \infty$ limit is taken of the operators and vectors before calculating expectation values, the usual mathematical tools associated with infinite-dimensional Hilbert space can be used for the analysis of a system involving a field mode. For example, the equation of motion can be expressed in terms of the standard quasiprobability distributions when the field is in a state represented in H. There is no need for truncating the number state expansion at a finite value of s. This gives the present formalism an important advantage—*it allows the use of conventional mathematical tools.*

We conclude this section with the observation that while the formalism presented here differs from both the NBP and Pegg-Barnett formalisms, nevertheless it can also be viewed as a hybrid containing various features of these related formalisms: an extended infinite-dimensional Hilbert space like that used in the former and the non-negative photon number spectrum and the recipe for canonical conjugate observables of the latter.

V. DISCUSSION AND CONCLUSION

We have introduced here a simple formalism for the quantum phase based on an infinite-dimensional Hilbert space H_{sym} and the strong limits of the Pegg-Barnett operator sequences. An interesting property of H_{sym} is that it is symmetrical in the sense that it contains vectors in the neighborhood of both the vacuum state and the infinite photon number state. This allows it to support Hermitian and unitary phase operators which are canonically conjugate to the photon number operator. Moreover, the space H_{sym} is spanned by non-negative photon number states only. Thus the Hermitian phase operator is the solution to the phase problem defined in the Introduction. The formalism also yields the polar decomposition of the annihilation operator as sought originally by Dirac. We noted how the unitary phase operator in the decomposition operates on a larger space than the annihilation operator itself. This is the reason why the search for unitary phase operators on the infinite-dimensional Hilbert space H conventionally used in quantum optics has been unsuccessful: a larger space is necessary.

In fact, essentially the only different piece here that is not in the conventional treatment of the single mode field is the particular method for dealing with states of infinite photon number. However, states with a divergent mean photon number are not new to quantum optics. We have already noted that the state $\sum(n+1)^{-1}|n\rangle$ belongs to the conventionally used Hilbert space H and has a divergent mean photon number. Also the eigenstates of the field quadrature operators have a divergent mean photon number. Thus our use of *extra* states with a divergent mean photon number is not exceptional [34].

Let us elaborate this point a little further since it is crucial for a complete understanding of the present formalism. Any given physical realization of a single-mode field in a cavity entails basic physical constraints that arise from considerations such as the physical structure of the cavity mirrors, etc., and which are important in the limit of extremely intense fields. These constraints limit the number of photons physically possible in a field mode because, e.g., exceeding the limit would damage the mirrors, etc. For the case of a single mode of the field propagating in free space (where periodic boundary conditions replace the mirrors) one may argue that there are physical constraints associated with, for example, the physical nature of the (remote) source of the energy of the field [13]. However, it is clear that these constraints, whatever their origin, are not included in the conventional theoretical model of the ideal single field mode because, from the commutator $[\hat{a},\hat{a}^\dagger]=1$, we find that the spectrum of the number operator $\hat{N}=\hat{a}^\dagger\hat{a}$ is unbounded above and the sequence of photon number states is infinite. This leaves us with two options: either (a) ignore the physical constraints and retain the conventional commutator $[\hat{a},\hat{a}^\dagger]=1$ or (b) include the physical constraints and adopt a modified commutator $[\hat{a},\hat{a}^\dagger]\neq 1$.

In option (b) the modified commutator gives a modified photon number operator. The phase operator is found as the operator which is canonically conjugate to the modified photon number operator. The Pegg-Barnett formalism, without the infinite-dimensional limit, provides operators of this type [13].

Alternatively, for option (a), which is the one implicitly chosen here, we must use an infinite-dimensional space that supports the commutator $[\hat{a},\hat{a}^\dagger]=1$, that is, a space that contains the domain of $\hat{a}^\dagger\hat{a}$. Many such spaces can be found with H and H_{sym} being the particular examples considered here. Which of these spaces is the more appropriate physically for the model of the field mode? The obvious difference between H and H_{sym} is that H_{sym} contains extra states, such as $|\widetilde{\infty}\rangle$, of infinite photon number. However, we note that the commutator itself implies that the spectrum of $\hat{N}$ is necessarily unbounded and also that both H and H_{sym} contain states, such as $\sum(1+n)^{-1}|n\rangle$, that have a divergent mean photon number. Thus the unboundedness of the photon number does not provide any grounds for regarding either H or H_{sym} as being the more appropriate space. On the other hand, there is an important physical distinction between these spaces on the basis of phase since H_{sym} supports phase operators that attribute the vacuum state to a random phase, whereas H does not support such operators. Indeed all evidence suggests that the space H supports only operators that attribute the vacuum with a nonrandom phase. Thus H_{sym} is clearly the more appropriate Hilbert space for the model of a single-mode field.

There is one last point we wish to make. The *lower bound* of the photon number spectrum has been held responsible for the difficulties previously encountered when attempting to define a well-behaved phase observable on H. Our analysis here (see also [27]) shows, however, that it is the *large photon number behavior* of H that forbids the definition of appropriate phase operators. The lower bound of the photon number spectrum is simply a physical feature that distinguishes the energy of the harmonic oscillator from, say, the angular momentum of a bead on a circular wire. In short, the inability to define appropriate phase operators on H results from the mathematical structure of H and *not* the physical nature of the energy spectrum or the field mode itself. The present formalism overcomes these difficulties by adopting the larger Hilbert space H_{sym} as the state space.

In conclusion, we have presented a simple formalism for describing the quantum phase that is more conventional than the NBP approach and less general than the Pegg-Barnett formalism. It supports the limit points of the Pegg-Barnett operators on an infinite-dimensional Hilbert space and thus allows the use of a Hermitian phase operator as well as the conventional mathematical tools for the analysis of the single-mode field.

ACKNOWLEDGMENTS

I wish to thank Professor H. Paul, Dr. U. Leonhardt, Dr. A. Orlowski, and Dr. A. Wünsche for helpful comments. I am also grateful to Professor D.T. Pegg and Dr. S.M. Barnett for previous discussions concerning the general nature of quantum phase and to Dr. R.F. Bonner for his help with various mathematical constructions associated with this subject. This work was supported by the Max-Planck-Gesellschaft.

[1] See, e.g., papers in Phys. Scr. **T48** (1993), special issue on quantum phase and phase dependent measurements, edited by W.P. Schleich and S.M. Barnett.
[2] P.A.M. Dirac, Proc. R. Soc. London Ser. A **114**, 243 (1927); see also F. London, Z. Phys. **37**, 915 (1926); **40**, 193 (1927).
[3] W.H. Louisell, Phys. Lett. **7**, 60 (1963).
[4] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[5] P. Carruthers and M.M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[6] E.C. Lerner, Nuovo Cimento B **56**, 183 (1968); **57**, 251 (1969).
[7] J.C. Garrison and J. Wong, J. Math. Phys. **11**, 2242 (1970).
[8] V.N. Popov and V.S. Yarunin, Leningrad Univ. J.: Phys. **22**, 7 (1973); J. Mod. Opt. **39**, 1525 (1992).
[9] H. Paul, Fortschr. Phys. **22**, 657 (1974).
[10] J.-M. Lévy Leblond, Ann. Phys. (N.Y.) **101**, 319 (1976).
[11] R.G. Newton, Ann. Phys. (N.Y.) **124**, 327 (1980).
[12] S.M. Barnett and D.T. Pegg, J. Phys. A **19**, 3849 (1986).
[13] D.T. Pegg and S.M. Barnett, Europhys. Lett. **6**, 483 (1988); Phys. Rev. A **39**, 1665 (1989); S.M. Barnett and D.T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[14] J.H. Shapiro and S.R. Shepard, Phys. Rev. **43**, 3795 (1991).
[15] M. Ban, J. Opt. Soc. Am. B **7**, 1189 (1992); Opt. Commun. **94**, 231 (1992).
[16] J.A. Vaccaro and R.F. Bonner, Phys. Lett. **198A**, 167 (1995).
[17] C.W. Helstrom, Int. J. Theor. Phys. **11**, 357 (1974).
[18] U. Leonhardt, J.A. Vaccaro, B. Böhmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).
[19] J.W. Noh, A. Fougères, and L. Mandel, Phys. Scr. **T48**, 29 (1993).
[20] We note that there are also methods that use a tensor product of the state space of the field mode with that of an apparatus or other device. However, we are restricting our attention here to descriptions of phase operators on the state space of the field mode alone.
[21] S. Stenholm, Phys. Scr. **T48**, 77 (1993).
[22] A. Luckš and V. Peřinová, Phys. Scr. **T48**, 94 (1993).
[23] The basic idea underlying this approach was suggested by S.M. Barnett (private communication).
[24] One can compare this vector with the point at the "north pole" of a sphere which maps to the point at infinity in the stereographic projection of the sphere onto the tangent plane at the "south pole."
[25] We note that the Pegg-Barnett formalism [13] itself does not suffer from this restriction since no infinite-dimensional Hilbert space is constructed nor is necessary in their formalism.
[26] For example, we can set the phase window to be $[\theta_0, \theta_0 + 2\pi)$ with $\theta_0 = 2\pi p/q$, where p and q are fixed positive integers, and then let $(s+1)$ be a multiple of $2q$. This allows θ_0 to be 2π times an *arbitrary* rational number. The expression for $\exp(i\hat{\phi})$ and the results which follow remain unchanged.
[27] J.A. Vaccaro and D.T. Pegg, Phys. Scr. **T48**, 22 (1993).
[28] I. M. Gel'fand and G.E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. 4.
[29] A. Böhm, *The Rigged Hilbert Space and Quantum Mechanics* (Springer-Verlag, Berlin, 1978).
[30] A concise introduction to the rigged Hilbert space is given in L.E. Ballentine, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1990).
[31] See, e.g., W.H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[32] D.T. Pegg, J.A. Vaccaro, and S.M. Barnett, J. Mod. Opt. **37**, 1703 (1990).
[33] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950), Chap. IV, Sec. 14.
[34] We should also mention that the state $|\widetilde{\infty}\rangle$ is, nevertheless, quite different from states in H with a divergent mean photon number such as $|B\rangle = \sum (n+1)^{-1}|n\rangle$. For example, in the Copenhagen interpretation each measurement of the photon number of the state $|B\rangle$ yields a finite number of photons, say n, with probability proportional to $(n+1)^{-2}$; the mean photon number of a finite sample of such measurements diverges as the sample size increases. In contrast, each corresponding measurement of $|\widetilde{\infty}\rangle$ is associated with an infinity of photons.
[35] We note that in Refs. [11,12] the polar decomposition is found for the operator equivalent to that given in Eq. (19), which operates on a dense subset of H_{sym}. This operator is very different from the annihilation operator $\hat{a}$, which operates on a dense subset of H.
[36] Newton also considered the case where the Hamiltonian has degenerate eigenvalues.