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Quantum theory of nonlinear fiber optics: Phase-space representations

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In this paper the equations for quantum optical pulse propagation in nonlinear and dispersive single-mode fibers are presented in terms of two phase-space formulations based on the positive-P and the Wigner distributions. Included are the effects due to the coupling of the electromagnetic modes to the vibrational states of a vitreous silica fiber. By making use of the well-known equivalence of Fokker-Planck and Ito stochastic equations, we demonstrate two alternative methods for formulating the equations of motion as coupled stochastic c-number equations for the propagating field. The first method involves a representation of the density operator in terms of the positive-P distribution function. This leads to *exact* stochastic equations of motion. The second method makes use of the Wigner distribution function. This method, which requires truncation of third-order derivative terms in the corresponding Fokker-Planck equation, is necessarily approximate. However, we discuss certain advantages to the Wigner approach that have made it the preferable method for exploratory work.

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I. INTRODUCTION

It is the purpose of this paper to consider the development of phase-space methods as an alternative to previous Heisenberg treatments of quantum pulse propagation in single-mode optical fibers [1–4]. It is our intention to present two practical methods of analysis for the quantum problem. At the very least, these offer straightforward means of numerical analysis when operator methods become intractable due to the nonlinearities encountered in the theory of strong fields in fibers. It is suggested that analytic methods based on the classical theory of inverse scattering may also be applied to the resulting equations. As yet, however, no such work has been carried out in that direction, though we see no reason why it should not.

A good deal of work on the problem of quantum pulse propagation in fibers has appeared over the last few years [1–6]. For a review of the subject the reader is referred to the article by Drummond *et al.,* in Nature [7]. However, except for the work originating from Carter *et al.* [5] all analyses have begun with the Heisenberg equations of motion. Some useful methods for solving these operator equations, which are always variants of the nonlinear Schrödinger (NLS) equation, have been developed. In particular, by using Bethe's ansatz [8], it is possible to obtain exactly the eigenstates of the simplest nonlinear Hamiltonian leading to the NLS equation. The states of this basis are the true quantum solitons of the NLS equation—they represent a bound cluster of photons of a definite number; they are number states. The NLS quantum soliton will maintain its identity while propagating in an ideal nonlinear and dispersive single-mode fiber and also while interacting with other quantum solitons it encounters. However, it is not a trivial matter to construct arbitrary states of the field from such a basis.

For this reason most methods for solving the Heisenberg NLS equation with arbitrary initial conditions usually rely on being able to linearize the equations of motion in the limit of large photon number. Currently, with the investigation of soliton pulses [9–11] typically containing $\sim 10^8$ photons, this approximation is quite a reasonable one. A more general formulation of the problem which does not rely on linearization would be useful. Phase-space treatments offer this possibility—in particular, through the use of the positive-P representation of the density operator. Moveover, they allow one to dispense with noncommuting operators and deal directly with standard c-number equations. The resulting equations appear similar to the classical nonlinear Schrödinger equation. The main difference is that they are actually multiplicative *stochastic* differential equations. Nevertheless, it is expected that the sophisticated tools of the inverse scattering transform [12] could be successfully applied to these equations. At the very least, direct numerical simulations of these nonlinear stochastic equations are not difficult to carry out [13].

We have also found that these phase-space methods allow the straightforward inclusion of excess thermal noise sources which are critical to any realistic appraisal of the nature of quantum effects in pulse propagation. An important class of such excess noise is the one due to the possible vibrational modes of a glass fiber. The high frequency oscillators in this class have a molecular origin—relatively localized groups of silicon and oxygen atoms in silica provide modes of vibrational motion which can couple to the electromagnetic field and cause Raman scattering. Low frequency oscillators are formed from the collective motion of the fiber atoms on the scale of the fiber diameter and generate what is termed guided acoustic wave Brillouin scattering, or GAWBS [14].

These vibrational modes can be included in one of two ways. They can either be regarded as a type of generalized reservoir which extracts energy from the electromagnetic field, as in [4], or we may treat them as oscillators which are coupled to the electromagnetic field and which are themselves coupled to a reservoir source [15]. We take the latter approach in this paper. We choose to explicitly include *two* types of reservoirs. One set of reservoirs models the small scattering losses of the electromagnetic field that are typical of silica fibers. The other set of reservoirs found here are those that couple to the primary phonon modes of the glass. The primary modes, in turn, are those that couple to the propagating electromagnetic field. The use of these phonon reservoirs allows us to associate with each primary phonon mode a finite linewidth. The existence of this linewidth is useful in the mathematical development of the theory of the vibrational response of the fiber and constitutes the main reason why it is retained in this paper. Thus the theory found here represents the rigorous derivations behind the results presented in Refs. [4,13,15]. New results are also presented. In particular, the correlation functions for the thermal noise sources of an optical fiber have been worked out in the *time* domain for both the positive-P and Wigner formulations. Also, a new result is presented for the Wigner treatment. This is an infinite vacuum noise term which was previously overlooked by Drummond and Hardman [13]. Clearly, neglect of this correction term may lead to serious implications for anyone attempting to use the Wigner theory.

Phase-space methods

Quantum mechanics need not always be phrased in the language of operators. The phase-space picture of quantum mechanics is an alternative to the commonplace method of representing quantum variables with noncommuting operators. Interestingly, the phase-space method in quantum theory is almost as old as the better known representations of Heisenberg (matrix mechanics) and Schrödinger (wave mechanics), being introduced by Wigner [16] in 1932 in connection with quantum corrections to classical thermodynamic formulas.

The basic idea is that the quantum density operator $\hat{\rho}$ can be replaced by a distribution function, P say, which contains the same (i.e., complete) information about the state of the system as does $\hat{\rho}$. Noncommuting operators and an equation for $\hat{\rho}$ (the master equation) are replaced by conventional c numbers (x and p, say) and a distribution function P with which moments of the c-number variables can be computed, just as in the *classical* theory of statistical mechanics. However, the peculiarities of nature described by the quantum theory do not allow a representation in terms of distribution functions as regular as those which may be used to describe the classical mechanics of systems with a large number of degrees of freedom. Trajectories in classical mechanics are replaced by diffuse curves in the phase space of quantum mechanics. The quantum distribution functions can rarely be interpreted directly as *positive definite* probability distributions, although, as we will see in the case of the positive-P representation [17], this can be acheived at the expense of *doubling* the number of phase-space dimensions which would be required to describe an equivalent system of classical variables.

One advantage which obtains from the representation method, and the one we shall be making use of, is that the phase-space picture allows us to *reintroduce* the idea of trajectories—albeit trajectories quite unlike their classical counterpart. The quantum trajectories must obey the Heisenberg uncertainty principle. This requirement can be achieved by introducing the idea of a stochastic equation as the fundamental equation of motion for the system variables. This idea, which relies on the well-known equivalence between Fokker-Planck equations and Ito's stochastic differential approach, is outlined in Appendix B.

II. THE INTERACTION HAMILTONIAN

The Hamiltonian for the system of electromagnetic and vibrational modes of a silica fiber has been outlined in detail elsewhere [4,15]. For completeness we shall briefly describe the various terms of which it is composed. For the sake of simplicity, the Hamiltonian is assumed to describe a polarization-preserving fiber. This allows us to dispense with the vector nature of the electromagnetic field. For the complete tensorial Hamiltonian treatment, see Ref. [4]. Our simplified Hamiltonian is meant to describe a nonlinear and dispersive single-mode fiber which supports vibrational states (i.e., phonons) at the molecular and macroscopic level. The coupling of phonons to photons of the electromagnetic field will be shown to produce an extra source of nonlinearity beyond the usual electronic one. Also included is a coupling of both the photons and the phonons to modes outside those of primary interest. These "reservoirs" introduce an element of damping into the problem. Our Hamiltonian is defined by $\hat{H} = \sum_{j=1}^{8} \hat{H}_j$, where

$$\begin{aligned}
\hat{H}_1 &= \hbar\omega_0 \sum_l \hat{\alpha}_l^\dagger \hat{\alpha}_l + \hbar \sum_{ll'} \omega_{ll'} \hat{\alpha}_l^\dagger \hat{\alpha}_{l'}, \\
\hat{H}_2 &= -\hbar\chi_\alpha \sum_l \hat{\alpha}_l^{\dagger\,2} \hat{\alpha}_l^2, \\
\hat{H}_3 &= \hbar \sum_{l\nu} \omega_\nu^\beta \hat{\beta}_{l\nu}^\dagger \hat{\beta}_{l\nu}, \\
\hat{H}_4 &= \hbar \sum_l \hat{\alpha}_l^\dagger \hat{\alpha}_l \sum_\nu g_\nu^\beta [\hat{\beta}_{l\nu}^\dagger + \hat{\beta}_{l\nu}], \\
\hat{H}_5 &= \hbar \sum_{l\mu} (\omega_0 + \omega_\mu) \hat{\alpha}_{l\mu}^{r\dagger} \hat{\alpha}_{l\mu}^r, \\
\hat{H}_6 &= \hbar \sum_{l\nu\mu} (\omega_\nu^\beta + \omega_{\nu\mu}) \hat{\beta}_{l\nu\mu}^{r\dagger} \hat{\beta}_{l\nu\mu}^r, \\
\hat{H}_7 &= \hbar \sum_{l\mu} g_\mu [\hat{\alpha}_l^\dagger \hat{\alpha}_{l\mu}^r + \hat{\alpha}_l \hat{\alpha}_{l\mu}^{r\dagger}], \\
\hat{H}_8 &= \hbar \sum_{l\nu\mu} g_{\nu\mu} [\hat{\beta}_{l\nu}^\dagger \hat{\beta}_{l\nu\mu}^r + \hat{\beta}_{l\nu} \hat{\beta}_{l\nu\mu}^{r\dagger}]. \qquad (1)
\end{aligned}$$

All operators have bosonic commutation relations of the form $[\hat{a}_m, \hat{a}_{m'}^\dagger] = \delta_{mm'}$, where m and m' denote different modes associated with the field described by the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. Note that the Hamiltonian has been written in terms of *local* operators which describe excitations at local spatial positions in the fiber, each position being labeled by a "cell index" l. For convenience the fiber has been partitioned into $2N+1$ cells, each of length $\Delta z = L/(2N+1)$. This procedure has been covered elsewhere, as in [4,6,15]. The first Hamiltonian term $\hat{H}_1$ describes the linear contribution to the energy of the electromagnetic field in a Kerr medium, and results from having transformed the free-field energy $\hbar \sum \omega_k \hat{a}_k^\dagger \hat{a}_k$, in terms of momentum-space operators, into one involving only the local operators $\hat{\alpha}_l^\dagger$ and $\hat{\alpha}_l$.

This configuration-space operator pair is defined explicitly by the following finite-Fourier expression of the equivalent k-space operators:

$$\hat{\alpha}_l = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \hat{a}_{k(n)}\, e^{in\Delta k\, l\Delta z}, \tag{2a}$$

$$\hat{\alpha}_l^\dagger = \frac{1}{\sqrt{2N+1}} \sum_{l=-N}^{N} \hat{a}_{k(n)}^\dagger\, e^{-in\Delta k\, l\Delta z}, \tag{2b}$$

where $\hat{a}_{k(n)}$ is the annihilation operator of the mode with wave number $k(n) = k_0 + n\Delta k$, and $\Delta k = 2\pi/L$. Note that we have also set the number of k-space modes equal to $2N+1$ (and later we will allow $N \to \infty$). We have chosen to use a carrier wave number k_0 in this transformation, where k_0 corresponds to the central wave number of interest, and have expanded the frequency ω_k as

$$\omega_k = \omega_0 + \Delta\omega + n\Delta k\omega' + \frac{(n\Delta k)^2}{2}\,\omega''. \tag{3}$$

The extra factor of $\Delta\omega$ has been included to allow for the expected frequency shift observed for fields propagating in Kerr media. We will choose the magnitude of this renormalization factor at a later stage when it can be used to simplify the solutions to the equations.

The second Hamiltonian term $\hat{H}_2$ is easily recognizable as the Kerr nonlinearity of the fiber, or the intensity-dependent refractive index term.

The third and fourth Hamiltonian terms $\hat{H}_3$ and $\hat{H}_4$, respectively, describe the free energy of the molecular (and also macroscopic) vibrational states of the glass and their coupling to the electromagnetic field. This coupling may also be thought of as changes induced in the refractive index of the fiber by the excitation of phonons. Thus $\hat{\beta}_{l\nu}$ is the operator which annilihates a phonon of frequency ω_ν^β, within the lth cell. The justification for the form of $\hat{H}_4$ is covered in [4].

The remaining terms, $\hat{H}_5$–$\hat{H}_8$, represent the non-Hamiltonian terms, i.e., they describe the irreversible aspects of the evolution of an electromagnetic wave propagating in a fiber. These are characterized by reservoir operators (superscripted with an r) and their associated coupling constants. $\hat{H}_5$ is the free energy of oscillation of photon reservoirs responsible for scattering losses in the fiber, while $\hat{H}_6$ represents the free energy of oscillation of the secondary phonon modes which act as reservoirs to the primary phonon modes. Note that ω_μ is the angular frequency of the photon reservoir operator $\hat{\alpha}_{l\mu}^r$ relative to the carrier frequency ω_0, and that $\omega_{\nu\mu}$ is the angular frequency of the phonon reservoir operator $\hat{\beta}_{l\nu\mu}^r$ relative to ω_ν^β, the angular frequency of the oscillation for the phonons in mode ν.

The last two Hamiltonian terms, $\hat{H}_7$ and $\hat{H}_8$, represent the physical couplings of the photons and phonons to their individual reservoir sources. Note also that the rotating wave approximation has been assumed so that counterrotating terms are ignored here. In summary the variables are the following.

$\hat{\alpha}_l \equiv$ photon, location l.

$\hat{\alpha}_{l\mu}^r \equiv$ photon scattering reservoir, location l, mode μ.

$\hat{\beta}_{l\nu} \equiv$ phonon, location l, mode ν.

$\hat{\beta}_{l\nu\mu}^r \equiv$ phonon reservoir, location l, mode μ, coupled to mode ν of phonons.

For a more comprehensive discussion of these eight Hamiltonian terms, see [4].

We intend to make use of an interaction picture, defining our interaction Hamiltonian $\hat{H}_I$ by the relation $\hat{H} = \hat{H}_0 + \hat{H}_I$, where the free Hamiltonian $\hat{H}_0$ is taken to be composed of the free-field terms for the photon field and the photon scattering reservoirs. The free-field part of the Hamiltonian is thus taken to be

$$\frac{1}{\hbar}\hat{H}_0 = (\omega_0 - \Delta\omega)\sum_l \hat{\alpha}_l^\dagger \hat{\alpha}_l + (\omega_0 - \Delta\omega)\sum_{l\mu} \hat{\alpha}_{l\mu}^{r\dagger}\hat{\alpha}_{l\mu}^r. \tag{4}$$

By employing the same frequency shift for the reservoir operators, both $\hat{\alpha}_l$ and $\hat{\alpha}_{l\nu}^r$ rotate with the same frequency so that terms of the form $\hat{\alpha}_l \hat{\alpha}_{l\nu}^{r\dagger}$ do not exhibit any rotation in the interaction picture. As a matter of convenience, we absorb the frequency shift term $\Delta\omega$ for the photon operators in the interaction Hamiltonian into the definition for $\omega_{ll'}$. Thus, in this paper the term is defined as

$$\omega_{ll'} = \sum_{n=-N}^{N} \frac{\{\omega'(n\Delta k) + \omega''(n\Delta k)^2/2\}}{2N+1}\, e^{in\Delta k(l-l')\Delta z} + \Delta\omega\, \delta_{ll'}. \tag{5}$$

III. THE POSITIVE-P METHOD

Here we extract the equations of motion defined upon the positive-P phase space. The most important result of this method, which was developed by Drummond and Gardiner [17], is that positive-definite diffusion coefficients can be constructed for Fokker-Planck processes which would ordinarily be negative for cases involving nonclassical photon statistics, as, for instance, would be the case if the Glauber-Sudarshan P representation was employed. In the theory of quantum pulse propagation the positive-P representation has been our standard choice for formulating the problem [5,6,15,18,19]. The following sections give the details of our analysis of a single pulse propagating under the influence of the Hamiltonian described by (1).

A. The Fokker-Planck equation in the positive-P representation

The basis for the positive-P representation is the assumption that the density operator can be expanded as a sum, or integral, of nondiagonal coherent-state projection operators $\hat{\Lambda}(\boldsymbol{\alpha})$ of the form

$$\hat{\Lambda}(\boldsymbol{\alpha}) = \prod_{l=-N}^{N} \frac{|\alpha_l\rangle\langle(\alpha_l^+)^*|}{\langle(\alpha_l^+)^*|\alpha_l\rangle}. \tag{6}$$

Here $\boldsymbol{\alpha} = (\alpha_{-N}, \alpha_{-N}^+, \alpha_{-N+1}, \ldots, \alpha_N^+)$ is the vector describing the $4(2N+1)$-dimensional phase space associated with $2N+1$ field modes, each of which has associated with it two complex-number field variables of the form α_l and α_l^+. The density operator is then written as

$$\hat{\rho} = \int P(\boldsymbol{\alpha})\, \hat{\Lambda}(\boldsymbol{\alpha})\, d\mu(\boldsymbol{\alpha}), \tag{7}$$

with $P(\boldsymbol{\alpha})$ the (possibly) complex distribution function that one wishes to solve for. The integration measure is simply an area measure over the complex phase space,

$$d\mu(\boldsymbol{\alpha}) = \prod_{l=-N}^{N} d^2\alpha_l\, d^2\alpha_l^+. \tag{8}$$

Noting that the projection operator is a product of terms that can be written as

$$\frac{|\alpha_l\rangle\langle(\alpha_l^+)^*|}{\langle(\alpha_l^+)^*|\alpha_l\rangle} = e^{\alpha_l \hat{a}_l^\dagger - \alpha_l \alpha_l^+} |0\rangle\langle 0| \, e^{\alpha_l^+ \hat{a}_l}, \tag{9}$$

it is easy to demonstrate that in operator products involving the mode operators $\hat{a}_l^\dagger$, $\hat{a}_l$, and $\hat{\Lambda}(\boldsymbol{\alpha})$, the mode operators can always be dispensed with by using the substitutions

$$\hat{a}_l \hat{\Lambda}(\boldsymbol{\alpha}) = \alpha_l \hat{\Lambda}(\boldsymbol{\alpha}), \tag{10a}$$

$$\hat{a}_l^\dagger \hat{\Lambda}(\boldsymbol{\alpha}) = \left(\alpha_l^+ + \frac{\partial}{\partial \alpha_l}\right)\hat{\Lambda}(\boldsymbol{\alpha}), \tag{10b}$$

$$\hat{\Lambda}(\boldsymbol{\alpha})\hat{a}_l^\dagger = \alpha_l^+ \hat{\Lambda}(\boldsymbol{\alpha}), \tag{10c}$$

$$\hat{\Lambda}(\boldsymbol{\alpha})\hat{a}_l = \left(\alpha_l + \frac{\partial}{\partial \alpha_l^+}\right)\hat{\Lambda}(\boldsymbol{\alpha}). \tag{10d}$$

Thus, when solving the equation of motion for $\hat{\rho}$, which depends on $[\hat{H}_I, \hat{\rho}]$, we can use the above operator correspondences to rewrite $[\hat{H}_I, \hat{\Lambda}]$ in a form which contains only the projection operator $\hat{\Lambda}$. In this way the master equation $i\hbar\, \partial\hat{\rho}/\partial t = [\hat{H}_I, \hat{\rho}]$ can often be transformed into an equation of the form

$$\int \hat{\Lambda}(\boldsymbol{\alpha}) \frac{\partial P(\boldsymbol{\alpha})}{\partial t}\, d\mu(\boldsymbol{\alpha}) = \int d\mu(\boldsymbol{\alpha})\, P(\boldsymbol{\alpha}) \left[\sum_{l,\mu} A_l^\mu(\boldsymbol{\alpha}) \frac{\partial}{\partial \alpha_l^\mu} + \frac{1}{2} \sum_{l,l',\mu,\nu} D_{l,l'}^{\mu\nu}(\boldsymbol{\alpha}) \frac{\partial^2}{\partial \alpha_l^\mu \partial \alpha_{l'}^\nu} \right] \hat{\Lambda}(\boldsymbol{\alpha}), \tag{11}$$

or, after integrating by parts with the assumption that the boundary terms vanish, the equivalent Fokker-Planck equation

$$\frac{\partial P(\boldsymbol{\alpha})}{\partial t} = \left[-\sum_{l,\mu} \frac{\partial}{\partial \alpha_l^\mu} A_l^\mu(\boldsymbol{\alpha}) + \frac{1}{2} \sum_{l,l',\mu,\nu} \frac{\partial^2}{\partial \alpha_l^\mu \partial \alpha_{l'}^\nu} D_{l,l'}^{\mu\nu}(\boldsymbol{\alpha}) \right] P(\boldsymbol{\alpha}). \tag{12}$$

The indices μ and ν are used to designate the two types of field variable associated with each mode, i.e., α_l and α_l^+. From this point the Ito stochastic equations can be written down and then converted to the Stratonovich form to which the normal rules of calculus may be applied. Applying this procedure to the full Raman Hamiltonian, where the phase-space vector $\boldsymbol{\alpha}$ is now extended to $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}^r$, $\boldsymbol{\beta}^r$, to include the phase-space dimensions of the phonons and the reservoirs, we arrive at

$$\begin{aligned}
\int d\mu\, \hat{\Lambda}\, \frac{\partial P}{\partial t} &= \frac{1}{i\hbar} \int d\mu\, P\, [\hat{H}_I, \hat{\Lambda}] \\
&= \int d\mu\, \hat{\Lambda} \Bigg\{ \sum_l -\frac{\partial}{\partial \alpha_l}\Big[-i \sum_{l'} \omega_{ll'} \alpha_{l'} + 2i\chi_\alpha \alpha_l^2 \alpha_l^+ - i\alpha_l \sum_\nu g_\nu^\beta (\beta_{l\nu}^+ + \beta_{l\nu}) - i \sum_\mu g_\mu \alpha_{l\mu}^r \Big] \\
&\quad + \sum_l -\frac{\partial}{\partial \alpha_l^+}\Big[i \sum_{l'} \omega_{l'l} \alpha_{l'}^+ - 2i\chi_\alpha {\alpha_l^+}^2 \alpha_l + i\alpha_l^+ \sum_\nu g_\nu^\beta (\beta_{l\nu}^+ + \beta_{l\nu}) + i \sum_\mu g_\mu \alpha_{l\mu}^{r+} \Big] \\
&\quad + \sum_{l\nu} -\frac{\partial}{\partial \beta_{l\nu}}\Big[-i\omega_\nu^\beta \beta_{l\nu} - i g_\nu^\beta \alpha_l^+ \alpha_l - i \sum_\mu g_{\nu\mu} \beta_{l\nu\mu}^r \Big] + \sum_{l\nu} -\frac{\partial}{\partial \beta_{l\nu}^+}\Big[+ i\omega_\nu^\beta \beta_{l\nu}^+ + i g_\nu^\beta \alpha_l^+ \alpha_l + i \sum_\mu g_{\nu\mu} \beta_{l\nu\mu}^{r+} \Big] \\
&\quad + \sum_{l\mu} -\frac{\partial}{\partial \alpha_{l\mu}^r}\Big[-i(\omega_\mu + \Delta\omega)\, \alpha_{l\mu}^r - i g_\mu \alpha_l \Big] + \sum_{l\mu} -\frac{\partial}{\partial \alpha_{l\mu}^{r+}}\Big[+ i(\omega_\mu + \Delta\omega)\, \alpha_{l\mu}^{r+} + i g_\mu \alpha_l^+ \Big]
\end{aligned}$$

$$+\sum_{l\nu\mu} -\frac{\partial}{\partial \beta^r_{l\nu\mu}}\Big[-i(\omega^\beta_\nu+\omega_{\nu\mu})\,\beta^r_{l\nu\mu} - ig_{\nu\mu}\beta_{l\nu}\Big] + \sum_{l\nu\mu} -\frac{\partial}{\partial \beta^{r+}_{l\nu\mu}}\Big[+i(\omega^\beta_\nu+\omega_{\nu\mu})\,\beta^{r+}_{l\nu\mu} + ig_{\nu\mu}\beta^+_{l\nu}\Big]$$
$$+\frac{1}{2}\sum_l \frac{\partial^2}{\partial {\alpha_l}^2}\Big[+2i\chi_\alpha {\alpha_l}^2\Big] + \frac{1}{2}\sum_{l\nu}\frac{\partial^2}{\partial\alpha_l\partial\beta_{l\nu}}\Big[-2ig^\beta_\nu\alpha_l\Big]$$
$$+\frac{1}{2}\sum_l \frac{\partial^2}{\partial {\alpha^+_l}^2}\Big[-2i\chi_\alpha {\alpha^+_l}^2\Big] + \frac{1}{2}\sum_{l\nu}\frac{\partial^2}{\partial\alpha^+_l\partial\beta^+_{l\nu}}\Big[+2ig^\beta_\nu\alpha^+_l\Big]\Bigg\} P, \tag{13}$$

from which the Fokker-Planck equation for $\partial P/\partial t$ can easily be made out.

B. The equivalent Ito stochastic equations

Using the rules for converting from a Fokker-Planck equation to equivalent Ito stochastic equations, which are outlined in Appendix B, we arrive at the following set of equations for the photon and phonon variables:

$$\frac{\partial\alpha_l}{\partial t} = -i\sum_{l'}\omega_{ll'}\alpha_{l'} + 2i\chi_\alpha{\alpha_l}^2\alpha^+_l - i\alpha_l\sum_\nu g^\beta_\nu(\beta^+_{l\nu}+\beta_{l\nu}) - i\sum_\mu g_\mu\alpha^r_{l\mu} + \eta_{\alpha_l}, \tag{14a}$$

$$\frac{\partial\alpha^+_l}{\partial t} = +i\sum_{l'}\omega_{l'l}\alpha^+_{l'} - 2i\chi_\alpha{\alpha^+_l}^2\alpha_l + i\alpha^+_l\sum_\nu g^\beta_\nu(\beta^+_{l\nu}+\beta_{l\nu}) + i\sum_\mu g_\mu\alpha^{r+}_{l\mu} + \eta_{\alpha^+_l}, \tag{14b}$$

$$\frac{\partial\beta_{l\nu}}{\partial t} = -i\omega^\beta_\nu\beta_{l\nu} - ig^\beta_\nu\alpha^+_l\alpha_l - i\sum_\mu g_{\nu\mu}\beta^r_{l\nu\mu} + \eta_{\beta_{l\nu}}, \tag{14c}$$

$$\frac{\partial\beta^+_{l\nu}}{\partial t} = +i\omega^\beta_\nu\beta^+_{l\nu} + ig^\beta_\nu\alpha^+_l\alpha_l + i\sum_\mu g_{\nu\mu}\beta^{r+}_{l\nu\mu} + \eta_{\beta^+_{l\nu}}. \tag{14d}$$

The last terms appearing on the far right of these equations represent the quantum fluctuations associated with the fields. Each of these terms has its origin in the diffusion coefficients (the second-order derivative terms) of the Fokker-Planck equation. Their exact characterization is the subject of the next section.

C. The noise sources

Note from the Fokker-Planck equation implied by (13) that α_l has noise due to the two terms

$$\frac{\partial^2}{\partial\alpha_l^2} \quad\text{and}\quad \frac{\partial^2}{\partial\alpha_l\partial\beta_{l\nu}}, \tag{15}$$

whilst the $\beta_{l\nu}$ variable has noise due only to the latter term. Instead of explicitly attempting to find a matrix $\boldsymbol{B}(\boldsymbol{\alpha},\boldsymbol{\alpha}^+,t)$ such that $\boldsymbol{B}\cdot\boldsymbol{B}^T = \boldsymbol{D}$, the diffusion matrix occurring in our Flokker-Planck equation, we will simply write down noise sources (corresponding to terms like $\boldsymbol{B}\cdot d\boldsymbol{W}$ in the theory of stochastic equations given in Appendix B) which have the correlation properties we desire. We can do this because the diffusion matrix $\boldsymbol{D}$ can be shown to be related to the two-time correlations of the field fluctuations by, for example,

$$\Big\langle[\alpha^\mu_l(t) - \langle\alpha^\mu_l(t)\rangle]\,[\alpha^\nu_{l'}(t') - \langle\alpha^\nu_{l'}(t')\rangle]\Big\rangle = D^{\mu\nu}_{\alpha_l,\alpha_{l'}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}^r,\boldsymbol{\beta}^r)\,\delta(t-t'), \tag{16}$$

or, say,

$$\Big\langle[\beta^\mu_{l\nu}(t) - \langle\beta^\mu_{l\nu}(t)\rangle]\,[\alpha^\epsilon_{l'}(t') - \langle\alpha^\epsilon_{l'}(t')\rangle]\Big\rangle = D^{\mu\epsilon}_{\beta_{l\nu},\alpha_{l'}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}^r,\boldsymbol{\beta}^r)\,\delta(t-t'). \tag{17}$$

Thus we construct the following noise sources (which are not unique, and may not even be optimal, but they are certainly easy to implement):

$$\eta_{\alpha_l}(t) = \Big\{\sqrt{+2i\chi_\alpha}\,\xi_l(t) - i\sqrt{t_0}\sum_\nu g^\beta_\nu\,\xi^\alpha_{l\nu}(t)\Big\}\alpha_l, \tag{18a}$$

$$\eta_{\alpha^+_l}(t) = \Big\{\sqrt{-2i\chi_\alpha}\,\xi^+_l(t) + i\sqrt{t_0}\sum_\nu g^\beta_\nu\,\xi^{\alpha+}_{l\nu}(t)\Big\}\alpha^+_l, \tag{18b}$$

$$\eta_{\beta_{l\nu}}(t) = \frac{1}{\sqrt{t_0}}\,\xi^\beta_{l\nu}(t), \tag{18c}$$

$$\eta_{\beta^+_{l\nu}}(t) = \frac{1}{\sqrt{t_0}}\,\xi^{\beta+}_{l\nu}(t). \tag{18d}$$

The variables ξ_l and ξ^+_l are independent real Gaussian stochastic functions (from the intrinsic $\chi^{(3)}$ nature of the medium due to electronic transitions) with "white noise" correlations

$$\langle\xi_l(t)\,\xi_{l'}(t')\rangle = \langle\xi^+_l(t)\,\xi^+_{l'}(t')\rangle = \delta_{ll'}\,\delta(t-t'). \tag{19}$$

The variables $\xi^\alpha_{l\nu}$ and $\xi^{\alpha+}_{l\nu}$ are independent *complex* Gaussian stochastic functions (complex because we require the autocorrelations of α_l and α^+_l to be independent of these noises), as are the pair $\xi^{\beta}_{l\nu}$ and $\xi^{\beta+}_{l\nu}$ (for similar reasons). However, these two pairs need to possess the following cross correlations:

$$\langle\xi^{\alpha}_{l\nu}(t)\,\xi^{\beta}_{l'\nu'}(t')\rangle = \langle\xi^{\alpha+}_{l\nu}(t)\,\xi^{\beta+}_{l'\nu'}(t')\rangle = \delta_{ll'}\,\delta_{\nu\nu'}\,\delta(t-t'). \tag{20}$$

When the fundamental noises are of this form then the noise sources appearing in the stochastic equations (14a) obey (16) and (17) which appear as

$$\langle\eta_{\alpha_{l'}}(t')\,\eta_{\alpha_l}(t)\rangle = 2i\chi_\alpha\,\alpha_l^2\,\delta_{ll'}\delta(t-t'), \tag{21a}$$

$$\langle\eta_{\alpha_{l'}}(t')\,\eta_{\beta_{l\nu}}(t)\rangle = -ig^{\beta}_{\nu}\alpha_l\,\delta_{ll'}\,\delta(t-t'). \tag{21b}$$

Note that, because the last equation of this pair corresponds to two off-diagonal matrix elements in the diffusion matrix, the coefficient in (21b) is half of that appearing explicitly in the Fokker-Planck equation (13). Similar correlations hold for the daggered variables with the appropriate conjugation. The introduction of the time scale t_0 is not specified as yet. It will turn out to be convenient to set this to the natural time scale for the system. Note the zero correlations:

$$\langle\xi^{\alpha}_{l\nu}(t)\,\xi^{\beta+}_{l'\nu'}(t')\rangle = \langle\xi^{\alpha}_{l\nu}(t)\,\xi^{\alpha+}_{l'\nu'}(t')\rangle = 0, \tag{22a}$$

$$\langle\xi^{\beta}_{l\nu}(t)\,\xi^{\beta+}_{l'\nu'}(t')\rangle = \langle\xi^{\alpha}_{l\nu}(t)\,\xi^{\alpha}_{l'\nu'}(t')\rangle = 0, \tag{22b}$$

$$\langle\xi_l(t)\,\xi^{+}_{l'}(t')\rangle = \langle\xi^{\alpha+}_{l\nu}(t)\,\xi^{\alpha+}_{l'\nu'}(t')\rangle = 0. \tag{22c}$$

This is all of the noise appearing in the problem. The remaining equations are for the scattering reservoirs; these do not contain explicit stochastic terms. However, their corresponding initial conditions do give rise to a source of noise. The equations are

$$\frac{\partial\alpha^{r}_{l\mu}}{\partial t} = -i(\omega_\mu+\Delta\omega)\,\alpha^{r}_{l\mu} - ig_\mu\alpha_l, \tag{23a}$$

$$\frac{\partial\alpha^{r+}_{l\mu}}{\partial t} = +i(\omega_\mu+\Delta\omega)\,\alpha^{r+}_{l\mu} + ig_\mu\alpha^{+}_l, \tag{23b}$$

$$\frac{\partial\beta^{r}_{l\nu\mu}}{\partial t} = -i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})\,\beta^{r}_{l\nu\mu} - ig_{\nu\mu}\beta_{l\nu}, \tag{23c}$$

$$\frac{\partial\beta^{r+}_{l\nu\mu}}{\partial t} = +i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})\,\beta^{r+}_{l\nu\mu} + ig_{\nu\mu}\beta^{+}_{l\nu}. \tag{23d}$$

D. Integrating the reservoirs

The elimination of the reservoir variables is identical to the procedure outlined in detail in Ref. [4] where it was carried out in the Heisenberg picture. As these equations contain *no* noise terms we need not treat them in any special way. Thus, ordinary direct integration gives for $\alpha^{r}_{l\mu}$

$$\alpha^{r}_{l\mu}(t) = \alpha^{r}_{l\mu}(t_i)\,e^{-i(\omega_\mu+\Delta\omega)(t-t_i)} + \int_{t_i}^{t} dt'\,e^{-i(\omega_\mu+\Delta\omega)(t-t')}\,[-ig_\mu\alpha_l(t')], \tag{24}$$

with corresponding initial conditions for reservoirs in the far past of $\langle\alpha^{r+}_{l\mu}(t_i)\,\alpha^{r}_{l\mu}(t_i)\rangle = n^{\mu}_{\text{th}}$, where $t_i \to -\infty$. The thermal photon number for mode μ is $n^{\mu}_{\text{th}} = [e^{\hbar(\omega_0+\omega_\mu)/kT}-1]^{-1}$. Note that this initial condition, with n^{μ}_{th} appearing instead of, say, $n^{\mu}_{\text{th}}+1$, is due to the positive-P representation being one which gives ensemble averages for *normally ordered* operator expressions. In contrast to the case for the Heisenberg operators, these variables like $\alpha^{r}_{l\mu}$ and $\alpha^{r+}_{l\mu}$ are ordinary complex numbers and hence *do commute.* Thus $\langle\alpha^{r}_{l\mu}\alpha^{r+}_{l\mu}\rangle = \langle\alpha^{r+}_{l\mu}\alpha^{r}_{l\mu}\rangle$ and the order of the variables in these moments is irrelevant (however, it should be remembered that they always correspond to the appropriate normally ordered operator expectation—so that $\langle\hat{a}^{r\dagger}_{l\mu}\hat{a}^{r}_{l\mu}\rangle$ is the quantity that is represented by the above positive-P moments.

The reservoir term in the equation for α_l in (14a) reduces to

$$\begin{aligned}-i\sum_\mu g_\mu\alpha^{r}_{l\mu}(t) &= -\int_{t_i}^{t} dt'\sum_\mu g_\mu^2\,e^{-i(\omega_\mu+\Delta\omega)(t-t')}\,\alpha_l(t') \\ &\quad -i\sum_\mu g_\mu\,e^{-i(\omega_\mu+\Delta\omega)(t-t_i)}\alpha^{s}_{l\mu}(t_i) \\ &= d^{\alpha}_l(t) + \Gamma^{\alpha}_l(t).\end{aligned} \tag{25}$$

Converting the discrete frequency sums to integral expressions by introducing a density of states function ρ about the carrier frequency ω_0 and replacing the discrete index μ by the continuous frequency offset Ω renders the result

$$\begin{aligned}&\sum_\mu g_\mu^2\,e^{-i(\omega_\mu+\Delta\omega)(t-t')} \\ &\to e^{-i\Delta\omega(t-t')}\int_{-\infty}^{+\infty} d\Omega\,\rho(\omega_0+\Omega)g^2(\omega_0+\Omega)\,e^{-i\Omega(t-t')} \\ &\approx\ 2\pi\,\rho(\omega_0)g^2(\omega_0)\,\delta(t-t').\end{aligned} \tag{26}$$

The extension of the lower limit on the integral from $-\omega_0$ to $-\infty$ introduces an error of infinitesimal magnitude. Hence,

$$d^{\alpha}_l(t) \approx -\frac{\kappa}{2}\,\alpha_l(t). \tag{27}$$

In the same manner as was described in Ref. [4], the second term, Γ^{α}_l, behaves like a stochastic quantity due to the random initial conditions for the reservoir amplitudes. This noise term scales with the loss parameter κ as is clear from the correlation function for Γ^{α}_l and the conjugate term $\Gamma^{\alpha+}_l$:

$$\begin{aligned}\langle\Gamma^{\alpha}_l(t)\,\Gamma^{\alpha+}_{l'}(t')\rangle &= \sum_{\mu\nu} g_\mu g_\nu\,e^{-i(\omega_\mu+\Delta\omega)(t-t_i)} \\ &\quad\times e^{i(\omega_\nu+\Delta\omega)(t'-t_i)}\langle\alpha^{r}_{l\mu}(t_i)\,\alpha^{r+}_{l'\nu}(t_i)\rangle \\ &\approx 2\pi\,\delta_{ll'}\,\rho(\omega_0)g^2(\omega_0)\,n_{\text{th}}(\omega_0)\,\delta(t-t') \\ &= \kappa n^{\alpha}_{\text{th}}\,\delta_{ll'}\,\delta(t-t').\end{aligned} \tag{28}$$

Here $n_{\rm th}^{\alpha} = n_{\rm th}(\omega_0) = [\,e^{\hbar\omega_0/kT} - 1\,]^{-1}$. The result is that the reservoir sum (25) is replaced by a continuous loss term and its associated noise:

$$-i\sum_{\mu} g_{\mu}\alpha^{r}_{l\mu}(t) \to -\frac{\kappa}{2}\,\alpha_l(t) \;+\; \Gamma^{\alpha}_{l}(t). \tag{29}$$

Note that the dimensions for the new variables are $[\kappa] = [\Gamma^{\alpha}_{l}] = s^{-1}$. The steps for the phonon reservoirs are virtually identical:

$$\beta^{r}_{l\nu\mu}(t) = \beta^{r}_{l\nu\mu}(t_i)\,e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t_i)} + \int_{t_i}^{t} dt'\,e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t')}\,[\,-ig_{\nu\mu}\beta_{l\nu}(t')\,]\,, \tag{30}$$

where $\langle\beta^{r+}_{l\nu\mu}(t_i)\,\beta^{r}_{l\nu\mu}(t_i)\rangle = n^{\nu\mu}_{\rm th}$, the thermal phonon number for reservoir mode μ coupled to phonon mode ν. Explicitly $n^{\nu\mu}_{\rm th} = [\,e^{\hbar(\omega^{\beta}_{\nu}+\omega_{\nu\mu})/kT} - 1\,]^{-1}$. Thus the reservoir sum appearing in the $\beta_{l\nu}$ equation in (14a) becomes

$$\begin{aligned}-i\sum_{\mu} g_{\nu\mu}\beta^{r}_{l\nu\mu}(t) &= -\sum_{\mu}\int_{t_i}^{t} dt'\,g^{2}_{\nu\mu}\,e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t')}\beta_{l\nu}(t') - i\sum_{\mu} g_{\nu\mu}e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t_i)}\beta^{r}_{l\nu\mu}(t_i)\\ &= d^{\beta}_{l}(t) \;+\; \Gamma^{\beta}_{l\nu}(t).\end{aligned} \tag{31}$$

Also,

$$\begin{aligned}\sum_{\mu} g^{2}_{\nu\mu}e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t')} &\to 2\pi\,\rho_{\nu}(\omega^{\beta}_{\nu})g^{2}_{\nu}(\omega^{\beta}_{\nu})\,e^{-i\omega^{\beta}_{\nu}(t-t')}\,\delta(t-t')\\ &\equiv 2\gamma'_{\nu}\,\delta(t-t'),\end{aligned} \tag{32}$$

$$d^{\beta}_{l}(t) = -\,\gamma'_{\nu}\,\beta_{l\nu}(t)\,, \tag{33}$$

and

$$\begin{aligned}\langle\Gamma^{\beta}_{l\nu}(t)\,\Gamma^{\beta+}_{l\nu}(t')\rangle &= \sum_{\mu} g^{2}_{\nu\mu}\,e^{-i(\omega^{\beta}_{\nu}+\omega_{\nu\mu})(t-t')}\,n^{\nu\mu}_{\rm th}\\ &\approx 2\pi\,\rho_{\nu}(\omega^{\beta}_{\nu})\,g^{2}_{\nu}(\omega^{\beta}_{\nu})\,n^{\nu}_{\rm th}(\omega^{\beta}_{\nu})\,e^{-i\omega^{\beta}_{\nu}(t-t')}\,\delta(t-t')\\ &\equiv 2\gamma'_{\nu}\,n^{\nu}_{\rm th}\,\delta(t-t')\end{aligned} \tag{34}$$

with $n^{\nu}_{\rm th}(\omega^{\beta}_{\nu} + \Omega_{\nu}) = [\,e^{\hbar(\omega^{\beta}_{\nu}+\Omega_{\nu})/kT} - 1\,]^{-1}$, and $n^{\nu}_{\rm th} \equiv [\,e^{\hbar\omega^{\beta}_{\nu}/kT} - 1\,]^{-1}$. Thus

$$-i\sum_{\mu} g_{\nu\mu}\,\beta^{s}_{l\nu\mu}(t) \to -\,\gamma'_{\nu}\,\beta_{l\nu}(t) \;+\; \Gamma^{\beta}_{l\nu}(t), \tag{35}$$

where $[\,\gamma'_{\nu}\,] = [\,\Gamma^{\beta}_{l\nu}\,] = s^{-1}$.

E. The traced equations

Replacing the reservoir terms in (14a) with the integrated expressions obtained in the last section leads to the following system of coupled *Ito stochastic* differential equations for the photon amplitudes:

$$\begin{aligned}\frac{\partial\alpha_l}{\partial t} &= -i\sum_{l'}\omega_{ll'}\alpha_{l'} + 2i\chi_{\alpha}{\alpha_l}^{2}\alpha^{+}_{l}\\ &\quad -i\alpha_l\sum_{\nu} g^{\beta}_{\nu}(\beta^{+}_{l\nu} + \beta_{l\nu})\\ &\quad -\frac{\kappa}{2}\,\alpha_l + \Gamma^{\alpha}_{l} + \Big\{\sqrt{2i\chi_{\alpha}}\,\xi_l - i\sqrt{t_0}\sum_{\nu} g^{\beta}_{\nu}\,\xi^{\alpha}_{l\nu}\Big\}\alpha_l,\end{aligned} \tag{36a}$$

$$\begin{aligned}\frac{\partial\alpha^{+}_{l}}{\partial t} &= i\sum_{l'}\omega_{l'l}\alpha^{+}_{l'} - 2i\chi_{\alpha}{\alpha^{+}_{l}}^{2}\alpha_l + i\alpha^{+}_{l}\sum_{\nu} g^{\beta}_{\nu}(\beta^{+}_{l\nu} + \beta_{l\nu})\\ &\quad -\frac{\kappa}{2}\,\alpha^{+}_{l} + \Gamma^{\alpha+}_{l} + \Big\{\sqrt{-2i\chi_{\alpha}}\,\xi^{+}_{l}\\ &\quad +i\sqrt{t_0}\sum_{\nu} g^{\beta}_{\nu}\,\xi^{\alpha+}_{l\nu}\Big\}\alpha^{+}_{l},\end{aligned} \tag{36b}$$

and the phonon amplitudes:

$$\frac{\partial\beta_{l\nu}}{\partial t} = -\gamma_{\nu}\,\beta_{l\nu} - ig^{\beta}_{\nu}\alpha^{+}_{l}\alpha_l + \Gamma^{\beta}_{l\nu} + \frac{1}{\sqrt{t_0}}\,\xi^{\beta}_{l\nu}, \tag{36c}$$

$$\frac{\partial\beta^{+}_{l\nu}}{\partial t} = -\gamma^{*}_{\nu}\,\beta^{+}_{l\nu} + ig^{\beta}_{\nu}\alpha^{+}_{l}\alpha_l + \Gamma^{\beta+}_{l\nu} \;+\; \frac{1}{\sqrt{t_0}}\,\xi^{\beta+}_{l\nu}. \tag{36d}$$

where we have defined the complex phonon damping parameter as $\gamma_{\nu} = \gamma'_{\nu} + i\omega^{\beta}_{\nu}$. Strictly speaking, these equations should be manipulated with the Ito rules for calculus. However, we shall scale these equations first and see (in Sec. III J) that the Ito corrections to the equivalent Stratonovich equations are tiny for the current experimental regime of large photon number. Thus we may ignore these corrections and treat the above equations with the normal rules of calculus without introducing any appreciable error in the process.

F. Scaled variables (the macroscopic fields)

By introducing a time scale t_0 and a length scale z_0 we can reduce the photon and phonon variables to a convenient dimensionless form. These scales will be explicitly defined in Sec. III H when we choose an appropriate coordinate system to simplify the form of the propagation equation for the photon field. The following definitions for the field variables include a factor of $(\Delta z)^{-1/2}$ which removes their previous dependence on the cell size. The newly scaled quantum fields are chosen as

$$\phi_l = \alpha_l\,\sqrt{\frac{\omega' t_0}{\bar{n}\Delta z}}, \tag{37}$$

$$b_{l\nu} = \beta_{l\nu}\,\sqrt{\frac{h_{\nu} z_0}{2\bar{n}\Delta z}}, \tag{38}$$

where the phonon coupling strength has been redefined as

$$h_\nu = \frac{2\Delta z (g_\nu^\beta)^2 \bar{n} z_0}{(\omega')^2}. \tag{39}$$

By introducing a convenient form of the nonlinear coefficient which is independent of the cell length, namely,

$$\chi_E = \frac{2\chi_\alpha \Delta z}{(\omega')^2}, \tag{40}$$

we find that, after rescaling, Eqs. (36d) become

$$\frac{\partial \phi_l}{\partial t} = -i\sum_{l'} \omega_{ll'}\phi_{l'} + i\chi_E \frac{\bar{n}\omega'}{t_0}\phi_l{}^2\phi_l^+ -i\phi_l\frac{\omega'}{z_0}\sum_\nu [b_{l\nu}^+ + b_{l\nu}] - \frac{\kappa}{2}\phi_l + \sqrt{\frac{\omega' t_0}{\bar{n}}}\frac{\Gamma_l^\alpha}{\sqrt{\Delta z}} +\omega'\left\{\sqrt{i\chi_E}\frac{\xi_l}{\sqrt{\Delta z}} - i\sum_\nu \sqrt{\frac{h_\nu t_0}{2\bar{n} z_0}}\frac{\xi_{l\nu}^\alpha}{\sqrt{\Delta z}}\right\}\phi_l, \tag{41a}$$

$$\frac{\partial \phi_l^+}{\partial t} = i\sum_{l'} \omega_{l'l}\phi_{l'}^+ - i\chi_E \frac{\bar{n}\omega'}{t_0}\phi_l^{+2}\phi_l +i\phi_l^+\frac{\omega'}{z_0}\sum_\nu [b_{l\nu}^+ + b_{l\nu}] - \frac{\kappa}{2}\phi_l^+ + \sqrt{\frac{\omega' t_0}{\bar{n}}}\frac{\Gamma_l^{\alpha+}}{\sqrt{\Delta z}} +\omega'\left\{\sqrt{-i\chi_E}\frac{\xi_l^+}{\sqrt{\Delta z}} + i\sum_\nu \sqrt{\frac{h_\nu t_0}{2\bar{n} z_0}}\frac{\xi_{l\nu}^{\alpha+}}{\sqrt{\Delta z}}\right\}\phi_l, \tag{41b}$$

$$\frac{\partial b_{l\nu}}{\partial t} = -\gamma_\nu b_{l\nu} - \frac{ih_\nu}{2t_0}\phi_l^+\phi_l + \sqrt{\frac{h_\nu z_0}{2\bar{n}}}\left\{\frac{\Gamma_{l\nu}^\beta}{\sqrt{\Delta z}} + \frac{\xi_{l\nu}^\beta}{\sqrt{t_0\,\Delta z}}\right\}, \tag{41c}$$

$$\frac{\partial b_{l\nu}^+}{\partial t} = -\gamma_\nu^* b_{l\nu}^+ + \frac{ih_\nu}{2t_0}\phi_l^+\phi_l + \sqrt{\frac{h_\nu z_0}{2\bar{n}}}\left\{\frac{\Gamma_{l\nu}^{\beta+}}{\sqrt{\Delta z}} + \frac{\xi_{l\nu}^{\beta+}}{\sqrt{t_0\,\Delta z}}\right\}. \tag{41d}$$

G. Continuum limit as $\Delta z \to 0$

As we now show, the equations readily go over to a continuous form by taking the limit as the cell size tends to zero. Thus stochastic terms of the form ϱ_l generate the equivalent continuum term $\varrho(z)$ as follows:

$$\langle \varrho(z)\,\varrho(z')\rangle = \delta(z-z'), \tag{42}$$

where

$$\varrho(z) = \lim_{\Delta z\to 0} \left.\frac{\varrho_l}{\sqrt{\Delta z}}\right|_{l\Delta z = z}. \tag{43}$$

The term in $\omega_{ll'}$ is shown in Appendix A to generate the spatial derivatives which are a consequence of the dispersive nature of the medium. Thus the continuous fields ϕ and b_ν now satisfy equations which look like

$$\frac{\partial \phi}{\partial t} = \left[-i\Delta\omega - \omega'\frac{\partial}{\partial z} + \frac{i\omega''}{2}\frac{\partial^2}{\partial z^2}\right]\phi + i\chi_E\frac{\bar{n}\omega'}{t_0}\phi^2\phi^+ -\frac{\kappa}{2}\phi - i\phi\frac{\omega'}{z_0}\sum_\nu [b_\nu^+ + b_\nu] + \sqrt{\frac{\omega' t_0}{\bar{n}}}\Gamma_L(z,t) +\omega'\left\{\sqrt{i\chi_E}\,\xi(z,t) - i\sum_\nu\sqrt{\frac{h_\nu t_0}{2\bar{n} z_0}}\xi_\nu^\phi(z,t)\right\}\phi, \tag{44a}$$

$$\frac{\partial b_\nu}{\partial t} = -\gamma_\nu b_\nu - \frac{ih_\nu}{2t_0}\phi^+\phi + \sqrt{\frac{h_\nu z_0}{2\bar{n}}}\left\{\Gamma_\nu^b(z,t) + \frac{\xi_\nu^b(z,t)}{\sqrt{t_0}}\right\}. \tag{44b}$$

The conjugate pair of equations for ϕ^+ and b_ν^+ in the positive-P representation, which we no longer write down explicitly, contain the other half of the information on the system evolution. The noise sources in the continuum limit have an almost identical appearance to the discrete results (28), (34), (20), and (19),

$$\langle \Gamma_L(z,t)\,\Gamma_L^+(z',t')\rangle = \kappa n_{\rm th}^\alpha\,\delta(z-z')\,\delta(t-t'), \tag{45a}$$

$$\langle \Gamma_\nu^b(z,t)\,\Gamma_\nu^{b+}(z',t')\rangle = 2\gamma_\nu' n_{\rm th}^\nu\,\delta(z-z')\,\delta(t-t'), \tag{45b}$$

$$\langle \xi_\nu^{\phi+}(z,t)\,\xi_\nu^{b+}(z',t')\rangle = \langle \xi_\nu^\phi(z,t)\,\xi_\nu^b(z',t')\rangle = \langle \xi^+(z,t)\,\xi^+(z',t')\rangle = \langle \xi(z,t)\,\xi(z',t')\rangle = \delta(z-z')\,\delta(t-t'). \tag{45c}$$

H. Dimensionless coordinates

In this section we reduce the space and time coordinates to dimensionless numbers so that the overall field equations are in a convenient dimensionless form. The way we shall rescale our coordinates is to normalize them with the quantities z_0 and t_0 that were introduced in Sec. III F but were not specified. We specify their exact values here. At the same time we introduce a coordinate system traveling at the speed of a pulse moving with the group velocity ω'. It is convenient to specify dimensionless space and time coordinates of the form

$$\zeta = z/z_0 \qquad \text{and} \qquad \tau = \frac{t - z/\omega'}{t_0}, \tag{46}$$

where

$$z_0 = \frac{t_0{}^2}{|k''|}, \qquad k'' = -\frac{\omega''}{(\omega')^3}. \tag{47}$$

With this choice of comoving coordinates, ζ represents the distance the field has propagated down the fiber, while τ specifies the time coordinate relative to the pulse profile, with $\tau = 0$ at the pulse peak. Notice that we define z_0 here in terms of the fundamental time scale t_0. In the case of pulse propagation, t_0 can be chosen to represent the pulse duration. We also introduce the effective photon number $\bar{n}$ for the problem,

$$\bar{n} = \frac{|k''|}{\chi_T\, t_0}. \tag{48}$$

χ_T is taken to be the total nonlinear coefficient for the medium and is made up of three contributions. The first is the fast electronic nonlinear response of the medium (χ_E), with a time response of the order of a few electromagnetic cycles, or $\sim 10^{-15}$–10^{-14} s. The second is a medium time scale ($\sim 10^{-13}$ s) nonlinear response due to the Raman interaction (χ_R). Finally the slowest of the nonlinear responses (10^{-9} s) is due to electrostriction, or GAWBS processes (χ_G).

The scaled equations (in which the photon field would evolve as a first-order equation in ζ if we dropped the mixed space and time derivatives introduced by the transformation to the comoving frame) is

$$\begin{aligned}\frac{\partial\phi}{\partial\zeta} &= -\frac{i}{2}\left[1 \pm \left\{\frac{\partial^2}{\partial\tau^2} - 2\delta\frac{\partial^2}{\partial\zeta\,\partial\tau} + \delta^2\frac{\partial^2}{\partial\zeta^2}\right\}\right]\phi \\ &\quad + i\left[\frac{\chi_E}{\chi_T}\right]\phi^2\phi^+ - \gamma\phi - i\phi\sum_\nu [b_\nu^+ + b_\nu] \\ &\quad + \Gamma_L(\zeta,\tau) + \sqrt{i}\phi\,\Gamma_E(\zeta,\tau) - i\phi\,\Gamma_V^\phi(\zeta,\tau), \end{aligned} \tag{49}$$

$$\frac{\partial b_\nu}{\partial\tau} = -\bar{\gamma}_\nu\, b_\nu - \frac{ih_\nu}{2}\,\phi^+\phi + \Gamma_V^\nu. \tag{50}$$

Note that as well as arbitrarily setting the free parameter $\Delta\omega$ to $\Delta\omega = \omega'|k''|/2t_0^2$, we have also defined the dimensionless complex phonon damping parameter as $\bar{\gamma}_\nu = t_0\gamma_\nu$, and the loss parameter $\gamma = \kappa z_0/2\omega'$. We have also let $\delta = \omega' t_0/z_0$. We shall neglect the terms in δ since they are typically very small (e.g., δ is of the order of 10^{-5} for pulse durations around the 100 fs mark). The precise nature of the scaled noises (which we have grouped and renamed) is discussed in Sec. III J.

I. The Fourier fields

Since we will denote the dimensionless frequency by $\bar{\omega} = \omega t_0$ it will be convenient to denote Fourier variables in this paper with a tilde (˜). Thus if $f(\tau)$ is a function of the time τ, our Fourier pair is defined by

$$\tilde{f}(\bar{\omega}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\tau\, e^{i\bar{\omega}\tau}\, f(\tau), \tag{51a}$$

for the forward transform and

$$f(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\bar{\omega}\, e^{-i\bar{\omega}\tau}\, \tilde{f}(\bar{\omega}) \tag{51b}$$

for the reverse.

J. Scaled noises

After keeping track of the scaling parameters the associated correlation functions at this stage are summarized again. First, Γ_L, the source of noise associated with the loss of light from the fiber due to scattering. This is a thermal effect involving the thermal occupation number ($n_{\rm th}^\alpha$) of the scattering reservoir. Because the autocorrelations of Γ_L, Γ_L^+ vanish, these must be treated as complex-valued sources with a nonvanishing intensity correlation. This correlation in the time and the frequency domains has the usual appearance for a white noise variable [and originates from (28)]:

$$\langle\Gamma_L(\zeta,\tau)\,\Gamma_L^+(\zeta',\tau')\rangle = \frac{2\gamma n_{\rm th}^\alpha}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\tau-\tau'), \tag{52a}$$

$$\langle\tilde{\Gamma}_L(\zeta,\bar{\omega})\,\tilde{\Gamma}_L^+(\zeta',\bar{\omega}')\rangle = \frac{2\gamma n_{\rm th}^\alpha}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}'). \tag{52b}$$

Next, Γ_E, which is the noise due to fast (virtual) two-photon electronic scatterings within individual atoms. This is the quantum manifestation of the intensity-dependent refractive index induced by the field as it propagates within the fiber. Because the cross correlations vanish here, and the autocorrelations are real valued (the noises are the scaled form of ξ_l, ξ_l^+), Γ_E, Γ_E^+ may be treated as independent real-valued sources with autocorrelations [cf. (19)]

$$\begin{aligned}\langle\Gamma_E(\zeta,\tau)\,\Gamma_E(\zeta',\tau')\rangle &= \frac{1}{\bar{n}}\left[\frac{\chi_E}{\chi_T}\right]\delta(\zeta-\zeta')\,\delta(\tau-\tau') \\ &= \langle\Gamma_E^+(\zeta,\tau)\,\Gamma_E^+(\zeta',\tau')\rangle,\end{aligned} \tag{53}$$

$$\begin{aligned}\langle\tilde{\Gamma}_E(\zeta,\bar{\omega})\,\tilde{\Gamma}_E(\zeta',\bar{\omega}')\rangle &= \frac{1}{\bar{n}}\left[\frac{\chi_E}{\chi_T}\right]\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}') \\ &= \langle\tilde{\Gamma}_E^+(\zeta,\bar{\omega})\,\tilde{\Gamma}_E^+(\zeta',\bar{\omega}')\rangle.\end{aligned} \tag{54}$$

Finally, the noise due to the coupling to the vibrational modes. There are two sources here. The first is the *cross correlation* between the source Γ_V^ϕ appearing in (49), and Γ_V^ν appearing in (50) [and originating as the composite noise term appearing at the far right of the corresponding Eq. (44a)],

$$\begin{aligned}\langle\Gamma_V^\phi(\zeta,\tau)\,\Gamma_V^\nu(\zeta',\tau')\rangle &= \langle\Gamma_V^{\phi+}(\zeta,\tau)\,\Gamma_V^{\nu+}(\zeta',\tau')\rangle \\ &= \frac{h_\nu}{2\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\tau-\tau').\end{aligned} \tag{55}$$

Compare this with the original correlations (20) which are the unscaled version of this result. This is a purely quantum effect which, like the correlations for the electronic noises, is independent of the temperature. Because the autocorrelation of Γ_V^ϕ vanishes [see (22c)] this must be treated as a complex-valued noise source.

The second vibrational correlation function of interest is the *intensity* correlation of Γ_V^ν, which is clearly a thermal effect involving the thermal occupation number of the associated vibrational mode. The nature of this source is the same as for Γ_L and so it too is complex valued:

$$\langle \Gamma_V^\nu(\zeta,\tau)\,\Gamma_V^{\nu+}(\zeta',\tau')\rangle = \frac{h_\nu \bar{\gamma}_\nu' n_{\rm th}^\nu}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\tau-\tau'). \tag{56}$$

Hence, left in this form, all noise sources are δ correlated. These equations may be solved together by integrating the phonon equations (50) in time (at each point in space) and then updating the photon variable ϕ. As it stands, the model represents the "homogeneous gain" problem if we assume only one vibrational mode. Thus by using appropriate values for h_ν, $\bar{\gamma}_\nu'$, and $\bar{\omega}_\nu^\beta$ we may solve the coupled photon-phonon equations, say, numerically. Of course, what we shall do instead is *eliminate* the phonon part of the problem completely by assuming that it can be modeled as a large number of vibrational modes, each with a narrow linewidth. This is a much more satisfactory approach since in reality the number of vibrational modes *is* large and much more closely approximates a continuum than a single broad line (as would be the case for the homogeneous gain model).

Notice that all of the correlation functions given in this section scale inversely with the photon number $\bar{n}$. This means that if we were to calculate Ito *drift* corrections of the form $B_{kj}\partial B_{ij}/\partial x_k$, as in (B9), for the equivalent Stratonovich equations describing our system, then these would scale as $\bar{n}^{-1}$. Since we are generally interested in cases when $\bar{n} \sim 10^9$ we will neglect the Ito corrections and interpret the equations from this point on in the Stratonovich sense, i.e., we will employ the normal rules of calculus. This is a valid approximation because the neglected corrections directly affect only the *deterministic* part of the evolution, and hence only the mean-field behavior (and *not* the quantum fluctuations themselves). Their omission will thus have a negligible effect on moments relative to the mean field, which is generally the principal quantity of interest. This procedure should of course be reexamined for the case when $\bar{n}$ is small, when corrections to the mean-field behavior may be significant. In this case one may either generate the (nontrivial) Ito corrections directly, or else interpret the equations strictly in the Ito sense, where the positive-P equations are *exact.*

K. Elimination of the phonon variables

To rid ourselves of the phonons and the corresponding term which goes as $[\beta_\nu^+ + \beta_\nu]$ in the photon equation, we write down the formal integral for β_ν and substitute it back into the photon equation. This procedure generates a response function for the medium with a characteristic time delay. This is the primary effect of the phononic scattering. Secondary effects are due to the phonon noise which has been introduced. The formal phonon integral (with the term due to the initial conditions in the far past) is

$$\begin{aligned} b_\nu(\zeta,\tau) &= \lim_{T\to\infty} b_\nu(\zeta,-T)\,e^{-\bar{\gamma}_\nu(\tau+T)} \\ &\quad + \int_{-\infty}^{\tau} d\tau' e^{-\bar{\gamma}_\nu(\tau-\tau')}\left\{-\frac{ih_\nu}{2}\phi^+(\zeta,\tau')\phi(\zeta,\tau') \right. \\ &\quad \left. +\Gamma_V^\nu(\zeta,\tau')\right\}. \end{aligned} \tag{57}$$

We neglect the initial condition term since it rapidly decays due to the coupling of the phonons to reservoirs. The final equation for our photon field is what we may term the Raman-modified nonlinear Schrödinger equation:

$$\begin{aligned} \frac{\partial\phi}{\partial\zeta} &= -\frac{i}{2}\left[1 \pm \frac{\partial^2}{\partial\tau^2}\right]\phi + i\left[\frac{\chi_E}{\chi_T}\right]\phi^2\phi^+ - \gamma\phi \\ &\quad + i\phi\int_{-\infty}^{\tau} d\tau'\,h(\tau-\tau')\,\phi^+(\tau')\,\phi(\tau') \\ &\quad + \Gamma_L + \sqrt{i}\,\phi\,\Gamma_E + i\phi\,\Gamma_V. \end{aligned} \tag{58}$$

When the last three terms in this equation are neglected (which represent the thermal and quantum noise of the system), the Hermitian equation for ϕ^+ reduces to the complex conjugate equation and the result is the classical equation of motion for the field postulated by Gordon [20] with a nonlinear response for the medium defined by

$$\begin{aligned} h(\tau) &= \frac{i}{2}\sum_\nu h_\nu\,[e^{-\bar{\gamma}_\nu\tau} - e^{-\bar{\gamma}_\nu^*\tau}] \\ &= \sum_\nu h_\nu\,e^{-\bar{\gamma}_\nu'\tau}\sin[\bar{\omega}_\nu\tau] \quad (\tau > 0), \end{aligned} \tag{59}$$

and

$$\begin{aligned} \Gamma_V(\zeta,\tau) &= -\Gamma_V^\phi(\zeta,\tau) \\ &\quad - \sum_\nu \int_{-\infty}^{\tau} d\tau'\left[e^{-\bar{\gamma}_\nu(\tau-\tau')}\,\Gamma_V^\nu(\zeta,\tau') \right. \\ &\quad \left. + e^{-\bar{\gamma}_\nu^*(\tau-\tau')}\,\Gamma_V^{\nu+}(\zeta,\tau')\right]. \end{aligned} \tag{60}$$

This stochastic term represents all of the noise effects due to the vibrational processes. This term is more complicated than the corresponding noise operator found in the Heisenberg treatment since there is also a conjugate partner Γ_V^+. In contrast to the Heisenberg picture Γ_V^+ is quite distinct from Γ_V. This means that the correlation functions which define this pair have a slightly more complex appearance. These correlations are defined by the following expressions for the time and frequency domains. Note that we have yet to go over to a continuum of vibrational modes. This will be done in the next section, so the following may be considered as intermediate results. For the time domain we have

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V(\zeta',\tau')\rangle = \langle\Gamma_V^+(\zeta,\tau)\,\Gamma_V^+(\zeta',\tau')\rangle^* = \frac{1}{2\bar{n}}\delta(\zeta-\zeta')\sum_\nu h_\nu\Big\{e^{-\bar{\gamma}_\nu|\tau-\tau'|} + 2n_{\rm th}^\nu\, e^{-\bar{\gamma}'_\nu|\tau-\tau'|}\cos[\bar{\omega}_\nu(\tau-\tau')]\Big\}, \tag{61a}$$

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V^+(\zeta',\tau')\rangle = \frac{1}{2\bar{n}}\delta(\zeta-\zeta')\sum_\nu h_\nu\left\{\begin{bmatrix} e^{-\bar{\gamma}_\nu(\tau'-\tau)}, & \tau<\tau' \\ e^{-\bar{\gamma}_\nu^*(\tau-\tau')}, & \tau'<\tau\end{bmatrix} + 2n_{\rm th}^\nu\, e^{-\bar{\gamma}'_\nu|\tau-\tau'|}\cos[\bar{\omega}_\nu(\tau-\tau')]\right\}. \tag{61b}$$

And for the Fourier domain where $\bar{\omega}$ replaces τ we get

$$\langle\tilde{\Gamma}_V(\zeta,\bar{\omega})\,\tilde{\Gamma}_V(\zeta',\bar{\omega}')\rangle = \langle\tilde{\Gamma}_V^+(\zeta,\bar{\omega})\,\tilde{\Gamma}_V^+(\zeta',\bar{\omega}')\rangle^* = \frac{1}{2\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\sum_\nu h_\nu\left\{\left[\frac{1}{\bar{\gamma}'_\nu+i(\bar{\omega}_\nu-\bar{\omega}')} + \frac{1}{\bar{\gamma}'_\nu+i(\bar{\omega}_\nu+\bar{\omega}')}\right] + 2n_{\rm th}^\nu(\bar{\omega}_\nu)\,\bar{\gamma}'_\nu\left[\frac{1}{(\bar{\gamma}'_\nu)^2+(\bar{\omega}_\nu-\bar{\omega}')^2} + \frac{1}{(\bar{\gamma}'_\nu)^2+(\bar{\omega}_\nu+\bar{\omega}')^2}\right]\right\}, \tag{61c}$$

$$\langle\tilde{\Gamma}_V(\zeta,\bar{\omega})\,\tilde{\Gamma}_V^+(\zeta',\bar{\omega}')\rangle = \frac{1}{2\bar{n}}\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\sum_\nu 2h_\nu\,\bar{\gamma}'_\nu\left\{\frac{[n_{\rm th}^\nu(\bar{\omega}_\nu)+1]}{(\bar{\gamma}'_\nu)^2+(\bar{\omega}_\nu-\bar{\omega}')^2} + \frac{n_{\rm th}^\nu(\bar{\omega}_\nu)}{(\bar{\gamma}'_\nu)^2+(\bar{\omega}_\nu+\bar{\omega}')^2}\right\}. \tag{61d}$$

Clearly we have *four* equations in the positive-P treatment as opposed to only two in the Heisenberg formulation [4]—a consequence of the double dimensionality of the positive-P representation.

L. The inhomogeneous gain model

The current form of the coupled photon-phonon equations contains the phonon contribution in terms of a sum over all of the possible modes of the phonon spectrum. We adopt the view that the number of modes is extremely large and calculate the effect of taking the limit as the individual linewidths shrink to zero. Compared to the Raman gain bandwidth of $\sim$ 10 THz this is actually a good approximation to the true situation. Because this model of the vibrational modes allows us to assign different coupling strengths to different parts of the spectrum, we will refer to this approximation as the inhomogeneous gain model.

Thus, to arrive at a final set of correlation functions for the noise sources of our glass fiber which will turn out to be satisfyingly compact, we shall assume a model in which the frequency spacing between modes is taken (for convenience) to be uniform and the linewidth vanishingly small. For simplicity we shall assume $\bar{\gamma}'_\nu = \bar{\gamma}$ for all modes, so that $h(\bar{\omega})$, which we get by taking the Fourier transform of (59), becomes

$$h(\bar{\omega}) = \frac{1}{\sqrt{2\pi}}\sum_\nu h_\nu\,\frac{\bar{\omega}_\nu}{(\bar{\gamma}-i\bar{\omega})^2+\bar{\omega}_\nu^2} = h'(\bar{\omega}) + ih''(\bar{\omega}). \tag{62}$$

Now we assume that the modes are finely spaced and turn the summation into an integral by introducing a convenient function $\alpha(\Omega)$ and making the replacement

$$\sum_\nu h_\nu \to \int_0^\infty d\Omega\,\frac{\alpha(\Omega)}{\pi} \quad \text{with} \quad \bar{\omega}_\nu \to \Omega. \tag{63}$$

The factor of π is introduced here so that $\alpha(\Omega)$ may later be identified with the conventional gain function found in the current literature, i.e., $\alpha_R(\Omega)$ or $\alpha_G(\Omega)$ depending upon the frequency region that interests us [or $\alpha_G(\Omega) + \alpha_R(\Omega)$ in general]. At this stage we merely assume that $\alpha(\Omega) > 0$, since it is effectively a coupling strength. Since we shall later take the limit as $\bar{\gamma} \to 0$, the form of $h(\bar{\omega})$ [determined by the choice of $\alpha(\Omega)$] is finally given by

$$h(\bar{\omega}) = \lim_{\bar{\gamma}\to 0}\frac{1}{\sqrt{2\pi}}\int_0^\infty d\Omega\,\frac{\alpha(\Omega)}{\pi}\left[\frac{\Omega}{(\bar{\gamma}-i\bar{\omega})^2+\Omega^2}\right]. \tag{64}$$

As we are interested in the limit as the linewidth $\bar{\gamma} \to 0$ we rewrite the square bracket in (64) and take the limit after real and imaginary components have been separated. It is not difficult to show that

$$\frac{1}{(\bar{\gamma}-i\bar{\omega})^2+\Omega^2} = R(\bar{\omega}) + iI(\bar{\omega}), \tag{65}$$

where

$$R(\bar{\omega}) = \frac{\bar{\gamma}^2-\bar{\omega}^2+\Omega^2}{[\bar{\gamma}^2+\bar{\omega}^2-\Omega^2]^2+[2\bar{\gamma}\Omega]^2}, \tag{66a}$$

$$I(\bar{\omega}) = \frac{\bar{\omega}}{[\bar{\gamma}^2+\bar{\omega}^2+\Omega^2]} \times\left[\frac{\bar{\gamma}}{(\bar{\omega}-\Omega)^2+\bar{\gamma}^2} + \frac{\bar{\gamma}}{(\bar{\omega}+\Omega)^2+\bar{\gamma}^2}\right]. \tag{66b}$$

Considered as an integrand in Ω it is easy to see that as $\bar{\gamma} \to 0$ we have

$$R \to \frac{1}{\Omega^2-\bar{\omega}^2}, \tag{67a}$$

$$I \to \frac{\bar{\omega}}{(\Omega^2+\bar{\omega}^2)}\,\pi\,\delta(\Omega-|\bar{\omega}|) = \frac{\pi}{2\bar{\omega}}\,\delta(\Omega-|\bar{\omega}|); \tag{67b}$$

hence

$$\sqrt{2\pi}\, h(\bar{\omega}) = \int_0^\infty d\Omega\, \frac{\alpha(\Omega)\,\Omega}{\pi\,(\Omega^2 - \bar{\omega}^2)} + \frac{i\,\mathrm{sgn}(\bar{\omega})}{2}\,\alpha(|\bar{\omega}|). \quad (68)$$

Note that for a particular form of $\alpha(\Omega)$ the integral with finite $\bar{\gamma}$ should be done first and then the limit taken, as given by (64). This is what the last equation means.

The above is a derivation of the Kramers-Kronig relation which states that $h'(\bar{\omega})$ (the dispersive part) may be obtained from $h''(\bar{\omega})$ (the absorptive or gain part) by an integration, namely,

$$h'(\bar{\omega}) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\Omega\, \frac{h''(\Omega)\,\Omega}{(\Omega^2 - \bar{\omega}^2)}. \quad (69)$$

This is purely the result of modeling our vibrational modes, to which the field may couple, as a continuum of harmonic oscillators with very narrow linewidths. The Kramers-Kronig expression usually turns up as a relation between the real and imaginary components of the frequency-dependent susceptibility in the theory of the refractive index of *linear* media (in which the polarization is strictly proportional to the incident electromagnetic field amplitude). In our context it turns up as a susceptibility relation due to the *nonlinear* part of the refractive index, since the quantity $\int d\zeta' h(\zeta-\zeta')\hat{\phi}^\dagger(\zeta')\hat{\phi}(\zeta')$ is effectively the Kerr component of the refractive index (as a function of the pulse profile) due to the "Raman" effect. By calculating the gain of a small signal in the presence of a strong quasi-cw pump field it is possible to show [4] that $\alpha(\Omega)$, the mode density function that we have introduced, can be identified with the "Raman" gain spectrum introduced by Gordon [20]. We note further that $h''(\bar{\omega})$ is an odd function and $h'(\bar{\omega})$ an even function.

M. The response function

A detailed summary of the properties and formulation of the response function is given in Ref. [4]. For that reason, only a brief review of the relevant points concerning $h(\tau)$ is presented here. As demonstrated in the previous section, we adopt the inhomogeneous gain model for the vibrational modes by introducing a continuum of vibrational angular frequencies instead of the discrete number which were indexed by the subscript ν. Thus we let $\bar{\omega}_\nu \to \Omega$, and introduce a density of modes function $\alpha(\Omega)$ for modes with an angular frequency near Ω. We also replace the linewidth $\bar{\gamma}_\nu$ of the modes uniformly with $\bar{\gamma}$ and allow this to tend to zero. In this case we can summarize the relevant properties of the response function in terms of $\alpha(\Omega)$:

$$h(\tau) \neq 0 \quad \text{only for } \tau > 0 \text{ (and decays as } \tau \to \infty), \quad (70a)$$

$$h(\tau) = \frac{1}{\pi}\int_0^\infty d\Omega\, \alpha(\Omega)\, \sin(\Omega\tau) \qquad (\tau > 0), \quad (70b)$$

$$\int_0^\infty d\tau\, h(\tau) = 1 - (\chi_E/\chi_T), \quad (70c)$$

$$h(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\bar{\omega}\, e^{-i\bar{\omega}\tau}\, \tilde{h}(\bar{\omega}), \quad (70d)$$

$$\begin{aligned}\sqrt{2\pi}\, \tilde{h}(\bar{\omega}) &= \int_0^\infty d\Omega\, \frac{\alpha(\Omega)\,\Omega}{\pi\,(\Omega^2 - \bar{\omega}^2)} + \frac{i\,\mathrm{sgn}(\bar{\omega})}{2}\,\alpha(|\bar{\omega}|) \quad (70e)\\ &\equiv \lim_{\bar{\gamma}\to 0} \frac{1}{\sqrt{2\pi}}\int_0^\infty d\Omega\, \frac{\alpha(\Omega)}{\pi}\left[\frac{\Omega}{(\bar{\gamma} - i\bar{\omega})^2 + \Omega^2}\right].\end{aligned}$$

The particular choice for $\alpha(\Omega)$ that we have made is one which can adequately model both the low and the high frequency sections of the experimentally observed gain. Its form is suggested by Eq. (59), which gives the response in the time domain for phonon linewidths of finite size — $\alpha(\Omega)$ is then proportional to the imaginary part of that response in the frequency domain. Thus we model $\alpha(\Omega)$ by

$$\alpha(\Omega) = 4\Omega \sum_{j=1}^{n} \frac{h_j\, \Omega_j\, \lambda_j}{[\lambda_j^2 - \Omega_j^2 + \Omega^2]^2 + [2\lambda_j\Omega_j]^2}. \quad (71)$$

Note that in this expression h_j, λ_j, and Ω_j are analogous, respectively, to the coupling strength, linewidth, and oscillation frequency of phonon modes in the homogeneous gain model. However, they appear here simply as fitting parameters in the inhomogeneous gain model. By using the expression for $\alpha(\Omega)$ given by (71) we can find $h(\bar{\omega})$ by doing a contour integral and taking the limit $\bar{\gamma} \to 0$. Thus from (70e)

$$\begin{aligned}\sqrt{2\pi}\, \tilde{h}(\bar{\omega}) &= \lim_{\bar{\gamma}\to 0} \frac{4}{\pi} \sum_{j=0}^{n} \int_0^\infty \frac{(h_j\Omega_j\lambda_j)\Omega^2 d\Omega}{\{[\lambda_j^2 - \Omega_j^2 + \Omega^2]^2 + [2\lambda_j\Omega_j]^2\}[(\bar{\gamma} - i\bar{\omega})^2 + \Omega^2]}\\ &= \sum_{j=0}^{n} \frac{h_j\, \Omega_j [(\lambda_j + i\bar{\omega})^2 + \Omega_j^2]}{[\lambda_j^2 - \Omega_j^2 + \bar{\omega}^2]^2 + [2\lambda_j\Omega_j]^2}.\end{aligned} \quad (72)$$

Thus the explicit forms for the real and imaginary parts of $\tilde{h}(\bar{\omega})$ are given by

$$\tilde{h}'(\bar{\omega}) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \frac{h_j\, \Omega_j\, [\lambda_j^2 + \Omega_j^2 - \bar{\omega}^2]}{[\lambda_j^2 - \Omega_j^2 + \bar{\omega}^2]^2 + [2\lambda_j\Omega_j]^2}, \quad (73a)$$

$$\tilde{h}''(\bar{\omega}) = \frac{2\bar{\omega}}{\sqrt{2\pi}} \sum_{j=0}^{n} \frac{h_j\, \Omega_j\, \lambda_j}{[\lambda_j^2 - \Omega_j^2 + \bar{\omega}^2]^2 + [2\lambda_j\Omega_j]^2}. \quad (73b)$$

The appropriate values for h_j, λ_j, and Ω_j are determined in Ref. [4]. However, since we have chosen a different set of comoving coordinates for the phase-space treatment, it is worth going over what these fitted parameters are.

In this paper where our *dimensionless* frequency variable is $\bar{\omega} = \omega t_0$, the physical (dimensional) parameters are obtained by dividing h_j, λ_j, and Ω_j by t_0. The parameters then obtained are *angular* frequencies with units of Trad/s. Particular values for these parameters, which specify the gain in the polarization *parallel* to the propagating electromagnetic field, can be found in Ref. [4] where, by trial and error, a direct fit to the experimental data for the high frequency portion of the gain curve, which defines Raman processes, was obtained using $j = 1, \ldots, 6$. The GAWBS results, for which the $j = 0$ contribution is reserved, will be considered separately and added in later (see Sec. III P). Assuming that these gain parameters are known, we could at this stage specify the response function in the time domain uniquely. It would have a form analogous to (59).

Note also that the fraction of the nonlinear response due to vibrational transitions in this model is given by

$$\frac{(\chi_G + \chi_R)}{\chi_T} = \sqrt{2\pi}\,\tilde{h}'(0) = \frac{1}{\pi}\int_0^\infty d\Omega \frac{\alpha(\Omega)}{\Omega} = \sum_{j=0}^{n} \frac{h_j\,\Omega_j}{[\lambda_j^2 + \Omega_j^2]}. \tag{74}$$

N. Behavior of the noise sources

Now we take a look at the correlation functions for the stochastic variables Γ, Γ^+. What we are interested in here is the behavior in both the time and frequency domains. The frequency domain results for $\Gamma_L, \Gamma_L^+, \Gamma_E, \Gamma_E^+$ are trivial:

$$\langle \tilde{\Gamma}_L(\zeta,\bar{\omega})\,\tilde{\Gamma}_L^+(\zeta',\bar{\omega}')\rangle = \frac{2\gamma n_{\rm th}^{\alpha}}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}'), \tag{75}$$

$$\langle \tilde{\Gamma}_E(\zeta,\bar{\omega})\,\tilde{\Gamma}_E(\zeta',\bar{\omega}')\rangle = \frac{1}{\bar{n}}\left[\frac{\chi_E}{\chi_T}\right]\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}'). \tag{76}$$

The result for the autocorrelation of Γ_V is obtained here by a sequence of steps. We begin with the expression (61c) and assume all linewidths to be uniformly given by $\bar{\gamma}'_\nu = \bar{\gamma}$,

$$\langle \tilde{\Gamma}_V(\zeta,\bar{\omega})\,\tilde{\Gamma}_V(\zeta',\bar{\omega}')\rangle = \frac{1}{2\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\sum_\nu h_\nu \left\{\left[\frac{1}{\bar{\gamma}+i(\bar{\omega}_\nu+\bar{\omega}')} + \frac{1}{\bar{\gamma}+i(\bar{\omega}_\nu-\bar{\omega}')}\right] + 2n_{\rm th}^{\nu}\left[\frac{\bar{\gamma}}{\bar{\gamma}^2+(\bar{\omega}_\nu+\bar{\omega}')^2} + \frac{\bar{\gamma}}{\bar{\gamma}^2+(\bar{\omega}_\nu-\bar{\omega}')^2}\right]\right\}. \tag{77}$$

Now we go over to a continuum of vibrational modes where we make the two replacements (i) $\bar{\omega}_\nu \to \Omega$ and (ii) $\sum_\nu h_\nu \to \int_0^\infty d\Omega\,\alpha(\Omega)/\pi$,

$$\begin{aligned}\langle \tilde{\Gamma}_V(\zeta,\bar{\omega})\,\tilde{\Gamma}_V(\zeta',\bar{\omega}')\rangle &= \frac{1}{2\pi\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\int_0^\infty d\Omega\,\alpha(\Omega)\left\{\frac{1}{\bar{\gamma}+i(\Omega+\bar{\omega}')} + \frac{1}{\bar{\gamma}+i(\Omega-\bar{\omega}')}\right.\\ &\quad\left. + 2n_{\rm th}(\Omega)\left[\frac{\bar{\gamma}}{\bar{\gamma}^2+(\Omega+\bar{\omega}')^2} + \frac{\bar{\gamma}}{\bar{\gamma}^2+(\Omega-\bar{\omega}')^2}\right]\right\}\\ &= \frac{1}{2\pi\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\sum_{j=1}^{4} I_j(\bar{\omega}'). \end{aligned}\tag{78}$$

Then we take the limit as $\bar{\gamma} \to 0$ for each of the I_j. Some care is necessary here. Strictly, for a given form of $\alpha(\Omega)$, the integrals should be carried out and then the limit taken. But it is possible to write down the general result in terms of $\alpha(\Omega)$ without doing this by using a seemingly obvious plausibility argument. First of all we rewrite I_1 as

$$I_1(\bar{\omega}') = \int_0^\infty d\Omega\,\alpha(\Omega)\left[\frac{\bar{\gamma}-i(\Omega+\bar{\omega}')}{\bar{\gamma}^2+(\Omega+\bar{\omega}')^2}\right]. \tag{79}$$

Now as $\bar{\gamma}$ becomes extremely small we may write this, depending on the sign of $\bar{\omega}'$, as

$$I_1(\bar{\omega}') = -i\int_0^\infty d\Omega\frac{\alpha(\Omega)}{\Omega+\bar{\omega}'} + \Theta(-\bar{\omega}')\,\alpha(-\bar{\omega}')\int_{-\infty}^{\infty} d\Omega\,\frac{\bar{\gamma}}{\bar{\gamma}^2+(\Omega+\bar{\omega}')^2}. \tag{80}$$

The Θ function is unity for positive arguments and zero elsewhere, i.e., $\Theta(x) = 1 \quad (x > 0)$. Likewise for I_2 we have

$$I_2(\bar\omega') = \int_0^\infty d\Omega\, \alpha(\Omega) \left[\frac{\bar\gamma - i(\Omega - \bar\omega')}{\bar\gamma^2 + (\Omega - \bar\omega')^2} \right]$$
$$= -i \int_0^\infty d\Omega\, \frac{\alpha(\Omega)}{\Omega - \bar\omega'}$$
$$+ \Theta(\bar\omega')\, \alpha(\bar\omega') \int_{-\infty}^\infty d\Omega\, \frac{\bar\gamma}{\bar\gamma^2 + (\Omega - \bar\omega')^2}. \quad (81)$$

The integrals containing $\bar\gamma$ as the parameter defining a Lorentzian line shape thus contribute a factor of π as $\bar\gamma$ goes to zero. Note that we may take the lower limit of the $\bar\gamma$ integrals to $-\infty$ without introducing any error here. In this way I_1 and I_2 can be combined to give

$$I_1(\bar\omega') + I_2(\bar\omega') = \pi\, \alpha(|\bar\omega'|) - 2i \int_0^\infty d\Omega \frac{\alpha(\Omega)\, \Omega}{\Omega^2 - (\bar\omega')^2}. \quad (82)$$

In the same manner one may straightforwardly show the contribution to I_3 and I_4 to be $2\pi n_{\rm th}(|\bar\omega|)\alpha(|\bar\omega|)$, so that the general expression for the correlation of $\tilde\Gamma_V$ with itself in the Fourier domain is

$$\langle \tilde\Gamma_V(\zeta,\bar\omega)\, \tilde\Gamma_V(\zeta',\bar\omega') \rangle = \frac{1}{\bar n}\, \delta(\zeta - \zeta')\, \delta(\bar\omega + \bar\omega') \left\{ [n_{\rm th}(|\bar\omega|) + 1/2]\, \alpha(|\bar\omega|) \;-\; \frac{i}{\pi} \int_0^\infty d\Omega\, \frac{\alpha(\Omega)\, \Omega}{\Omega^2 - \bar\omega^2} \right\} \quad (83a)$$
$$= \frac{1}{\bar n}\, \delta(\zeta - \zeta')\, \delta(\bar\omega + \bar\omega') \left\{ [n_{\rm th}(|\bar\omega|) + \Theta(-\bar\omega)]\, \alpha(|\bar\omega|) \;-\; i\sqrt{2\pi}\, \tilde h(\bar\omega) \right\}. \quad (83b)$$

Next we need to find the cross correlation between $\tilde\Gamma_V$ and $\tilde\Gamma_V^+$. The same sequence of steps is carried out here as for the autocorrelation. Thus, beginning with (61d),

$$\langle \tilde\Gamma_V(\zeta,\bar\omega)\, \tilde\Gamma_V^+(\zeta',\bar\omega') \rangle = \frac{1}{2\bar n}\, \delta(\zeta - \zeta')\, \delta(\bar\omega + \bar\omega') \sum_\nu h_\nu \left\{ \frac{2\bar\gamma(n_{\rm th}^\nu + 1)}{\bar\gamma^2 + (\bar\omega_\nu - \bar\omega')^2} + \frac{2\bar\gamma n_{\rm th}^\nu}{\bar\gamma^2 + (\bar\omega_\nu + \bar\omega')^2} \right\}$$
$$\to \frac{1}{\pi\bar n}\, \delta(\zeta - \zeta')\, \delta(\bar\omega + \bar\omega') \int_0^\infty d\Omega\, \alpha(\Omega) \left\{ \frac{\bar\gamma[n_{\rm th}(\Omega) + 1]}{\bar\gamma^2 + (\Omega + \bar\omega)^2} + \frac{\bar\gamma n_{\rm th}(\Omega)}{\bar\gamma^2 + (\Omega - \bar\omega)^2} \right\}. \quad (84)$$

As $\bar\gamma \to 0$ this correlation function behaves differently depending on whether $\bar\omega'$ (or $\bar\omega$) is positive or negative since either one or the other Lorentzian contributes, but not both, giving

$$\int_0^\infty d\Omega\, \alpha(\Omega) \left\{ \frac{\bar\gamma[n_{\rm th}(\Omega) + 1]}{\bar\gamma^2 + (\Omega + \bar\omega)^2} + \frac{\bar\gamma n_{\rm th}(\Omega)}{\bar\gamma^2 + (\Omega - \bar\omega)^2} \right\}$$
$$= \alpha(|\bar\omega|)\, [n_{\rm th}(|\bar\omega|) + \Theta(-\bar\omega)] \int_{-\infty}^\infty d\Omega\, \frac{\bar\gamma}{\bar\gamma^2 + (\Omega - |\bar\omega|)^2}. \quad (85)$$

The integrated Lorentzians have a value of π and so finally

$$\langle \tilde\Gamma_V(\zeta,\bar\omega)\, \tilde\Gamma_V^+(\zeta',\bar\omega') \rangle = \frac{1}{\bar n}\, \delta(\zeta - \zeta')\, \delta(\bar\omega + \bar\omega') \times [n_{\rm th}(|\bar\omega|) + \Theta(-\bar\omega)]\, \alpha(|\bar\omega|). \quad (86)$$

O. Noise correlations in the time domain

Let us now return to the time domain by inverse Fourier transforming the results of the last section. Then we can check the result with that obtained by beginning with the original (finite $\bar\gamma$) time correlations, given by (61a) and (61b), and directly taking the limit as the linewidth $\bar\gamma \to 0$. If the expressions derived in the frequency domain are correct then the results for the time domain will agree for the two methods. Thus inverse Fourier transformation of the autocorrelation (83a) gives

$$\langle \Gamma_V(\zeta,\tau)\, \Gamma_V(\zeta',\tau') \rangle$$
$$= \frac{\delta(\zeta - \zeta')}{2\pi\bar n} \int_{-\infty}^\infty d\bar\omega\, e^{-i\bar\omega(\tau - \tau')} \left\{ [n_{\rm th}(|\bar\omega|) + 1/2]\, \alpha(|\bar\omega|) - \frac{i}{\pi} \int_0^\infty d\Omega \frac{\alpha(\Omega)\, \Omega}{(\Omega^2 - \bar\omega^2)} \right\} \quad (87)$$

or

$$\langle \Gamma_V(\zeta,\tau)\, \Gamma_V(\zeta',\tau') \rangle$$
$$= \frac{\delta(\zeta - \zeta')}{2\pi\bar n} \left\{ 2 \int_0^\infty d\Omega\, \alpha(\Omega)\, n_{\rm th}(\Omega) \cos[\Omega(\tau - \tau')] + \int_0^\infty d\Omega\, e^{-i\Omega|\tau - \tau'|} \alpha(\Omega) \right\}. \quad (88)$$

Note that for the last term in (87) we have made use of the exact definition of this term,

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}d\bar{\omega}\,e^{-i\bar{\omega}\tau}\int_{0}^{\infty}d\Omega\frac{\alpha(\Omega)\Omega}{\Omega^2-\bar{\omega}^2}$$

$$\equiv\int_{-\infty}^{\infty}d\bar{\omega}\,e^{-i\bar{\omega}\tau}\Bigg\{-\frac{1}{2}\,\mathrm{sgn}(\bar{\omega})\,\alpha(|\bar{\omega}|)-\frac{i}{\pi}\lim_{\gamma\to 0}\int_{0}^{\infty}d\Omega\frac{\alpha(\Omega)\Omega}{[(\gamma-i\bar{\omega})^2+\Omega^2]}\Bigg\}. \quad (89)$$

This last equality allows us to exchange the order of integration in (87) and carry out a straightforward contour integral which gives (88). We note that by beginning with (61a), the correlation in the time domain, the same result is obtained rather quickly. For the generic fit to $\alpha(\Omega)$ that we have chosen, i.e., (71), it is possible to calculate the temporal correlation function explicitly. To do this we rewrite the thermal phonon number using the following identity:

$$n_{\rm th}(\Omega)=\frac{1}{e^{\Delta\Omega}-1}=-\frac{1}{2}+\frac{\Delta\Omega}{4\pi^2}\sum_{m=-\infty}^{\infty}\frac{1}{m^2+\left[\frac{\Delta\Omega}{2\pi}\right]^2}. \quad (90)$$

Substituting this into (88), with (71) for $\alpha(\Omega)$, we obtain the following result after performing contour integrations:

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V(\zeta',\tau')\rangle=\frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\sum_{j=0}^{n}\Bigg\{-\frac{ih_j}{2}\,e^{\lambda_j|\tau-\tau'|}\sin[\Omega_j|\tau-\tau'|]-\frac{4\pi h_j\lambda_j\Omega_j}{\Delta^2}B^j(\tau-\tau',\Delta)$$
$$+\frac{h_j\Omega_j}{\Delta}\,C^j_+(\Delta)\,e^{\lambda_j|\tau-\tau'|}\cos[\Omega_j(\tau-\tau')]+\frac{h_j\lambda_j}{\Delta}\,C^j_-(\Delta)\,e^{\lambda_j|\tau-\tau'|}\sin[\Omega_j|\tau-\tau'|]\Bigg\}. \quad (91)$$

As a consequence of the expansion (90), the coefficients $C^j_\pm(\Delta)$ and the function $B^j(\tau-\tau',\Delta)$, which depend on the thermal parameter $\Delta\equiv\hbar/kTt_0$, are expressed in terms of infinite series as

$$B^j(\tau,\Delta)\equiv\sum_{m=-\infty}^{\infty}\frac{|m|\,e^{-2\pi|m\tau|/\Delta}}{\left[\lambda_j^2-\Omega_j^2-(2\pi m/\Delta)^2\right]^2+[2\lambda_j\Omega_j]^2}, \quad (92)$$

$$C^j_\pm(\Delta)\equiv\sum_{m=-\infty}^{\infty}\frac{\lambda_j^2+\Omega_j^2\pm(2\pi m/\Delta)^2}{\left[\lambda_j^2-\Omega_j^2-(2\pi m/\Delta)^2\right]^2+[2\lambda_j\Omega_j]^2}. \quad (93)$$

In general these series require numerical summation, but are highly convergent for all but cryogenic temperatures.

Employing the same procedure for the cross correlation, the Fourier transform of (86) gives

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V^+(\zeta',\tau')\rangle=\frac{\delta(\zeta-\zeta')}{2\pi\bar{n}}\int_{-\infty}^{\infty}d\bar{\omega}\,e^{-i\bar{\omega}(\tau-\tau')}\,\alpha(|\bar{\omega}|)\times[n_{\rm th}(|\bar{\omega}|)+\Theta(-\bar{\omega})], \quad (94)$$

or

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V^+(\zeta',\tau')\rangle=\frac{\delta(\zeta-\zeta')}{2\pi\bar{n}}\Bigg\{2\int_0^{\infty}d\Omega\,\alpha(\Omega)\,n_{\rm th}(\Omega)\cos[\Omega(\tau-\tau')]+\int_0^{\infty}d\Omega\,e^{i\Omega(\tau-\tau')}\alpha(\Omega)\Bigg\}. \quad (95)$$

This is also in agreement with the answer obtained by beginning directly in the time domain with (61b) and letting $\bar{\gamma}\to 0$. Performing direct integration of this function with the same substitutions that led to (91) leads to

$$\langle\Gamma_V(\zeta,\tau)\,\Gamma_V^+(\zeta',\tau')\rangle=\frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\sum_{j=0}^{n}\Bigg\{+\frac{ih_j}{2}\,e^{\lambda_j|\tau-\tau'|}\sin[\Omega_j(\tau-\tau')]-\frac{4\pi h_j\lambda_j\Omega_j}{\Delta^2}B^j(\tau-\tau',\Delta)$$
$$+\frac{h_j\Omega_j}{\Delta}\,C^j_+(\Delta)\,e^{\lambda_j|\tau-\tau'|}\cos[\Omega_j(\tau-\tau')]+\frac{h_j\lambda_j}{\Delta}\,C^j_-(\Delta)\,e^{\lambda_j|\tau-\tau'|}\sin[\Omega_j|\tau-\tau'|]\Bigg\}. \quad (96)$$

Note that this is different from the autocorrelation (91) only in the imaginary part, where the result now depends on the sign of $(\tau-\tau')$.

We now have the correct behavior of the stochastic variables in both the time and frequency representation. This is important if we wish to do calculations analytically in certain simplifying cases, or model the noise sources numerically for the more general case when this is the only option available to us.

P. GAWBS correlations

Our treatment of the excess noise has so far been quite general and includes both the Raman spectral features relevant at high frequencies of the order of THz and the low frequency noise, on the GHz scale, known as guided acoustic wave Brillouin scattering. In this section we consider exclusively the GAWBS component of the noise since this is distinct from the Raman contribution (GAWBS has been discussed thoroughly in earlier references [4,14,18]). For completeness we include here the same model that we adopted in [4] and write down the equivalent GAWBS correlation functions for the positive-P representation, and then later the Wigner representation also.

Before we consider the time or frequency behavior of the low frequency noise, we should perhaps address the question of the validity of the Dirac δ-function approximation for the spatial correlation functions that we have assumed for the phonon modes to this point. Certainly for the high frequency Raman modes of oscillation, which can be highly localized in space and contain as little as a few dozen neighboring atoms, the assumption of independent thermal phonon noise at different spatial locations seems a reasonable one.

The question arises as to whether this approximation also holds good for the GAWBS modes, which represent the collective motion of vast numbers of atoms on the scale of the fiber diameter. Currently we have no available experimental information to suggest what the correlation length l_G for this noise should be set at. A rough estimate based on a coherence time suggested by the linewidth of a typical GAWBS mode ($\sim 10^6\,\mathrm{s}^{-1}$) and the group velocity of sound along the fiber (say $\sim 10^3$ m/s) would give $l_G \simeq 1$ mm. As this length is much much smaller than typical pulse reshaping lengths in the fiber (of the order of meters) the approximation of independent thermal noise at different spatial locations along the fiber would also appear to be justifiable in the low frequency GAWBS region. That the GAWBS correlation length l_G is at least known to be much shorter than typical fiber lengths of the order of a meter or more is established from the observation that GAWBS noise power spectra scale directly with the fiber length, and not, say, with the square of the length as would be the case if l_G was of the order of the fiber length itself [21]. For these reasons we assume here that the approximation $l_G = 0$ is not a serious simplification of the theory.

The model for the low frequency portion of the gain spectrum that we have chosen is obtained from the general fitting function (71) by setting $\lambda_0 = \Omega_0$,

$$\alpha_G(\bar\omega) = \frac{4h_0\lambda_0^2\bar\omega}{(4\lambda_0^4 + \bar\omega^4)}. \tag{97}$$

This section of the gain rises linearly and then tails off to zero rapidly at about 1 GHz or so. More precisely, the GAWBS bandwidth is given [4] approximately by $3\lambda_0/2$ where $\lambda_0/t_0 \simeq (0.81\text{ GHz}) \times 2\pi$ rad. The effective coupling strength of the noise is given by the parameter $h_0/t_0 \simeq (0.132\text{ GHz}) \times 2\pi$ rad. These parameters are for typical silica fibers with a cladding diameter $\sim 125\ \mu$m and a core diameter which supports a beam waist for the fundamental transverse mode $\sim 4\ \mu$m. Of course, for the positive-P equations, there are two different correlation functions of interest, expressed in either the time or frequency domains. The first of these (given in the time domain) is found by taking $\alpha(\Omega) = \alpha_G(\Omega)$ in (88),

$$\begin{aligned}\langle\Gamma_G(\zeta,\tau)\,\Gamma_G(\Gamma',\tau')\rangle &= \frac{1}{2\pi\bar n}\delta(\zeta-\zeta')\int_0^\infty d\Omega\,\frac{4h_0\lambda_0^2\Omega}{(4\lambda_0^4+\Omega^4)}\left\{\frac{2}{(e^{\Delta\Omega}-1)}\cos[\Omega(\tau-\tau')]+e^{-i\Omega|\tau-\tau'|}\right\}\\ &\approx \frac{h_0}{2\bar n}\delta(\zeta-\zeta')\,e^{-\lambda_0|\tau-\tau'|}\left\{\frac{1}{\lambda_0\Delta}\Big(\cos[\lambda_0(\tau-\tau')]+\sin[\lambda_0|\tau-\tau'|]\Big)-i\sin[\lambda_0|\tau-\tau'|]\right\}.\end{aligned} \tag{98}$$

The thermal phonon number has been approximated above using

$$n_{\rm th}(\Omega) = \frac{1}{(e^{\Delta\Omega}-1)} \approx \frac{1}{\Delta\Omega} - \frac{1}{2}. \tag{99}$$

The equivalent Fourier expression for this correlation function, given by (83b) with $\alpha_G(\Omega)$ as the gain function, is

$$\begin{aligned}\langle\tilde\Gamma_G(\zeta,\bar\omega)\,\tilde\Gamma_G(\zeta',\bar\omega')\rangle &= \frac{1}{\bar n}\,\delta(\zeta-\zeta')\,\delta(\bar\omega+\bar\omega')\left\{[n_{\rm th}(|\bar\omega|)+\Theta(-\bar\omega)]\,\alpha_G(|\bar\omega|)\;-\frac{i}{\pi}\lim_{\gamma\to0}\int_0^\infty\frac{\alpha_G(\Omega)\,\Omega\,d\Omega}{[(\gamma-i\bar\omega)^2+\Omega^2]}\right\}\\ &= \frac{1}{\bar n}\,\delta(\zeta-\zeta')\,\delta(\bar\omega+\bar\omega')\left\{[n_{\rm th}(|\bar\omega|)+1/2]\,\alpha_G(|\bar\omega|)\;+ih_0\lambda_0\,\frac{(\bar\omega^2-2\lambda_0^2)}{(\bar\omega^4+4\lambda_0^4)}\right\}\\ &\approx \frac{h_0\lambda_0}{\bar n}\,\delta(\zeta-\zeta')\,\delta(\bar\omega+\bar\omega')\left\{\frac{4\lambda_0/\Delta+i(\bar\omega^2-2\lambda_0^2)}{4\lambda_0^4+\bar\omega^4}\right\}.\end{aligned} \tag{100}$$

Similarly we find the cross-correlation functions, beginning with (95),

$$\langle\Gamma_G(\zeta,\tau)\,\Gamma_G^+(\zeta',\tau')\rangle = \frac{1}{2\pi\bar{n}}\,\delta(\zeta-\zeta')\int_0^\infty d\Omega\,\frac{4h_0\lambda_0^2\Omega}{(4\lambda_0^4+\Omega^4)}\left\{\frac{2}{(e^{\Delta\Omega}-1)}\cos[\Omega(\tau-\tau')]+e^{i\Omega(\tau-\tau')}\right\}$$
$$\approx \frac{h_0}{2\bar{n}}\,\delta(\zeta-\zeta')\,e^{-\lambda_0|\tau-\tau'|}\left\{\frac{1}{\lambda_0\Delta}\Big(\cos[\lambda_0(\tau-\tau')]+\sin[\lambda_0|\tau-\tau'|]\Big)+i\sin[\lambda_0(\tau-\tau')]\right\}, \quad (101)$$

and then for (86)

$$\langle\tilde{\Gamma}_G(\zeta,\bar{\omega})\,\tilde{\Gamma}_G^+(\zeta',\bar{\omega}')\rangle = \frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\tilde{\omega}+\tilde{\omega}')[n_{\mathrm{th}}(|\bar{\omega}|)+\Theta(-\bar{\omega})]\,\alpha_G(|\bar{\omega}|)$$
$$\approx \frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\,\frac{4h_0\lambda_0^2}{(4\lambda_0^4+\bar{\omega}^4)}\left[\frac{1}{\Delta}-\frac{\bar{\omega}}{2}\right]. \quad (102)$$

All of these correlation functions could in fact have been obtained from the more general expressions of the previous section by realizing that the thermal approximation (99) is equivalent to taking only the $m=0$ term in the thermal expansion (90). Note that these GAWBS correlation functions do not consist only of a pure thermal term, scaling as $1/\Delta \sim T$, but have a temperature-independent component which has to be viewed as manifestly quantum mechanical in origin. This brings us to the end of our consideration of the problem from the point of view of the positive-P representation.

IV. THE WIGNER METHOD

In this part of the paper we turn to the question of whether the Wigner distribution function can be used as an alternative to our formulation based upon the positive-P function. Obviously, since the physical results are independent of the representation chosen, an exact employment of these two methods must give identical answers. However, it turns out that approximations have to be made when attempting to generate simple mathematical solutions from the Wigner representation. Thus the two methods in fact *may* give rise to differing predictions. Also one method might be favored over the other depending on the sort of problem we are trying to solve. It is the aim here to compare and contrast these two representations in terms of the possibility of utilizing stochastic equations.

In the case of the Wigner distribution, this representation is well known in quantum theory and is used by many workers. As can be inferred from the relatively simple appearance of the resulting equations in this representation, this method has advantages due to the reduced dimensionality of the Wigner phase space compared to the positive-P phase space. However, when attempting to derive stochastic equations from the equation of motion for the Wigner distribution we find that the equation is itself *not* a true Fokker-Planck equation. This can be remedied in a brute-handed fashion by simply truncating the offending higher-order derivative terms in the corresponding "Fokker-Planck-like" equation (that is to say, terms higher than second order in the field derivatives). As we shall see, this truncation procedure does generate viable stochastic equations, but at the expense of accuracy in the long-term evolution of the fields. In particular, the truncated Wigner equations do not contain the necessary information to predict the correct behavior of third- (and higher-) order correlation functions. But for large photon number and relatively short propagation distances, the truncation does not affect calculations dependent upon second-order correlation functions (as would be the case when one considered squeezing experiments).

We note that the conditions for the reliability of the Wigner method are basically those for which linearization of the Heisenberg equations is valid. Thus, in contrast to the positive-P treatment, the Wigner method is no more general than the Heisenberg method, although it does offer an alternative treatment which allows for direct numerical simulation of the corresponding field equations.

The use of a truncated Wigner equation was first demonstrated by Graham [22] in 1973 to predict the tunneling rate between the two above-threshold steady states of the degenerate parametric oscillator (DPO) due to quantum noise. The truncated Wigner analysis presented in this paper was first carried out by Drummond and Hardman [13]. However, an exact derivation of this result is given here, which points out the existence of a frequency renormalization term in the equations which was previously overlooked. In the limit as the spatial cell size is taken to zero this renormalization term cannot be neglected. Indeed, it becomes infinite in this limit, compensating for the infinite vacuum noise which enters the Wigner problem from each of the (infinitely many) frequency modes.

A. The Wigner function

Just as it was for the positive-P case, the Wigner representation of the density operator also leads to correspondences between raising and lowering operators acting on the density operator and differential operators acting on the Wigner function itself. But before we can write down these correspondences we first need a definition of the Wigner function.

For this, we begin with a representation in terms of the quantum characteristic function, as given for example in

Gardiner's *Quantum Noise* ([23], Sec. 4.4.4). Then we shall transform this expression in different ways in order to bring out various features of the Wigner function.

Thus we begin with the following definition for $W(\alpha,\alpha^*)$, the Wigner function defined over the complex phase space of the variable α:

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda\, e^{-\lambda\alpha^*+\lambda^*\alpha}\,\mathrm{Tr}\left\{\hat{\rho}e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}\right\}. \quad (103)$$

The trace is over some complete set of basis states; we shall specify three different types of trace in order to bring out one aspect or another of W in this section and the next. To begin with, we can represent the density operator in (103) by the positive-P function as defined by (7). Then by taking a trace over the coherent-state basis we can show how W is related to the positive-P distribution used in the first part of the paper. When we do this the appearance of W changes to

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int\int P(\beta,\beta^+)\, e^{-2(\alpha-\beta)(\alpha^*-\beta^+)}d^2\beta\, d^2\beta^+. \quad (104)$$

We can see directly from this result that the Wigner function is obtained from the positive-P function by a complex Gaussian convolution. Thus W is always less singular, or more smeared out, than the corresponding P result. For example, if the field is initially in a coherent state, i.e., $\hat{\rho} = |\alpha_c\rangle\langle\alpha_c|$, we have for the two distributions

$$P(\alpha,\alpha^+) = \delta^2(\alpha-\alpha_c)\,\delta^2(\alpha^+-\alpha_c^*) \quad (105)$$

and

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\,e^{-2|\alpha-\alpha_c|^2}. \quad (106)$$

We shall see that this fuzziness of the Wigner function arises because it is the distribution most suited to calculating *symmetrically ordered* moments of the field operators $\hat{a}$ and $\hat{a}^\dagger$. These symmetrically ordered moments include the vacuum fluctuations (responsible for the extra width of the distribution), in contrast to the P function which is associated with normally ordered moments in which the vacuum has been factored out.

To manipulate the Wigner function it is useful to have a list of operator correspondences similar to those described for the P function earlier. The easiest way to find these correspondences is to work with a form for the Wigner function which brings out most clearly the dependence of the function on the real and imaginary components of the variable α. Thus we shall write the annihilation and creation operators ($\hat{a}$ and $\hat{a}^\dagger$, respectively) in the following way:

$$\hat{a} = \hat{x} + i\hat{p} \quad \text{and} \quad \hat{a}^\dagger = \hat{x} - i\hat{p}, \quad (107)$$

so that $\hat{x}$ and $\hat{p}$ are the noncommuting Hermitian quadrature operators with corresponding commutator $[\hat{x},\hat{p}] = i/2$.

If the numbers x and p represent the results of a measurement of the variables corresponding to the operators $\hat{x}$ and $\hat{p}$ then we can reexpress the definition (103) for $W(\alpha,\alpha^*)$ in the form $W(x,p)$. We do this by performing the trace with the state vectors $|x\rangle$ in the "position" basis. These are the eigenstates of the operator $\hat{x}$. They are related to the eigenstates of the "momentum" operator $\hat{p}$ by

$$|x\rangle = \frac{1}{\sqrt{\pi}}\int dp\, e^{-2ixp}\,|p\rangle \quad (108)$$

and

$$|p\rangle = \frac{1}{\sqrt{\pi}}\int dx\, e^{2ixp}\,|x\rangle. \quad (109)$$

This relationship follows from the commutator for $\hat{x}$ and $\hat{p}$, as discussed in Ref. [24], Sec. 23. The momentum operator $\hat{p}$ can also be expressed as $\hat{p} = -(i/2)\partial/\partial x$. By using these results it is straightforward to derive the following expression for the Wigner function in terms of the position basis:

$$W(x,p) = \frac{2}{\pi}\int dy\langle x+y|\hat{\rho}|\,x-y\rangle e^{-4iyp}. \quad (110)$$

This form for $W(x,p)$ is close to the original distribution function introduced by Wigner [16].

B. Operator correspondences

Now we can state what we mean by operator correspondences for W. If, for instance, we replace $\hat{\rho}$ by $\hat{x}\hat{\rho}$ in expression (110) for $W(x,p)$, it is easy to show that the result is equivalent to the expression

$$\left(x + \frac{i}{4}\frac{\partial}{\partial p}\right)W(x,p). \quad (111)$$

In similar fashion we get results for the operator $\hat{p}$ by making use of the relations (108) and (109), rewriting W in terms of a trace over the momentum states when convenient. The complete table of correspondences for $\hat{x}$ and $\hat{p}$ is given by

$$\hat{x}\hat{\rho} \rightarrow \left(x + \frac{i}{4}\frac{\partial}{\partial p}\right)W(x,p), \quad (112a)$$

$$\hat{p}\hat{\rho} \rightarrow \left(p - \frac{i}{4}\frac{\partial}{\partial x}\right)W(x,p), \quad (112b)$$

$$\hat{\rho}\hat{x} \rightarrow \left(x - \frac{i}{4}\frac{\partial}{\partial p}\right)W(x,p), \quad (112c)$$

$$\hat{\rho}\hat{p} \rightarrow \left(p + \frac{i}{4}\frac{\partial}{\partial x}\right)W(x,p). \quad (112d)$$

In practice, however, what we deal with usually is a master equation composed of various powers of the operators $\hat{a}$ and $\hat{a}^\dagger$. Now, since the eigenvalue of the operator $\hat{a} = \hat{x} + i\hat{p}$ is $\alpha = x + ip$ then the following is true from standard complex analysis:

$$\frac{\partial}{\partial\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial p}\right),\quad \frac{\partial}{\partial\alpha^*}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial p}\right). \tag{113}$$

Combining this with the correspondences above in terms of x and p, we get [in terms of α and α^*, and remembering also that $W(x,p)\equiv W(\alpha,\alpha^*)$]

$$\hat{a}\hat{\rho}\rightarrow\left(\alpha+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha,\alpha^*), \tag{114a}$$

$$\hat{a}^\dagger\hat{\rho}\rightarrow\left(\alpha^*-\frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha,\alpha^*), \tag{114b}$$

$$\hat{\rho}\hat{a}^\dagger\rightarrow\left(\alpha^*+\frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha,\alpha^*), \tag{114c}$$

$$\hat{\rho}\hat{a}\rightarrow\left(\alpha-\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha,\alpha^*). \tag{114d}$$

Now we have a set of operator correspondences which enable us to write down the evolution equation for $W(\alpha,\alpha^*)$ directly from the equation for $\partial\hat{\rho}/\partial t$. This is the route we shall take. Another approach might be to begin with the evolution equation for $P(\alpha,\alpha^+)$ that was obtained in the first part of the paper; an equation for $W(\alpha,\alpha^*)$ can then be generated by making use of expression (104) which relates the Wigner function to the equivalent positive-P result.

Finally, notice that in the above operator–c number correspondences, there is always a derivative term on the right-hand side connected with W. This is in contrast to the positive-P case in which derivative terms are absent in the first and third correspondences of (10). This means that the order of the derivative terms appearing in the equation of motion for the Wigner distribution will generally be higher than that for the equivalent evolution equation for the positive-P distribution. In particular, if the latter is a true Fokker-Planck equation (i.e., second order in the derivative terms) the former will not be.

C. Example

As an example let us consider a relatively trivial Hamiltonian which includes a $\chi^{(3)}$ type nonlinearity plus the free field, namely $\hat{H}=\hat{H}_0+\hat{H}_I$ where

$$\hat{H}_0=\hbar\omega_0\,\hat{a}^\dagger\hat{a}\quad\text{and}\quad\hat{H}_I=-\hbar\chi\,\hat{a}^{\dagger 2}\hat{a}^2. \tag{115}$$

We shall apply the Wigner formalism to this example just to see how it works. This will be useful since it will allow us to identify the main features of the method without having to worry about the sort of complications which arise in the more realistic Raman Hamiltonian and to which we will return later. Since we can remove the free-field part of $\hat{H}$ by moving to an interaction picture, the density equation associated with (115) becomes

$$\frac{\partial\hat{\rho}}{\partial t}=\frac{-i}{\hbar}[\hat{H}_I,\hat{\rho}]=i\chi\left[\hat{a}^{\dagger 2}\hat{a}^2\hat{\rho}-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2\right], \tag{116}$$

which, after using the operator correspondences given above in (114) results in

$$\begin{aligned}\frac{\partial W(\alpha,\alpha^*)}{\partial t}&=i\chi\left[\left(\alpha^*-\frac{1}{2}\frac{\partial}{\partial\alpha}\right)^2\left(\alpha+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)^2-\left(\alpha-\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)^2\left(\alpha^*+\frac{1}{2}\frac{\partial}{\partial\alpha}\right)^2\right]W(\alpha,\alpha^*)\\&=2i\chi\left[\frac{\partial}{\partial\alpha^*}(\alpha^*\alpha-1)\alpha^*-\frac{\partial}{\partial\alpha}(\alpha^*\alpha-1)\alpha+\frac{1}{4}\frac{\partial^3}{\partial\alpha^*\partial\alpha^2}\alpha-\frac{1}{4}\frac{\partial^3}{\partial\alpha^{*2}\partial\alpha}\alpha^*\right]W(\alpha,\alpha^*).\end{aligned} \tag{117}$$

D. Approximate Wigner stochastic equations

The final equation obtained in the last section has (except for the third-order derivative terms) the appearance of a Fokker-Planck equation with zero coefficients for the diffusive terms. For intense fields these third-order terms may be neglected for the early development of the Wigner function W. This is because they are small in comparison to the first-order terms—at least to begin with. The main reason for this is that the first derivative terms scale roughly with the mean photon number while the third-order derivative terms scale inversely with the same quantity. This certainly suggests that truncation of the higher-order terms for short times would be justifiable. However, for much longer time scales significant quantum correlations that would ordinarily be carried by these third derivative terms can cause appreciable error. For instance, the truncation of third-order terms means that if the Wigner function starts off as a positive function then it will remain that way. However, it is often the case that in order for the full quantum character of a system to manifest itself the Wigner function is required to take on negative values at times. This cannot happen with truncation. The process of truncation must thus be viewed from the start as one which is inevitably approximate at some stage. Nevertheless, it may be a good approximation for long enough times to justify its consideration as an alternative calculational method to the positive-P treatment. In fact, as has been shown by direct numerical comparison of the two methods [13], the Wigner representation can generate stochastic ensembles with a sampling error significantly smaller than that for the corresponding positive-P ensemble, particularly for cases involving highly nonclassical light.

At this stage it is worth discussing how the Wigner function is related to moments of the field. Whereas for the case of the positive-P function, in which normally ordered moments of the field operators are represented by similarly appearing moments of the complex field variables, the moments taken over W represent *symmetrically ordered* combinations of the field operators. For instance, if the subscript "sym" stands for symmetric ordering, then we have (as an example) the following symmetric orderings rewritten in terms of normal order:

$$\{\hat{a}^\dagger\hat{a}^2\}_{\text{sym}} = \frac{1}{3}\{\hat{a}^\dagger\hat{a}^2 + \hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^2\hat{a}^\dagger\} = \hat{a}^\dagger\hat{a}^2 + \hat{a}, \tag{118}$$

where the factor of three is due to averaging over the three possible orderings of the operators, as explained in Ref. [23], Eq. 4.4.63. Likewise,

$$\{\hat{a}^\dagger\hat{a}\}_{\text{sym}} = \frac{1}{2}\{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\} = \hat{a}^\dagger\hat{a} + \frac{1}{2}. \tag{119}$$

Field moments in the Wigner representation are thus given by

$$\begin{aligned}\langle\{\hat{a}^n(\hat{a}^\dagger)^m\}_{\text{sym}}\rangle &= \int d^2\alpha\, \alpha^n(\alpha^*)^m\, W(\alpha,\alpha^*)\\ &\equiv \langle\alpha^n(\alpha^*)^m\rangle_W.\end{aligned} \tag{120}$$

The proof of this relation is most easily given by beginning with the Wigner function expressed in terms of its characteristic function, as in (103). This question is treated in Ref. [23], Sec. 4.4.4. Thus $W(\alpha,\alpha^*)$ is a quasiprobability function. That it is real valued may be seen by taking the conjugate of (110). However, as noted before, it is not positive definite in general. It may assume negative values.

Stochastic equations in α may be derived which are equivalent to the truncated form of the equation for $W(\alpha,\alpha^*)$, where the third-order derivative terms have been neglected. Thus if we take $W_T(\alpha,\alpha^*)$ to be the truncated Wigner distribution for the example Hamiltonian of the previous section, we have

$$\frac{\partial W_T}{\partial t} = 2i\chi\left[\frac{\partial}{\partial\alpha^*}(\alpha^*\alpha - 1)\alpha^* - \frac{\partial}{\partial\alpha}(\alpha^*\alpha - 1)\alpha\right]W_T, \tag{121}$$

from which the equivalent Ito equation for α is just

$$\frac{\partial\alpha}{\partial t} = 2i\chi(\alpha^*\alpha - 1)\alpha. \tag{122}$$

Note the factor of -1 in the intensity term here. This is due to the symmetric ordering and represents a correction to the Wigner-averaged estimate of the intensity of the field, which includes vacuum fluctuations which are not present in a normally ordered treatment (as for the positive-P method considered earlier in this paper). This field equation is lacking an explicit noise term. However, a stochastic element arises due to the initial conditions α must satisfy, since the initial distribution function is in general smeared out. Whereas for the P representations a coherent state is represented by a singular distribution function with no spread, the Wigner distribution for a coherent state exists on an extended region in phase space because it is, as previously mentioned, a complex Gaussian convolution of the positive-P representation. If the field is in a coherent state specified by α_c, and $\delta\alpha$ represents the initial displacement from α_c (at $t = 0$) for one member of a Wigner ensemble of trajectories governed by (122), then

$$\langle(\delta\alpha)^2\rangle_W = 0\,,\quad \langle|\delta\alpha|^2\rangle_W = 1/2, \tag{123}$$

where the subscript W indicates that these averages have the meaning defined in (120). These conditions (123) on the input fluctuations are required so that for a coherent state in the Wigner representation

$$\begin{aligned}\langle\{\hat{a}^\dagger\hat{a}\}_{\text{sym}}\rangle &= \langle(\alpha_c^* + \delta\alpha^*)(\alpha_c + \delta\alpha)\rangle_W\\ &= |\alpha_c|^2 + \langle|\delta\alpha|^2\rangle_W = |\alpha_c|^2 + 1/2.\end{aligned} \tag{124}$$

For $\alpha_c = 0$, the field is in a vacuum state and the initial spread of trajectories (equivalent to the factor of 1/2) is seen to represent vacuum noise. Hence for the truncated Wigner equations a stochastic element originates in the initial vacuum noise of the field. As we increase the number of frequency modes later on, this contribution of 1/2 will begin to add up. In the limit where the number of modes goes to infinity (the continuum limit) this will lead to a term in the equations which behaves like a renormalization factor, counterbalancing the infinite noise contribution. Further noise has its source in the coupling of the field to reservoirs, as we shall see.

E. The Raman evolution equation

The intention here is to derive the equivalent evolution equations for the Hamiltonian introduced in Sec. II. Thus, using a straightforward generalization of the results for the single-mode definition for $W(\boldsymbol{\alpha})$ given in the previous section (where $\boldsymbol{\alpha}$ is a vector which stands for *all* variables involved in the system) we obtain from the master equation:

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [\hat{H}_I, \hat{\rho}] \tag{125}$$

the following *exact* Wigner evolution equation:

$$\frac{\partial W}{\partial t} = \Bigg\{ \sum_l - \frac{\partial}{\partial \alpha_l}\Big[-i\sum_{l'} \omega_{ll'}\alpha_{l'} + 2i\chi_\alpha(\alpha_l\alpha_l^* - 1)\alpha_l - i\alpha_l \sum_\nu g_\nu^\beta(\beta_{l\nu}^* + \beta_{l\nu}) - i\sum_\mu g_\mu \alpha_{l\mu}^r \Big]$$
$$+ \sum_l - \frac{\partial}{\partial \alpha_l^*}\Big[+i\sum_{l'} \omega_{l'l}\alpha_{l'}^* - 2i\chi_\alpha(\alpha_l^*\alpha_l - 1)\alpha_l^* + i\alpha_l^* \sum_\nu g_\nu^\beta(\beta_{l\nu} + \beta_{l\nu}^*) + i\sum_\mu g_\mu \alpha_{l\mu}^{r*} \Big]$$
$$+ \sum_{l\nu} - \frac{\partial}{\partial \beta_{l\nu}}\Big[-i\omega_\nu^\beta \beta_{l\nu} - ig_\nu^\beta(\alpha_l^*\alpha_l - 1/2) - i\sum_\mu g_{\nu\mu}\beta_{l\nu\mu}^r \Big]$$
$$+ \sum_{l\nu} - \frac{\partial}{\partial \beta_{l\nu}^*}\Big[+i\omega_\nu^\beta \beta_{l\nu}^* + ig_\nu^\beta(\alpha_l^*\alpha_l - 1/2) + i\sum_\mu g_{\nu\mu}\beta_{l\nu\mu}^{r*} \Big]$$
$$+ \sum_{l\mu} - \frac{\partial}{\partial \alpha_{l\mu}^r}\Big[-i\omega_\mu \alpha_{l\mu}^r - ig_\mu\alpha_l \Big] + \sum_{l\nu\mu} - \frac{\partial}{\partial \beta_{l\nu\mu}^r}\Big[-i\omega_{\nu\mu}\beta_{l\nu\mu}^r - ig_{\nu\mu}\beta_{l\nu} \Big]$$
$$+ \sum_{l\mu} - \frac{\partial}{\partial \alpha_{l\mu}^{r*}}\Big[+i\omega_\mu \alpha_{l\mu}^{r*} + ig_\mu\alpha_l^* \Big] + \sum_{l\nu\mu} - \frac{\partial}{\partial \beta_{l\nu\mu}^{r*}}\Big[+i\omega_{\nu\mu}\beta_{l\nu\mu}^{r*} + ig_{\nu\mu}\beta_{l\nu}^* \Big]$$
$$+ \frac{i\chi_\alpha}{2} \sum_l \left[\frac{\partial^3 \alpha_l}{\partial\alpha_l^*\partial\alpha_l^2} - \frac{\partial^3 \alpha_l^*}{\partial\alpha_l\partial\alpha_l^{*2}} \right] + \frac{ig_\nu^\beta}{4}\sum_{l\nu}\left[\frac{\partial^3}{\partial\alpha_l\partial\alpha_l^*\partial\beta_{l\nu}^*} - \frac{\partial^3}{\partial\alpha_l^*\partial\alpha_l\partial\beta_{l\nu}} \right] \Bigg\} W. \quad (126)$$

F. The stochastic equations

The equations for individual trajectories, derived from the *truncated* form of the Wigner evolution equation (i.e., ignoring the third-order derivative terms), are just

$$\frac{\partial \alpha_l}{\partial t} = -i\sum_{l'} \omega_{ll'}\alpha_{l'} + 2i\chi_\alpha(\alpha_l^*\alpha_l - 1)\,\alpha_l - i\alpha_l \sum_\nu g_\nu^\beta(\beta_{l\nu}^* + \beta_{l\nu}) - i\sum_\mu g_\mu \alpha_{l\mu}^r, \quad (127a)$$

$$\frac{\partial \beta_{l\nu}}{\partial t} = -i\omega_\nu^\beta\beta_{l\nu} - ig_\nu^\beta(\alpha_l^*\alpha_l - 1/2) - i\sum_\mu g_{\nu\mu}\beta_{l\nu\mu}^r, \quad (127b)$$

$$\frac{\partial \alpha_{l\mu}^r}{\partial t} = -i\omega_\mu\,\alpha_{l\mu}^r - ig_\mu\alpha_l, \quad (127c)$$

$$\frac{\partial \beta_{l\nu\mu}^r}{\partial t} = -i\omega_{\nu\mu}\,\beta_{l\nu\mu}^r - ig_{\nu\mu}\beta_{l\nu}. \quad (127d)$$

Except for the lack of an explicit noise term here, which (as noted before) manifests itself in the Wigner case as initial vacuum noise, these equations are very similar to those obtained for the positive-P representation. However, note the factor $(\alpha_l^*\alpha_l - 1)\,\alpha_l$ in the equation for $\partial\alpha_l/\partial t$, instead of the term $\alpha_l^2\alpha_l^+$ which arose in the positive-P case. This is because for the Wigner representation we have

$$\langle(\alpha_l^*\alpha_l - 1)\,\alpha_l\rangle_W = \langle\{\hat\alpha_l^2\hat\alpha_l^\dagger - \hat\alpha_l\}_{\rm sym}\rangle = \langle\hat\alpha_l^\dagger\hat\alpha_l^2\rangle = \langle\alpha_l^+\alpha_l^2\rangle_P, \quad (128)$$

i.e., these terms have the same average behavior in the two representations. Similarly we have $\langle\alpha_l^*\alpha_l - 1/2)\rangle_W = \langle\alpha_l^+\alpha_l\rangle_P$, for the term appearing in the equation for $\partial\beta_{l\nu}/\partial t$.

The description is now complete once we specify the initial states of the field variables α_l and $\beta_{l\nu}$ and their respective reservoirs $\alpha_{l\mu}^r$ and $\beta_{l\nu\mu}^r$. For thermal states, at a temperature of T K, the mean number of quanta occupying a mode of frequency ω is given by $n_{\rm th} = 1/[\exp(\hbar\omega/kT) - 1]$. We shall take all the phonon modes and the reservoir modes to be initially in their respective thermal states (at time t_i) with the thermal occupation number appropriate to that state. Hence we have for the reservoirs

$$\langle|\alpha_{l\mu}^r(t_i)|^2\rangle_W = n_{\rm th}^\mu + 1/2, \quad (129a)$$

$$\langle|\beta_{l\nu\mu}^r(t_i)|^2\rangle_W = n_{\rm th}^{\nu\mu} + 1/2, \quad (129b)$$

where $n_{\rm th}^\mu$ and $n_{\rm th}^{\nu\mu}$ were defined for the positive-P treatment in Sec. III D. Likewise, the phonon modes would initially obey $\langle|\beta_{l\nu}(t_i)|^2\rangle_W = n_{\rm th}^\nu + 1/2$ since they are in equilibrium with the reservoirs. However, we will soon eliminate the reservoir variables and this will have the effect of introducing a rapid damping on the initial phonon amplitudes so that we may ignore the initial values completely.

G. Tracing and scaling the equations

Following the same procedure set out in the positive-P treatment, the results of tracing over the reservoir variables are outlined briefly here. The equations for the photon and the phonon fields, respectively, become, after the tracing,

$$\frac{\partial \alpha_l}{\partial t} = -i\sum_{l'} \omega_{ll'}\alpha_{l'} + 2i\chi_\alpha(\alpha_l^*\alpha_l - 1)\alpha_l - i\alpha_l \sum_\nu g_\nu^\beta(\beta_{l\nu}^* + \beta_{l\nu}) - \frac{\kappa}{2}\,\alpha_l + \Gamma_l^\alpha, \quad (130a)$$

$$\frac{\partial \beta_{l\nu}}{\partial t} = -\gamma_\nu \beta_{l\nu} - i g_\nu^\beta (\alpha_l^* \alpha_l - 1/2) + \Gamma_{l\nu}^\beta, \tag{130b}$$

where again we have defined $\gamma_\nu = \gamma'_\nu + i\omega_\nu$. The thermal noise terms are similar to those found for the positive-P case, except that the thermal occupation numbers in the Wigner representation are augmented by 1/2:

$$\langle \Gamma_l^\alpha(t)\, \Gamma_{l'}^{\alpha *}(t') \rangle = \kappa \,[n_{\rm th}^\alpha + 1/2]\, \delta_{ll'}\, \delta(t-t'), \tag{131a}$$

$$\langle \Gamma_{l\nu}^\beta(t)\, \Gamma_{l'\nu}^{\beta *}(t') \rangle = 2\gamma'_\nu \,[n_{\rm th}^\nu + 1/2]\, \delta_{ll'}\, \delta(t-t'). \tag{131b}$$

After making the transformation to the macroscopically defined fields of Sec. III F, the relevant equations look like

$$\frac{\partial \phi_l}{\partial t} = -i \sum_{l'} \omega_{ll'} \phi_{l'} + i\chi_E \frac{\bar n \omega'}{t_0} \left[\phi_l^* \phi_l - \frac{\omega' t_0}{\bar n \Delta x} \right] \phi_l - \frac{\kappa}{2}\phi_l - i\phi_l \frac{\omega'}{z_0} \sum_\nu [\, b_{l\nu}^* + b_{l\nu}\,] + \sqrt{\frac{\omega' t_0}{\bar n}}\, \frac{\Gamma_l^\alpha}{\sqrt{\Delta x}}, \tag{132}$$

$$\frac{\partial b_{l\nu}}{\partial t} = -\gamma_\nu\, b_{l\nu} - \frac{i h_\nu}{2 t_0} \left[\phi_l^* \phi_l - \frac{\omega' t_0}{2\,\bar n \Delta x} \right] + \sqrt{\frac{h_\nu z_0}{2\bar n}}\, \frac{\Gamma_{l\nu}^\beta}{\sqrt{\Delta x}}. \tag{133}$$

Now, as $\Delta x \to 0$ the extra Wigner terms in the large square brackets diverge. This is because they represent vacuum noise in each of the "frequency" modes appropriate to 1/2 a photon. This diverges as the number of modes goes to infinity. This well-known effect in the quantum theory of bosonic fields states that the variance of the field at a point in space and time is unbounded. To measure this variance as $\Delta x \to 0$ (which translates into the temporal step size $\Delta\tau = \Delta x/\omega' t_0$ in the comoving frame) we would have to employ a detector with a bandwidth of $1/\Delta\tau$. No real device has infinite bandwidth, and so a real measurement is one which necessarily averages the field over the time resolution appropriate to the measurement apparatus. Thus the infinite zero-point fluctuations do not cause problems, even though they are quite real and give rise to the Lamb shift and the natural linewidth of atoms. This problem of the infinite vacuum noise (or zero-point motion) is discussed by Louisell [25], Sec. 4.7.

It is convenient to leave this divergent term as a discrete component of the equations which can be adequately taken into account when real numerical modeling is performed with discrete-sized steps. Keeping this in mind we can write down the final *dimensionless* form of the equations (in the comoving frame as in Sec. III H) with the divergent Wigner term characterized by the temporal step size $\Delta\tau = \Delta x/\omega' t_0$.

$$\frac{\partial \phi}{\partial \zeta} = -\frac{i}{2}\left[1 \pm \frac{\partial^2}{\partial \tau^2} \right] \phi + i \left[\frac{\chi_E}{\chi_T} \right] \left(|\phi|^2 - \frac{1}{\bar n\, \Delta\tau} \right) \phi - \gamma\phi - i\phi \sum_\nu [b_\nu^* + b_\nu] + \sqrt{2\gamma}\, \Gamma_L(\zeta,\tau), \tag{134a}$$

$$\frac{\partial b_\nu}{\partial \tau} = -\tilde\gamma_\nu\, b_\nu - \frac{i h_\nu}{2}\left(|\phi|^2 - \frac{1}{2\bar n\, \Delta\tau} \right) + \sqrt{h_\nu \tilde\gamma'}\, \Gamma_V^\nu(\zeta,\tau). \tag{134b}$$

The rescaled noise sources are totally defined by the correlations

$$\langle \Gamma_L(\zeta,\tau)\, \Gamma_L^*(\zeta',\tau') \rangle = \frac{n_{\rm th}^\alpha + 1/2}{\bar n}\, \delta(\zeta-\zeta')\, \delta(\tau-\tau'), \tag{135a}$$

$$\langle \Gamma_V^\nu(\zeta,\tau)\, \Gamma_V^{\nu *}(\zeta',\tau') \rangle = \frac{n_{\rm th}^\nu + 1/2}{\bar n}\, \delta(\zeta-\zeta')\, \delta(\tau-\tau'). \tag{135b}$$

H. Elimination of the phonon variables

If we assume that the width of the damped phonon modes is sufficiently small then the initial value of the phonon amplitudes, $b_\nu(\zeta = 0, \tau = -\infty)$, decays away rather quickly. This is the assumption we make when we adopt the point of view that the Raman gain profile is due to the contribution of a very large number of inhomogeneous modes which effectively generate a continuum in the frequency domain (as described in detail in the positive-P part of this paper). Thus

$$b_\nu(\zeta,\tau) = \int_{-\infty}^{\tau} d\tau' e^{-\tilde\gamma_\nu(\tau-\tau')} \left\{ -\frac{i h_\nu}{2} \left(|\phi(\zeta,\tau')|^2 - \frac{1}{2\bar n \Delta\tau} \right) + \sqrt{h_\nu \tilde\gamma'}\, \Gamma_V^\nu(\zeta,\tau') \right\}. \tag{136}$$

We use this to evaluate the relevant term in the equation for ϕ,

$$-\sum_\nu [b_\nu^* + b_\nu] = \int_{-\infty}^{\tau} d\tau'\, h(\tau-\tau')\, |\phi(\tau')|^2 + \Gamma_V(\zeta,\tau) - \frac{(\chi_G + \chi_R)}{2\bar n \chi_T\, \Delta\tau}, \tag{137}$$

where

$$\Gamma_V(\zeta,\tau) = -\sum_\nu \sqrt{h_\nu \tilde\gamma'} \int_{-\infty}^{\tau} d\tau' \left[e^{-\tilde\gamma_\nu(\tau-\tau')}\, \Gamma_V^\nu(\zeta,\tau') + e^{-\tilde\gamma_\nu^*(\tau-\tau')}\, \Gamma_V^{\nu *}(\zeta,\tau') \right], \tag{138}$$

and $h(\tau)$ is the same response function which occurred in the positive-P formulation. The final "phonon-free" equation for the photon flux amplitude is

$$\frac{\partial \phi}{\partial \zeta} = -\frac{i}{2}\left[1 + \epsilon \pm \frac{\partial^2}{\partial \tau^2} \right] \phi + i \left[\frac{\chi_E}{\chi_T} \right] |\phi|^2 \phi - \gamma\phi + i\phi \int_{-\infty}^{\tau} d\tau'\, h(\tau-\tau')\, |\phi(\tau')|^2 + \Gamma_\gamma + i\phi\,\Gamma_V, \tag{139}$$

where $\epsilon = (1 + \chi_E/\chi_T)/(\bar n\, \Delta\tau)$ is an effective frequency shift introduced by the explicit presence of vacuum noise. This is actually an unbounded term in the continuum limit and what we really have in mind is a large but finite frequency cutoff in the theory, for which the above equation closely characterizes the behavior of the field. The term Γ_V is real valued so that the only interesting correlation function is the autocorrelation,

$$\langle \Gamma_V(\zeta,\tau)\,\Gamma_V(\zeta',\tau')\rangle = \frac{1}{\pi\,\bar{n}}\,\delta(\zeta-\zeta')\int_0^\infty d\Omega\,\alpha(\Omega)\,[n_{\rm th}(\Omega)+1/2]\,\cos[\Omega(\tau-\tau')]. \tag{140}$$

We may integrate this using the thermal expansion (90) and our representation (71) of $\alpha(\Omega)$ to obtain a result in terms of the coefficients $C^j_\pm(\Delta)$ of (93), and functions $B^j(\tau)$ of (92),

$$\langle \Gamma_V(\zeta,\tau)\,\Gamma_V(\zeta',\tau')\rangle = \frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\sum_{j=0}^{n}\left\{\frac{h_j\Omega_j}{\Delta}\,C^j_+(\Delta)\,e^{\lambda_j|\tau-\tau'|}\cos[\Omega_j(\tau-\tau')] + \frac{h_j\lambda_j}{\Delta}\,C^j_-(\Delta)\,e^{\lambda_j|\tau-\tau'|}\sin[\Omega_j|\tau-\tau'|]\right.$$
$$\left. -\frac{4\pi h_j\lambda_j\Omega_j}{\Delta^2}\,B^j(\tau-\tau',\Delta)\right\}. \tag{141}$$

Expressed in the Fourier domain,

$$\langle \tilde{\Gamma}_V(\zeta,\bar{\omega})\,\tilde{\Gamma}_V(\zeta',\bar{\omega}')\rangle$$
$$= \langle \tilde{\Gamma}^*_V(\zeta,-\bar{\omega})\,\tilde{\Gamma}_V(\zeta',\bar{\omega}')\rangle$$
$$= \frac{1}{\bar{n}}\,\delta(\zeta-\zeta')\,\delta(\bar{\omega}+\bar{\omega}')\,[n_{\rm th}(|\bar{\omega}|)+1/2]\,\alpha(|\bar{\omega}|). \tag{142}$$

The initial vacuum fluctuations must of course also be taken into account to describe the problem completely. Thus, to represent that aspect of the problem adequately, let us write the propagating field as the sum of an ensemble-averaged component (which we call the mean field) and a term which changes randomly from one member of the ensemble to the next, i.e.,

$$\phi(\zeta,\tau) = \bar{\phi}(\zeta,\tau) + \delta\phi(\zeta,\tau). \tag{143}$$

For a coherent state the vacuum noise in the Wigner representation thus enters the picture in the initial intensity correlations at the input to the fiber as

$$\langle \delta\phi(0,\tau)\,\delta\phi^*(0,\tau')\rangle_W = \frac{1}{2\bar{n}}\delta(\tau-\tau'). \tag{144}$$

This follows from the scaled stochastic variables α_l since by (37) and (124)

$$\langle \phi(0,\tau)\,\phi^*(0,\tau')\rangle_W = \frac{\omega' t_0}{\bar{n}\Delta x}\langle \alpha_l\alpha^*_{l'}\rangle_W$$
$$= \frac{1}{\bar{n}\Delta\tau}\left\{\langle \hat{\alpha}^\dagger_{l'}\hat{\alpha}_l\rangle + \frac{\delta_{l'l}}{2}\right\}$$
$$= \phi^*_c(0,\tau')\phi_c(0,\tau) + \frac{1}{2\bar{n}}\delta(\tau-\tau'). \tag{145}$$

I. GAWBS correlations

Lastly, we explicitly consider the low frequency portion of the gain curve, which is due primarily to the process of guided acoustic wave Brillouin scattering. Thus, using the same gain function $\alpha_G(\Omega)$ given by (97) from Sec. III P, and substituting into the general time domain correlation function (140), we obtain for the GAWBS noise

$$\langle \Gamma_G(\zeta,\tau)\,\Gamma_G(\zeta',\tau')\rangle$$
$$\approx \frac{1}{\pi\,\bar{n}}\,\delta(\zeta-\zeta')\left[\frac{4h_0\lambda_0^2}{\Delta}\right]\int_0^\infty d\Omega\,\frac{\cos[\Omega(\tau-\tau')]}{(4\lambda_0^4+\Omega^4)}$$
$$= \frac{kTh_0t_0}{2\lambda_0\hbar\bar{n}}\,\delta(\zeta-\zeta')\,e^{-\lambda_0|\tau-\tau'|}$$
$$\times\Big(\cos[\lambda_0(\tau-\tau')] + \sin[\lambda_0|\tau-\tau'|]\Big) \tag{146}$$

since $\Delta \equiv \hbar/kTt_0$. Thus it is clear from this expression that in the Wigner case the additional GAWBS noise term entering into the stochastic evolution equation for the field is purely thermal in nature, scaling directly with the temperature T. The GAWBS expression in the Fourier domain corresponding to (142) is

$$\langle \tilde{\Gamma}_G(\zeta,\bar{\omega})\,\tilde{\Gamma}_G(\zeta',\bar{\omega}')\rangle$$
$$= \langle \tilde{\Gamma}^*_G(\zeta,-\bar{\omega})\,\tilde{\Gamma}_G(\zeta',\bar{\omega}')\rangle$$
$$\approx \frac{4kTh_0\lambda_0^2t_0}{\hbar\bar{n}\,(4\lambda_0^4+\bar{\omega}^4)}\,\delta(\bar{\omega}+\bar{\omega}')\,\delta(\zeta-\zeta'). \tag{147}$$

For the more general case of vibrational noise, and in particular for the Raman portion of the gain curve, there will be *small* additional *nonthermal* contributions to the noise process, as the lack of them in the GAWBS case is due to the linearized approximation of the thermal factor spelled out by (99).

V. SUMMARY

In this paper we have made use of two phase-space representations to treat the problem of quantum pulse propagation in an optical fiber. The first of these was based on the positive-P representation, which is a nondiagonal coherent-state basis expansion of the density operator. This allowed us to write down a Fokker-Planck equation describing the motion of the distribution function for the photon and phonon variables. It is always possible in this representation to find a distribution function which remains positive, provided it was initially. Equivalent Ito stochastic differential equations were then obtained from the Fokker-Planck drift and diffusion coefficients. These ordinary c-number equations, which offer an *exact* formulation of the nonlinear quantum propagation problem, offer the possibility of direct numerical simulation in a straightforward way.

Using the second phase-space method, which involved a truncated form of the equation of motion for the Wigner function, and is therefore an *approximate* formulation of the problem which is valid under the usual conditions for linearization, we showed that it is possible to obtain a *single* stochastic equation for the field amplitude ϕ (along with appropriate initial conditions). This is in contrast to the equivalent pair of positive-P equations which are defined on a phase space with twice the number of dimensions. In practice, this means that the Wigner phase space can be sampled by a given number of stochastic trajectories more thoroughly than for the positive-P space. The result is that the sampling error in the Wigner case can be smaller, particularly for cases of highly nonclassical light, for which the positive-P trajectories need to sample the extra dimensions of the positive-P phase space in order to generate the correct statistics.

Of particular interest in the Wigner equations is the appearance of corrections (ϵ) to the deterministic equations that depend on the *frequency cutoff.* These act like the renormalization terms in Feynman-style perturbation theory, in the sense that they are infinite at infinite cutoff. These will cause severe problems in higher-order calculations if carried out analytically. Of course, they can be included in lattice calculations, and become increasingly significant as $\Delta\tau \rightarrow 0$.

Finally, although we have presented equations of motion for the field in both the positive-P and Wigner representations, we have not solved these equations for general input fields (such as solitons) except by direct numerical simulation of the relevant stochastic equations. However, it should be possible to reformulate the problem in terms of the equations of motion for the correlation functions themselves. In this manner one might expect that the analytic methods of inverse scattering could be applied profitably to the solution of the phase-space equations.

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APPENDIX A: CONTINUUM LIMIT OF $\omega_{ll'}$

In this section we demonstrate how the wave equation is developed in the continuum limit, i.e., we show how the term $\omega_{ll'}$ is replaced by ordinary differential operators. This is demonstrated here for an arbitrary function which we denote as u_l in the case of finite cell size, i.e., $u_l = u(z = l\Delta z)$. In order to show how the second derivative is related to the finite difference terms encountered when we express the fields as local functions of space, we start with the following well-known calculus definition:

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{l\Delta z} = \lim_{\Delta z \to 0} \frac{u_{l+1} + u_{l-1} - 2u_l}{(\Delta z)^2}. \tag{A1}$$

We suppose that the local field u_l can be written as a discrete Fourier relation in the usual way as

$$u_l = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} u_n \, e^{in\Delta k \, l\Delta z}, \tag{A2a}$$

$$u_n = \frac{1}{\sqrt{2N+1}} \sum_{l=-N}^{N} u_l \, e^{-in\Delta k \, l\Delta z}. \tag{A2b}$$

The numerator in terms of the Fourier components is

$$u_{l+1} + u_{l-1} - 2u_l$$

$$= \sum_n \frac{u_n \, e^{in\Delta k \, l\Delta z}}{\sqrt{2N+1}} \left\{ e^{in\Delta k\Delta z} + e^{-in\Delta k\Delta z} - 2 \right\}. \tag{A3}$$

Expanding out the terms in the large brackets gives the negative factor $-n^2(\Delta k\Delta z)^2 + O(n^4)$. Since $\Delta k\Delta z = 2\pi/(2N+1)$ we can obviously neglect the higher-order terms if u_n drops off sufficiently rapidly with increasing $|n|$, which we will assume is the case (and would only have to be reconsidered in the circumstance that the function u was extremely well localized in space, so that it was δ-function-like). Putting these pieces together, we can rewrite the second derivative term as

$$\begin{aligned}\frac{\partial^2 u(z)}{\partial z^2} &\simeq -(\Delta k)^2 \lim_{N\to\infty} \frac{1}{\sqrt{2N+1}} \sum_n n^2 e^{inl\Delta k\Delta z} u_n \\ &= \lim_{N\to\infty} \frac{-(\Delta k)^2}{2N+1} \sum_n n^2 e^{inl\Delta k\Delta z} \\ &\quad \times \sum_{l'} u_{l'} e^{-inl'\Delta k\Delta z},\end{aligned} \tag{A4}$$

or

$$\frac{\partial^2 u(z)}{\partial z^2} \simeq -\lim_{N\to\infty} \sum_{l'} u_{l'} \left\{ \frac{(\Delta k)^2}{(2N+1)} \times \sum_n n^2 e^{in\Delta k\Delta z(l-l')} \right\}. \tag{A5}$$

In the same way we can also show that the relation for the first derivative is given by

$$\frac{\partial u(z)}{\partial z} \simeq \lim_{N\to\infty} i \sum_{l'} u_{l'} \left\{ \frac{\Delta k}{(2N+1)} \sum_n n \, e^{in\Delta k\Delta z(l-l')} \right\}. \tag{A6}$$

When we put these last two results together and use the definition for $\omega_{ll'}$, which is

$$\omega_{ll'} = \sum_{n=-N}^{N} \frac{\{\omega'(n\Delta k) + \omega''(n\Delta k)^2/2\}}{2N+1} e^{in\Delta k(l-l')\Delta z} + \Delta\omega\, \delta_{ll'}, \tag{A7}$$

we find that the action of $\omega_{ll'}$ on an arbitrary function u_l in the limit as the cell size $\Delta z \to 0$ is

$$\sum_{l'} \omega_{ll'} u_{l'} \to \left\{ \Delta\omega - i\omega' \frac{\partial}{\partial z} - \frac{\omega''}{2} \frac{\partial^2}{\partial z^2} \right\} u(z). \tag{A8}$$

This is true whether we choose u_l to be of the form α_l, which are the basic Hamiltonian variables that we chose to begin our derivation of the NLS equation, or u_l is of the form ϕ_l, which are the scaled macroscopic fields.

APPENDIX B: STOCHASTIC EQUATIONS: ITO AND STRATONOVICH

Suppose we have a system of c-number variables $\boldsymbol{x}$ (a vector) for which the statistical behavior is known to be governed by a genuine (i.e., positive-definite) probability distribution $P(\boldsymbol{x})$. Instead of dealing with $P(\boldsymbol{x})$ and its time development, can we deal with $\boldsymbol{x}$ directly? In other words, can we write down an equation for $\boldsymbol{x}$ which generates the same moments as determined by $P(\boldsymbol{x})$? In some instances the answer is yes. In particular, for distribution functions $P(\boldsymbol{x})$ which satisfy a Fokker-Planck equation for the time development, i.e.,

$$\frac{\partial P(\boldsymbol{x},t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_i} A_i(\boldsymbol{x},t)\, P(\boldsymbol{x},t) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [\boldsymbol{B}(\boldsymbol{x},t)\boldsymbol{B}^T(\boldsymbol{x},t)]_{ij}\, P(\boldsymbol{x},t), \tag{B1}$$

where $\boldsymbol{A}$ is a vector and $\boldsymbol{B}$ a matrix, an equivalent process in terms of the behavior of individual trajectories of the vector $\boldsymbol{x}$ can be defined when the diagonal elements of the matrix $\boldsymbol{B}\cdot\boldsymbol{B}^T$ are positive definite. Such a process is given by the *stochastic* differential of the components of this vector,

$$dx_i(t) = A_i(\boldsymbol{x},t)\, dt + \sum_j B_{ij}(\boldsymbol{x},t)\, dW_j(t), \tag{B2}$$

the interpretation of which was first spelt out by Ito [26]. The first term on the right-hand side of (B2) is a purely deterministic, or drift, term governing the evolution of $\boldsymbol{x}$, as in a classically defined trajectory. The second term, involving the increments $dW_j(t)$, represents the indeterminacy responsible for the diffusive nature of the Fokker-Planck equation (B1). This term can be used to model the intrinsic unpredictability associated with quantum outcomes (which manifests itself as a kind of noise). The increments $dW_j(t)$ are taken to be independent real Gaussian-distributed random variables with the so called "white noise" correlations

$$\langle dW_i(t)\, dW_j(t')\rangle = 0 \qquad (t \neq t'), \tag{B3a}$$

$$\langle dW_i(t)\, dW_j(t)\rangle = \delta_{ij}\, dt, \tag{B3b}$$

$$\langle dW_i(t)\rangle = 0. \tag{B3c}$$

Note that these conditions imply that the dW_j are differentials of order $\sqrt{dt}$ and that they are highly *irregular* functions, being completely independent at every point in time.

Some care needs to be taken in the manipulation of quantities based on the stochastic differential (B2). In fact, the ordinary rules of calculus applied to this expression do not in general give the correct results for fundamental operations like changes of variable [i.e., what is the differential of $\boldsymbol{y}$ given $\boldsymbol{y} = f(\boldsymbol{x},t)$] and integration by parts. The reason for this is that the stochastic integral $\int_{t_0}^t B(x,s)dW(s)$, defined as the limit of the partial sums

$$\int_{t_0}^t B(x,s)\, dW(s) \equiv \lim_{\delta_n \to 0} \sum_{i=1}^{n} B[x(\tau_i),\tau_i]\, [W(t_i) - W(t_{i-1})] \tag{B4}$$

for $(t_0 \le t_1 \le t_2 \le \cdots \le t_{n-1} \le t)$ and $(t_{i-1} \le \tau_i \le t_i)$, does not converge (as $\delta_n = \max[t_i - t_{i-1}] \to 0$) to a unique integral independent of the value of the midpoints τ_i at which the integrand B is evaluated. The integral expression of (B2) is

$$x_i(t) = x_i(t_0) + \int_{t_0}^t A_i(\boldsymbol{x},s)\, ds + \sum_j \int_{t_0}^t B_{ij}(\boldsymbol{x},s)\, dW_j(s) \tag{B5}$$

where the first integral can be considered as an ordinary Riemann-Stieltjes integral independent of the midpoint values τ_i, while the second is of the *stochastic integral* form (B4). Ito's choice of midpoint was to take $\tau_i = t_{i-1}$ so that the integrand is evaluated at the beginning of each of the time intervals occurring in the partial sum expression. The appropriate rules for variable manipulation based on this choice are known as *Ito calculus.* These rules ensure that the diffusion process governed by the Fokker-Planck equation (B1) is reproduced in moments by (B2), or equivalently (B5), interpreted in the Ito sense.

The motivation for Ito's definition of the stochastic integral (based on $\tau_i = t_{i-1}$) is largely that it renders perturbative expansions in mathematical proofs much simpler than would be the case with any other choice of τ_i. Another choice would be to take $\tau_i = \{t_i + t_{i-1}\}/2$. This possibility was considered by Stratonovich [27] who was motivated by a desire for a definition of the stochastic integral which *did* formally obey the usual rules of calculus. However, difficulties arise in proving convergence of the resulting stochastic integral with this *symmetric* choice for τ_i. Nevertheless, Stratonovich *did* succeed in formulating a definition of the stochastic integral (given below) which does allow one to use all the usual rules of calculus, such as integration by parts, and so on. Thus the two most commonly considered definitions of the stochastic integral are the limiting forms

$$\int_{t_0}^{t} B(x,s)\,dW(s) \equiv \lim_{\delta_n\to 0}\sum_{i=1}^{n} B[x(t_{i-1}),t_{i-1}]\,[W(t_i)-W(t_{i-1})] \quad \text{(Ito)} \tag{B6a}$$

and

$$\int_{t_0}^{t} B(x,s)\,dW(s) \equiv \lim_{\delta_n\to 0}\sum_{i=1}^{n} B[\{x(t_{i-1})+x(t_i)\}/2,t_{i-1}]\,[W(t_i)-W(t_{i-1})] \quad \text{(Stratonovich).} \tag{B6b}$$

In general there is no relationship connecting the two limits, so that the stochastic integral equation (B5) gives rise to two *different* types of random process, depending on the sense in which the stochastic integral on the far right of the equation is interpreted. If, however, we ask the question "What is the stochastic differential (or integral) equation which, when interpreted in the Stratonovich sense, gives rise to the same process as does (B2), namely,

$$d\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{x},t)\,dt + \boldsymbol{B}(\boldsymbol{x},t)\cdot d\boldsymbol{W}(t), \tag{B7}$$

interpreted in the Ito sense?" then we can relate the Ito and Stratonovich forms. Relegating the proof to either Gardiner's *Handbook of Stochastic Methods* [28] or Arnold's *Stochastic Differential Equations* [29], we make the following assertion.

The solution $x_i(t)$ to the stochastic equation

$$dx_i(t) = A_i(\boldsymbol{x},t)\,dt + \sum_j B_{ij}(\boldsymbol{x},t)\,dW_j(t), \tag{B8}$$

interpreted in the Ito sense, is also a solution to the equation

$$dx_i(t) = \left\{ A_i(\boldsymbol{x},t) - \frac{1}{2}\sum_k\sum_j B_{kj}(\boldsymbol{x},t)\,\frac{\partial B_{ij}(\boldsymbol{x},t)}{\partial x_k}\right\}dt + \sum_j B_{ij}(\boldsymbol{x},t)\,dW_j(t)\,, \tag{B9}$$

when interpreted in the Stratonovich sense. Thus it is not necessary to learn the unfamiliar rules of Ito calculus. As long as we make the appropriate change to the drift vector we may utilize all the familiar rules of ordinary calculus when dealing with our (Stratonovich) stochastic equations.

The Stratonovich interpretation may also be considered as the white noise limit of the case in which a physical process $dx_i(t)$ is driven by noise sources $\zeta_j(t)dt$ which, instead of being δ-correlated "white noises," as in (B8), have finite correlation times. In this case ζ_j is said to be colored and to obey a relationship of the form

$$\langle \zeta_i(t)\,\zeta_j(t')\rangle = \delta_{ij}\,C_j(t-t'),$$

where the $C_j(t)$ are supposed to be regular functions [unlike the singular white noise δ-function correlations $\delta(t)$]. On the basis of the continuity of the ζ_j we may apply the classical rules of calculus to the differential relation in (B9) with $dW_j(t)$ replaced by $\zeta_j(t)dt$ so that $\zeta_j(t)$ is regarded as the derivative of the *regular* noise increment $W_j(t)$. If we solve for $x_i(t)$ and then allow the colored noise sources $\zeta_j(t)$ to approach the white noise form $\xi_j(t)$, so that $C_j(t) \to \delta(t)$, one finds that $x_i(t)$ converges to the same solution arrived at by solving the white noise equation (B9) directly interpreted in the Stratonovich sense. Applying the same (Stratonovich) limiting procedure to the Ito equation (B8) does *not* of course lead to the same result, since that equation must necessarily be interpreted in the Ito sense for it to be equivalent to the Stratonovich form (B9).

Thus, in general, a stochastic differential equation derived as the "white noise" (or wideband) limit of a physical process with a realizable (i.e., nonwhite) noise source should be interpreted in the Stratonovich sense and a transformation to the Ito form is required if we are to select out the drift and diffusion coefficients necessary to construct the equivalent Fokker-Planck equation corresponding to (B8) or (B9)—i.e., (B1). Our approach is the reverse of this procedure. We *begin* with a Fokker-Planck equation and ask what is the equivalent *Stratonovich* stochastic differential equation which corresponds to this. Ordinarily this necessitates the construction of Ito corrections to the drift term as in (B9). However, in our case it can be seen that these corrections (which scale inversely with the photon number $\bar{n}$) are tiny and we neglect them in comparison to the large Ito drift term.

APPENDIX C: GENERATING THE POSITIVE-*P* NOISE SOURCES NUMERICALLY

The correlation functions for the positive-P noise sources Γ_R, Γ_R^+ are not those of the white noise variety. Nevertheless, an arbitrary noise source can be constructed in the time domain if its spectrum is known. We consider here the more complicated case of constructing the noise sources appropriate to numerical simulations of the positive-P equations. The Wigner procedure is, of course, somewhat simpler. Suppose we have the following correlations:

$$\langle \tilde{\Gamma}(\bar{\omega})\,\tilde{\Gamma}(\bar{\omega}')\rangle = U(\bar{\omega})\,\delta(\bar{\omega}+\bar{\omega}'), \tag{C1a}$$

$$\langle \tilde{\Gamma}(\bar{\omega})\,\tilde{\Gamma}^+(\bar{\omega}')\rangle = Z(\bar{\omega})\,\delta(\bar{\omega}+\bar{\omega}'), \tag{C1b}$$

$$\langle \tilde{\Gamma}^+(\bar{\omega})\,\tilde{\Gamma}^+(\bar{\omega}')\rangle = U^*(\bar{\omega})\,\delta(\bar{\omega}+\bar{\omega}'). \tag{C1c}$$

For the purposes of this paper, we shall take $U(\bar{\omega})$ to be complex valued and symmetric and $Z(\bar{\omega}) \geq 0$ real and asymmetric. The results can then be used to model Γ_R, Γ_R^+.

We wish to present *one* way in which this might be done; it is not necessarily the most efficient one, but it illustrates the idea.

To begin, we discretize the problem for numerical analysis. The continuous frequency variable $\bar{\omega}$ is replaced by one of fixed increments $\Delta\bar{\omega}$ and the Dirac δ function becomes a Kronecker δ. Thus for $j = 1, \ldots, N = 2^m$,

$$\bar{\omega}_j = -\bar{\omega}_{\max} + (j-1)\,\Delta\bar{\omega}, \quad \Delta\bar{\omega} = \frac{2\bar{\omega}_{\max}}{N-1}, \tag{C2a}$$

$$\delta(\bar{\omega}_j + \bar{\omega}'_k) \to \frac{\delta_{k,N+1-j}}{\Delta\bar{\omega}}, \quad \tilde{\Gamma}(\bar{\omega}_j) \to \tilde{\Gamma}_j. \tag{C2b}$$

In this way the correlations look like

$$\langle \tilde{\Gamma}_j\, \tilde{\Gamma}_k \rangle = \frac{U_j}{\Delta\bar{\omega}}\, \delta_{k,N+1-j}, \tag{C3a}$$

$$\langle \tilde{\Gamma}_j\, \tilde{\Gamma}^+_k \rangle = \frac{Z_j}{\Delta\bar{\omega}}\, \delta_{k,N+1-j}, \tag{C3b}$$

$$\langle \tilde{\Gamma}^+_j\, \tilde{\Gamma}^+_k \rangle = \frac{U^*_j}{\Delta\bar{\omega}}\, \delta_{k,N+1-j}. \tag{C3c}$$

We require a sequence of random numbers which will represent the statistics of these noise sources. To accomplish this we will work with three independent *complex* Gaussian processes u_j, u^+_j, and z_j defined on the half-range $j = 1, \ldots, N/2$. We will construct linear combinations of these three sequences such that u_j and u^+_j will only contribute to the autocorrelations and z_j to the cross correlation. We take these complex random sequences to have unit variance for the modulus of each:

$$\langle u_j\, u^*_k \rangle = \langle u^+_j\, u^{+*}_k \rangle = \langle z_j\, z^*_k \rangle = \delta_{j,k}, \tag{C4a}$$

$$\langle u_j\, u^+_k \rangle = \langle u_j\, z_k \rangle = \langle u^+_j\, z_k \rangle = 0. \tag{C4b}$$

We are now in a position to construct sequences for $\tilde{\Gamma}_j$ and $\tilde{\Gamma}^+_k$. Suppose we try linear sequences of the form (for $j = 1, \ldots, N/2$)

$$\tilde{\Gamma}_j = A_j u_j + B_j z_j, \tag{C5a}$$

$$\tilde{\Gamma}_{N+1-j} = A_{N+1-j} u^*_j + B_{N+1-j} z^*_j, \tag{C5b}$$

$$\tilde{\Gamma}^+_j = A^*_j u^+_j + B_{N+1-j} z_j, \tag{C5c}$$

$$\tilde{\Gamma}^+_{N+1-j} = A^*_{N+1-j} u^{+*}_j + B_j z^*_j. \tag{C5d}$$

What then are the complex coefficients A_i and real coefficients B_i necessary to give the appropriate correlations? To begin with (for $j = 1, \ldots, N$) we find for the correlation functions of (C3)

$$\langle \tilde{\Gamma}_j\, \tilde{\Gamma}_k \rangle = (A_j A_{N+1-j} + B_j B_{N+1-j})\, \delta_{k,N+1-j}, \tag{C6a}$$

$$\langle \tilde{\Gamma}_j\, \tilde{\Gamma}^+_k \rangle = B^2_j\, \delta_{k,N+1-j}, \tag{C6b}$$

$$\langle \tilde{\Gamma}^+_j\, \tilde{\Gamma}^+_k \rangle = (A^*_j A^*_{N+1-j} + B_j B_{N+1-j})\, \delta_{k,N+1-j}. \tag{C6c}$$

Hence we may define the real coefficients B_j by

$$B_j = \sqrt{\frac{Z_j}{\Delta\bar{\omega}}} \geq 0. \tag{C7}$$

Note that the coefficients Z_i in our case are *asymmetric* in frequency (the index j) so the B_j are as well. The form of the coefficients of the autocorrelation functions is already symmetric, as is required if it is to represent the symmetric coefficients U_j. If we let $A_{N+1-j} = A_j$ then we can find the A_j from the Kronecker δ condition

$$A_j\, A_{N+1-j} + B_j\, B_{N+1-j} = \frac{U_j}{\Delta\bar{\omega}}, \tag{C8}$$

or

$$A^2_j = \frac{1}{\Delta\bar{\omega}} \left[U_j - \sqrt{Z_j Z_{N+1-j}} \right]. \tag{C9}$$

From this specification of the coefficients A_j and B_j it clearly follows also that $\langle \tilde{\Gamma}^+_j\, \tilde{\Gamma}^+_k \rangle = \delta_{k,N+1-j}\, U^*_j / \Delta\bar{\omega}$. Thus the noise sources on the full frequency range ($j = 1, \ldots, N$) can be constructed from three complex random sequences defined on the half-range ($j = 1, \ldots, N/2$).

The resulting sequences of random numbers $(\tilde{\Gamma}_j, \tilde{\Gamma}^+_j)$ then satisfy the original correlations in frequency and by Fourier transforming them to the time domain we generate the noise sources required for the stochastic nonlinear Schrödinger equation by an appropriate choice of the coefficients U_j and Z_j.

To evaluate these coefficients in the positive-P treatment for a given form of $\alpha(\Omega)$ it can be seen from (83a) that we need to perform the integral

$$\int_0^\infty d\Omega\, \frac{\alpha(\Omega)\, \Omega}{\pi\,(\Omega^2 - \bar{\omega}^2)} = \sqrt{2\pi}\, \tilde{h}'(\bar{\omega}). \tag{C10}$$

This quantity is given by (73a) since both the GAWBS and the Raman gain can be modeled using the form of $\alpha(\Omega)$ given by (71). Hence, using both (71) and (73a), the coefficients we seek for $U(\bar{\omega})$ and $Z(\bar{\omega})$ can be seen from (83a) and (86) to be given explicitly in the inhomogeneous gain model by

$$U(\bar{\omega}) = \frac{1}{\bar{n}} \sum_{j=1}^{n} \frac{h_j\, \Omega_j \left\{ 4\lambda_j\, |\bar{\omega}|\, [n_{\rm th}(|\bar{\omega}|) + 1/2] - i[\Omega^2_j + \lambda^2_j - \bar{\omega}^2] \right\}}{[\lambda^2_j - \Omega^2_j + \bar{\omega}^2]^2 + [2\lambda_j \Omega_j]^2}, \tag{C11a}$$

$$Z(\bar{\omega}) = \frac{[n_{\rm th}(|\bar{\omega}|) + \Theta(-\bar{\omega})]}{\bar{n}} \sum_{j=1}^{n} \frac{4|\bar{\omega}|\, h_j\, \Omega_j\, \lambda_j}{[\lambda^2_j - \Omega^2_j + \bar{\omega}^2]^2 + [2\lambda_j \Omega_j]^2}, \tag{C11b}$$

where the h_j, λ_j, Ω_j are specified in Ref. [4].

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