Micromaser as a maser without inversion

Paweł Masiak and Kazimierz Rzążewski
Centrum Fizyki Teoretycznej, Polskiej Akademii Nauk, Al. Lotników 32/46, 02-668 Warsaw, Poland

Luis Roso
Departamento de Fisica Aplicada, Universidad de Salamanca, 37008 Salamanca, Spain
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We show that the micromaser, widely studied for its distinct quantum features, has one more remarkable property: unlike conventional lasers and masers it works without population inversion. We study the time-averaged inversion of a single atom passing through a cavity with a field in its stationary state. The result is shown to hold also for the regularly pumped micromaser and for the two-photon micromaser. It even remains partially valid for a semiclassical description of the field.

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I. INTRODUCTION

In ordinary masers and lasers, lasing occurs only if atoms have population inversion. However, many papers in past few years have been published, suggesting a possibility of amplification without inversion and lasing without inversion [1–3]. In fact, recently, a possibility of lasing without inversion was confirmed exprimentally [4]. Subtle quantum effects are at work in all proposed schemes of amplification without inversion. It is clear that these effects will be difficult to observe in standard lasers or masers. Micromaser, on the other hand, is a remarkable device, which is simple to analyze and which has already shown many quantum effects [5,6]. Some examples include demonstration of quantum revivals of the atomic inversion [7] and production of nonclassical states of light [8]. The micromaser may also help to produce a pure number state of electromagnetic field [9,10]. The purpose of this paper is to show that the micromaser has one more interesting property: In a stationary state it works without population inversion. A more precise meaning of this statement will be given below. We show that this fact is independent of the pumping statistics or the detailed properties of the atomic transition. This is the general property of one- and two-photon micromasers.

We organize our paper as follows. In Sec. II, we present results of analytical and numerical calculations for the one-photon micromaser pumped by Poissonian distributed atomic beam. In Sec. III, we show that inversionless lasing occurs also when micromaser is pumped regularly. In Sec. IV, we extend our analysis to a two-photon micromaser, and in Sec. V, we consider a semiclassical model of the micromaser, in which an electromagnetic field is treated classically and atoms injected into the cavity are described by the Bloch equations.

II. ONE-PHOTON MICROMASER

Following Filipowicz *et al.* [11], we consider a single-mode cavity into which excited two-level atoms are injected at a rate low enough that at most one atom at a time is inside the resonator. It means that $t_p \gg \tau$, where t_p is a time distance between two atoms arriving in the resonator, and τ is the atom-field interaction time. In addition, we assume that τ is much shorter than the cavity damping time $t_{\rm cav} = \kappa^{-1}$. Under these conditions we may consider separately the cavity damping and the pumping by interaction with passing atoms.

At time t_i the ith atom enters the cavity. After the interaction time τ the atom exits the resonator, and the field in the cavity evolves freely in the interval $[t_i+\tau, t_{i+1}]$ damped at the rate κ. This condition is realized in experiments with quite good accuracy. At time t_{i+1} the density operator of the field is given by

$$\rho_f(t_{i+1}) = \exp(Lt_p)F(\tau)\rho_f(t_i), \tag{1}$$

where $t_p = t_{i+1} - t_i - \tau \approx t_{i+1} - t_i$, and L is the Liouville operator describing the damping of the field in the cavity. The operator $F(\tau)$ characterizing the atom-field interaction is given by

$$\begin{aligned}\langle n|F(\tau)\rho(t_i)|n\rangle &= p_n(t_i+\tau)\\ &= (1-\beta_{n+1})p_n(t_i) + \beta_n p_{n-1}(t_i),\end{aligned} \tag{2}$$

$$p_n(t) = \langle n|\rho_f(t)|n\rangle, \tag{3}$$

$$\beta_n = \frac{n\Omega^2}{\Delta^2 + n\Omega^2}\sin^2(\tfrac{1}{2}\sqrt{\Delta^2+n\Omega^2}\,\tau), \tag{4}$$

where Ω is the Rabi frequency and $\Delta = \omega - \omega_0$ is atom-field detuning [11]. The damping of the field in the cavity is described by the standard equation [11,12]

$$\dot{p}_n = \kappa[(n+1)p_{n+1} - np_n]. \tag{5}$$

Since we assume in this section that the atoms enter the cavity according to a Poisson process with mean spacing $1/R$ between events, we have to average the Eq. (1) over the exponential distribution $P(t_p) = R\,\exp(-Rt_p)$ of the intervals between atoms. We get the following equation:

$$\bar{\rho}_f(t_{i+1}) = (1 - L/R)^{-1} F(\tau) \bar{\rho}_f(t_i), \tag{6}$$

where R is the atomic flux.

We seek the field in a steady state. It means that the elements of density matrix have to satisfy the following condition:

$$(1 - L/R)\bar{\rho}_{f,\mathrm{st}} = F(\tau)\bar{\rho}_{f,\mathrm{st}}. \tag{7}$$

The analytic expression for the solution of the steady-state equation for the occupation number probability is given by

$$p_n = C \prod_{k=1}^{n} \left(\frac{N_{\mathrm{ex}} \beta_k}{k} \right), \tag{8}$$

where C is the normalization constant, and $N_{\mathrm{ex}} = R/\kappa$ is the average number of atoms that traverse the cavity during the lifetime of the field [11]. Unfortunately, this expression is not useful for further analytic calculations. The average photon number, dispersion, and atomic inversion will be calculated numerically. The final atomic inversion is given by the following expression:

$$\begin{aligned} w(\tau) &= |c_1(\tau)|^2 - |c_0(\tau)|^2 \\ &= \sum_{n=0}^{\infty} p_n^\tau [|c_{n,1}(\tau)|^2 - |c_{n,0}(\tau)|^2], \end{aligned} \tag{9}$$

where $c_{n,(0,1)}(\tau)$ are amplitudes of conditional probability to find the atom in the ground and excited states under the condition that the electromagnetic field was in the n-photon state. A relation $|c_{n,0}(\tau)|^2 + |c_{n,1}(\tau)|^2 = 1$ is fulfilled for all $n \in N$, thus we may rewrite Eq. (9) in the form,

$$w(\tau) = \sum_{n=0}^{\infty} p_n^\tau [1 - 2|c_{n,0}(\tau)|^2] = \sum_{n=0}^{\infty} p_n^\tau w_n(\tau), \tag{10}$$

$$\begin{aligned} w_n(\tau) &= 1 - 2|c_{n,0}(\tau)|^2 \\ &= 1 - 2\frac{n\Omega^2}{\Delta^2 + n\Omega^2} \sin^2[\tfrac{1}{2}\sqrt{\Delta^2 + (n+1)\Omega^2}\,\tau]. \end{aligned} \tag{11}$$

We will assume from this point to the end of the paper that $\Delta = 0$. For easier comparison of our results with others, presented in early papers, we introduce a dimensionless "time" $\theta = \theta(\tau)$ defined as

$$\theta(\tau) = \tfrac{1}{2}\Omega\sqrt{N_{\mathrm{ex}}}\tau.$$

For zero detuning and by using different variables, Eq. (11) becomes

$$w_n^\theta = 1 - 2\sin^2\left(\frac{\sqrt{n+1}}{\sqrt{N_{\mathrm{ex}}}}\,\theta \right). \tag{12}$$

Atoms leave the cavity with the final inversion w^θ. However, we may also trace the time dependence of the inversion while the atom is inside the cavity $w(\theta(t)) = \sum_{n=0}^{\infty} p_n^\theta w_n(\theta(t))$ for all moments between 0 and $\theta(\tau)$. The probability p_n^θ depends on θ only parametrically and is time t independent. We define a mean inversion as a time average of the time dependent inversion:

$$\begin{aligned} \bar{w}^\theta = \frac{1}{\theta}\int_0^\theta w(\theta')d\theta' &= \frac{1}{\theta}\sum_{n=0}^{\infty} p_n^\theta \int_0^\theta w_n(\theta')d\theta' \\ &= \sum_{n=0}^{\infty} p_n^\theta \frac{\sin(2\frac{\sqrt{n+1}}{\sqrt{N_{\mathrm{ex}}}}\theta)}{2\frac{\sqrt{n+1}}{\sqrt{N_{\mathrm{ex}}}}\theta}. \end{aligned} \tag{13}$$

This is the most important quantity in this paper. We will show that for most values of parameter θ, the mean inversion is negative. Unfortunately, it is not possible to get an effective analytical expressions for both $\bar{w}_\theta$ and $w(\theta)$, because the expression for photon number distribution is too complicated, numerical calculations, however, are rather simple.

A. Approximate description by Fokker-Planck equation

Before we present results of our numerical calculations, we consider an approximate approach to our problem in terms of a Fokker-Planck equation. In this approach, we follow Ref. [11] closely.

This procedure is correct only if we assume that the pumping is strong, cf. when $N_{\mathrm{ex}} \gg 1$. We consider the evolution of the photon number in the cavity over a time T, long compared to the interaction time τ, yet so short that the occupation number of the field does not change too much. During the time T, a random number of excited atoms with a Poissonian distribution of mean $\bar{N} = RT$ traverse the cavity. For precisely N atoms, the average change in photon number $\langle n(N) \rangle_g$ and the average of the square of this change $\langle n^2(N) \rangle_g$, are

$$\langle n(N)\rangle_g = PN, \text{ and } \langle n^2(N)\rangle_g = (P - P^2)N + P^2N^2, \tag{14}$$

respectively. Here P is the probability of adding one photon to the field by every atom that traverses the cavity.

$$P = \sin^2\left(\frac{\sqrt{n}}{\sqrt{N_{\mathrm{ex}}}}\theta \right) \tag{15}$$

After averaging Eqs. (14) over Poissonian distribution of the arrival times of the atoms we get

$$\langle n(N)\rangle_g = P\bar{N}, \text{ and } \langle n^2(N)\rangle_g = P\bar{N} + P^2\bar{N}^2. \tag{16}$$

The average changes of photon number and of its square due to cavity damping over the time T are

$$\langle n(N)\rangle_d = -T\kappa n, \text{ and } \langle n^2(N)\rangle_d = T\kappa n. \tag{17}$$

The changes in $\langle n \rangle$ and $\langle n^2 \rangle$ can be added, due to our assumption that gain and decay act independently. In the lowest order in T these quantities are given as

$$\langle \dot{n} \rangle = \langle n \rangle_g + \langle n \rangle_d = TQ(n),$$

$$(\Delta n)^2 = \langle n^2 \rangle_g - \langle n \rangle_g^2 + \langle n^2 \rangle_d - \langle n \rangle_d^2 = TG(n),$$

and

$$Q(n) = \kappa \left[N_{\text{ex}} \sin^2 \left(\frac{\sqrt{n}}{\sqrt{N_{\text{ex}}}} \theta \right) - n \right], \tag{18}$$

$$G(n) = \kappa \left[N_{\text{ex}} \sin^2 \left(\frac{\sqrt{n}}{\sqrt{N_{\text{ex}}}} \theta \right) + n \right]. \tag{19}$$

We replace the discrete distribution p_n by a continuous function $p(n)$, which fulfills the Fokker-Planck equation:

$$\frac{\partial p(n,\theta)}{\partial \theta} = -\frac{\partial}{\partial n}[Q(n)p(n,\theta)] + \tfrac{1}{2}\frac{\partial^2}{\partial n^2}[G(n)p(n,\theta)], \tag{20}$$

and correctly describes the photon number for large n. This approximation is given by the extrapolation of results (14)–(20), which are true for short times. The stationary solution of the Fokker-Planck Eq. (20) is

$$p(n) = C \frac{1}{G(n)} \exp \left(2 \int \frac{Q(n)}{G(n)} dn \right), \tag{21}$$

where C is a normalization constant.
Formally, this result is incorrect because $p(n)$ is not normalizable in the interval $[0,\infty]$—the Fokker-Planck approach breaks down at $n \ll 1$—but it is not important in our considerations. The maximum of the probability distribution $p(n)$ corresponds to the global minimum of the effective potential

$$V(n) = -2 \int \frac{Q(n)}{G(n)} dn = -2 \int \frac{\sin^2(\frac{\sqrt{n}}{\sqrt{N_{\text{ex}}}}\theta) - \frac{n}{N_{\text{ex}}}}{\sin^2(\frac{\sqrt{n}}{N_{\text{ex}}}\theta) + \frac{n}{N_{\text{ex}}}} dn, \tag{22}$$

and it is obtained as a solution of the algebraic equation:

$$0 = V'(n_0) = -\frac{Q(n_0)}{G(n_0)} \iff \sin^2 \left(\frac{\sqrt{n_0}}{\sqrt{N_{\text{ex}}}} \theta \right) = \frac{n_0}{N_{\text{ex}}}. \tag{23}$$

Because $n_0 \gg 1$, we can replace $\sqrt{n_0+1}$ with $\sqrt{n_0}$ and calculate the final inversion for n_0 corresponding to a maximum of the photon number distribution:

$$w_{n_0}(\tau) = 1 - 2\sin^2 \left(\frac{\sqrt{n_0+1}}{\sqrt{N_{\text{ex}}}} \theta \right) \approx 1 - 2\frac{n_0}{N_{\text{ex}}}, \tag{24}$$

The inversion in this expression depends on parameter θ by condition $n_0 = n_0(\theta)$. The mean inversion is given by

$$\bar{w}^\theta_{n_0} = \frac{\sin(\frac{\sqrt{n_0+1}}{\sqrt{N_{\text{ex}}}}\theta)\cos(\frac{\sqrt{n_0+1}}{\sqrt{N_{\text{ex}}}}\theta)}{\frac{\sqrt{n_0+1}}{\sqrt{N_{\text{ex}}}}\theta} \approx -\frac{\sqrt{1-\frac{n_0}{N_{\text{ex}}}}}{\theta}, \tag{25}$$

where sign "−" was chosen in order that this result is in agreement with the exact graphical solution of the Eq. (23). When the distribution p^θ_n is narrowly concentrated around the maximum of the probability n_0, then the dominant contribution to the mean inversion $\bar{w}^\theta = \sum_{n=0}^{\infty} p^\theta_n w^\theta_n$, comes from $p^\theta_{n_0} w^\theta_{n_0}$ term. It is easy to see that the mean inversion shall be negative. This result is in very good agreement with the exact numerical calculations, presented in the next section. Of course, this is true for not too large θ, when the potential $V(n)$ has only a few minima.

B. Numerical results

Now we present the results of numerical calculations. First, we show in Fig. 1 the function $w(t)$ for $N_{\text{ex}} = 200$, $\theta = 200$ and in an inset for $N_{\text{ex}} = 200$, $\theta = 2.91$. The second graph looks like a sine function and it is obvious that for these values of arguments, integral in the definition of the mean inversion is negative, but for the first figure the negative mean inversion may seem surprising. Modulated oscillations suggest that the sign of the mean inversion may be arbitrary, but it is negative too. This result is exact, because we have done the integration analytically. As we will see later, this simple picture breaks down, because of strongly nonclassical properties of the probability of the photon number distribution. In Figs. 2 and 3 we show the final and mean inversion as a function of θ for $N_{\text{ex}} = 20$ and 200. If we compare pictures presented in the Ref. [11], an anticorrelation between average number of photons and final inversion shall be evident. This feature is characteristic for the micromaser and has a simple physical reason. The maximum of the photon number coincides with the minimal value of the final inversion. The field in the cavity increases, when consecutive atoms, after depositing its energy to the field will be leaving the cavity in a state close to the lower state. Moreover, for some values of interaction time τ or in other words pumping parameter θ, mean inversion is very close to one. This is a characteristic for trapping states. In this case, atoms interact with the electromagnetic field leaving the cavity excited. These effects are known and agree with our expectations. The next result is more interesting. After averaging, a complicated dependence of final inversion upon θ practically vanishes, except for a quite deep minimum — corresponding to lasing without inversion — and very small peaks for $N_{\text{ex}} = 20$, which represent trapping states. Except in a perturbative region corresponding to a very short interaction time, mean

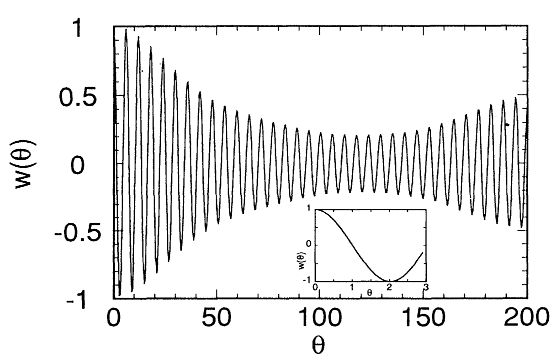

FIG. 1. One-photon micromaser. Time evolution of the atomic inversion $w(\theta(t))$ for $N_{\text{ex}} = 200$, and a pumping parameter $\theta = 200$. In the inset, $w(\theta(t))$ for $N_{\text{ex}} = 200$ and $\theta = 2.91$.

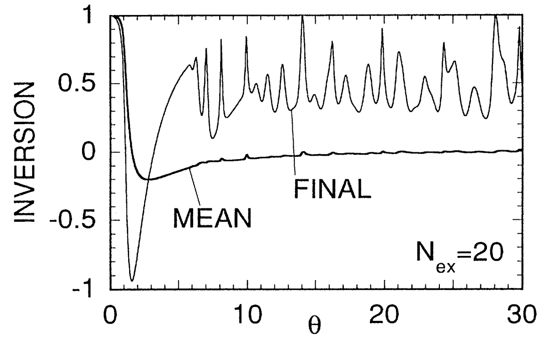

FIG. 2. One-photon micromaser. Final inversion w^θ and mean inversion $\bar{w}^\theta$ as a function of pumping parameter θ for $N_{\rm ex} = 20$.

inversion is negative for almost all values of parameter θ, and tends asymptotically to zero for θ going to large values. When θ crosses some critical value, a sign of the mean inversion becomes erratic, and $\bar{w}_\theta$ oscillates around zero. We can explain this effect in the following way. The probability distributions of the photon number has many peaks even for relatively weak pumping, when θ is very large. In this case, the distance between two neighboring paeks becomes smaller than their width, and they overlap. It means, that n-photon states with relatively large probability may correspond not only to minima of the $V(n)$ potential, but maxima too. In this case, the potential does not determine correctly positions of maxima of the photon number probability distribution. The mean inversion may be positive for these states.

Position and shape of the minimum of mean inversion is independent on pumping parameter $N_{\rm ex}$ in θ parametrization. This scaling law is one more interesting feature of the micromaser. On the other hand, the minimum of mean inversion $\bar{w}_\theta \approx -0.21$ for $\theta \approx 2.91$ is specially interesting because, as we show below it occurs in all considered cases. The negative value of the mean inversion suggests that emission is more probable than absorption during the flight of atoms across the cavity. This is the consequence of destructive interference between probability amplitudes of photon emission for different Fock-states of the field in the cavity. This effect in unexpected and does not occur in ordinary masers. The agreement between both exact numerical and approximate analytical results is very good.

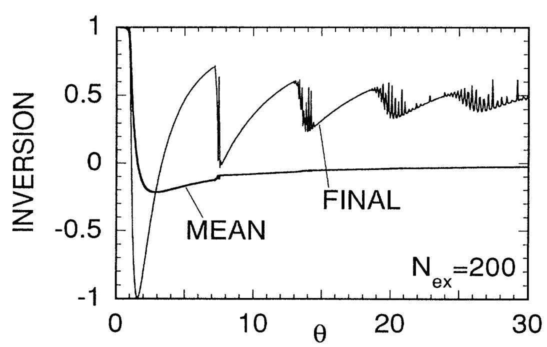

FIG. 3. One-photon micromaser. Final inversion w^θ and mean inversion $\bar{w}^\theta$ as a function of pumping parameter θ for $N_{\rm ex} = 200$.

III. REGULARLY PUMPED ONE-PHOTON MICROMASER

An interesting question is, how general is this feature? Now we present similar analysis, in the case when micromaser is pumped regularly. In this model, two-levels atoms are injected in a regular way, with a rate $r = \frac{1}{t_p}$, and leave the cavity after time τ. The population changes after the passage of one atom was found by Guerra *et al.* [13] and is given by

$$p_n(t+t_p) = e^{-\alpha n} \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} (1-e^{-\alpha})^k \Gamma_{k+n}(t), \quad (26)$$

where

$$\Gamma_n = (1-\beta_{n+1}) p_n + \beta_n p_{n-1}, \quad \alpha = \frac{1}{N_{\rm ex}}. \quad (27)$$

The resulting expression is not useful for analytical calculations. In order to use this expression for numerical calculation [13], we suppose that for some sufficiently large photon number n_0, $p_{n_0} = 0$. Under this condition, it is possible to calculate probability distribution for $0 < n < n_0 - 1$. After proper normalization, we are obtaining a very good approximation of photon number probability distribution, because for a sufficiently large n, p_n is equal to zero with a good numerical precision. To find the final and mean inversion, we insert p_n's into Eq. (26). For the most part, these results resemble the ones for Poissonian statistics. Only for strong pumping it is possible to notice quantitative but not qualitative differences.

It shows that the micromaser works without inversion, not only in a special case of pumping by the atomic beam with the Poissonian distribution. This feature occur in the regular pumping case as well and we think that it is independent of the statistics of atomic beam.

IV. TWO-PHOTON MICROMASER

Until now we have carried out calculations for one-photon transitions. Now we are going to the two-photon transition model of the micromaser. The system considered is a three-level cascade, coupled to a single mode of the electromagnetic field of frequency $\omega = (\omega_e - \omega_g)/2$; that is, exact two-photon resonance is assumed throughout. The detuning of the intermediate level $|i\rangle$ from exact one-photon resonance is

$$\delta = \omega - (\omega_e - \omega_g),$$

where $\hbar\omega_g$, $\hbar\omega_i$, $\hbar\omega_e$ are energies of ground, intermediate and excited levels, respectively. We suppose for simplicity that one-photon coupling constants are equal $\Omega_{ig} = \Omega_{ei} = \Omega$. This model describes the true two-photon micromaser, when we put detuning δ much larger than the coupling Ω, like it was presented in Ref. [14]. We assume that no more than one atom is present in the cavity at the same time. All atoms are injected in the upper state $|e\rangle$, and during the time τ interact with the electromagnetic field. The Hamiltonian in the interaction picture is

$$V_I = \Omega\{[e^{i\delta t}a|e\rangle\langle i| + e^{-i\delta t}a^\dagger|i\rangle\langle e|] + [e^{i\delta t}a|i\rangle\langle g| + e^{-i\delta t}a^\dagger|g\rangle\langle i|]\}, \quad (28)$$

where a, $a^\dagger$ are photon annihilation and creation operators. From the Schrödinger equation for a state vector

$$|\Psi_n(t)\rangle = C_{e,n}(t)|e,n\rangle + C_{i,n+1}(t)|i,n+1\rangle + C_{g,n+2}(t)|g,n+2\rangle, \quad (29)$$

we get the following system of equations:

$$\dot{C}_{e,n}(t) = -i\Omega e^{-i\delta t}\sqrt{n+1}C_{i,n+1}, \quad (30)$$

$$\dot{C}_{i,n+1}(t) = -i\Omega e^{i\delta t}\sqrt{n+1}C_{e,n} - i\Omega e^{i\delta t}\sqrt{n+2}C_{g,n+2}, \quad (31)$$

$$\dot{C}_{g,n+2}(t) = -i\Omega e^{-i\delta t}\sqrt{n+2}C_{i,n+1}. \quad (32)$$

We restrict our considerations to a case, when $\frac{\Omega^2 n}{\delta^2} \ll 1$. This condition is fulfilled for large detuning and weak pumping. In this limit the amplitude $C_{i,n}$ is much smaller than remaining amplitudes, and may by omitted. This means that the probability to find atom in the intermediate state is neglected, and we have effectively a two-level atom case, described by the following equations:

$$C_{e,n}(t_0+\tau) = 1 + \frac{n+1}{2n+3}(e^{i\frac{2n+3}{N_{\rm ex}}\eta} - 1), \quad (33)$$

$$C_{g,n+2}(t_0+\tau) = \frac{\sqrt{(n+1)(n+2)}}{2n+3}(e^{i\frac{2n+3}{N_{\rm ex}}\eta} - 1), \quad (34)$$

where in analogy to θ we have introduced the parameter $\eta = \frac{1}{2}\Omega^2 N_{\rm ex}\tau/\delta$. The equation describing the evolution of the reduced density matrix for the field obtained by tracing over the atomic states has the form

$$[\rho^f(t_0+\tau)]_{nm} = [\rho^f(t_0)]_{nm}C_{e,n}(t_0+\tau)C^*_{e,n}(t_0+\tau) + [\rho^f(t_0)]_{n-2,m-2}C_{g,n}(t_0+\tau) \times C^*_{g,m}(t_0+\tau). \quad (35)$$

Like in standard laser theory, we work out from Eq. (35) a differential equation for ρ^f_{nm}

$$\dot{\rho}^f_{nm} = \frac{\rho^f_{nm}(t+\tau) - \rho^f_{nm}(t)}{\Delta t} = r_a\delta\rho^f_{nm}, \quad (36)$$

where $\delta\rho^f_{nm} = \rho^f_{nm}(t+\tau) - \rho^f_{nm}(t)$, and $r_a = 1/\Delta t$ is the rate of injection. The total rate of change of the field density matrix is given by adding the terms describing the cavity losses

$$\dot{\rho}^f_{nm} = \tilde{a}_{nm}\rho^f_{nm} + \tilde{b}_{n-2,m-2}\rho^f_{n-2,m-2} + \tilde{c}_{n+1,m+1}\rho^f_{n+1,m+1}. \quad (37)$$

In this equation, we introduced the following notation:

$$\tilde{a}_{nm} = r_a(C_{e,n}C^*_{e,n} - 1) - \kappa\tfrac{1}{2}(n+m),$$
$$\tilde{b}_{nm} = r_a C_{g,n+2}C^*_{g,n+2}, \quad (38)$$
$$\tilde{c}_{nm} = \kappa\sqrt{nm},$$

and κ is the cavity-loss rate. We assume moreover, that the average number of photons in thermal equilibrium is equal to zero. At the steady state, the probability $p_n = \rho_{nn}$ to find n photons in the cavity satisfies the equation:

$$a_n p_n + b_{n-2}p_{n-2} + c_{n+1}p_{n+1} = 0, \quad (39)$$

where

$$a_{nm} = N_{\rm ex}(C_{e,n}C^*_{e,n} - 1) - \tfrac{1}{2}(n+m),$$
$$b_{nm} = N_{\rm ex}C_{g,n+2}C^*_{g,n+2}, \quad (40)$$
$$c_{nm} = \sqrt{nm} \qquad (a_n = a_{nn}, \text{ and so forth}).$$

Solution of this equation may by written as a product of continued fractions, like it was done in Ref. [14]:

$$p_n = p_0 \prod_{k=1}^{n} \frac{1}{a_n}\left[d_{k-1} + \cfrac{d_{k-2}c_{k-1}}{d_{k-2} + \cfrac{d_{k-3}c_{k-2}}{\ddots_{d_1 + \frac{d_0 c_1}{d_0}}}}\right]. \quad (41)$$

This recursive relation is very simple for numerical calculations. We insert evaluated values of probability into functions for final and mean inversions. In Fig. 4 we show the final inversion and in the inset the mean inversion for an arbitrarily chosen value of the parameter $N_{\rm ex} = 200$. Three characteristics are important in this case: a periodicity of the function, the anticorrelation between the average photon number, and the final inversion described earlier, and the trapping states, typical for the micromaser in general. We see that for two-photon micromaser the mean inversion has the same features as for the one-photon micromaser. The periodic structure of the interaction time dependence is not visible after averaging, except for the position of the trapping states, which are not so evident but are present, for large η. The mean inversion is negative for almost all values of the interaction time. It is interesting that these properties of the two-photon micromaser are insensitive to the change of detuning, as long as we fulfill conditions, which must be fulfilled for validity of this model. As before the mean inversion is non negative in the perturbative regime of

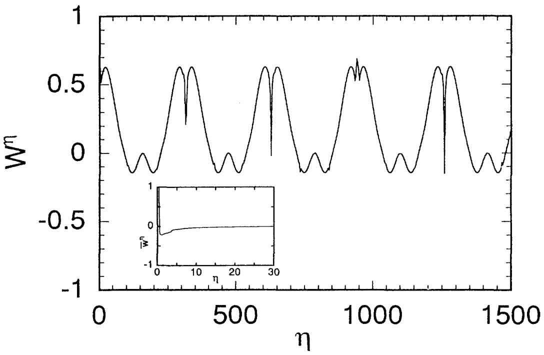

FIG. 4. Two-photon micromaser. Final inversion w^θ as a function of pumping parameter η for $N_{ex} = 200$, and detuning $\delta = 50$. In the inset, mean inversion for this same value of the parameter.

very short interaction time and in the vicinity of trapping states.

V. SEMICLASSICAL MODEL OF THE MICROMASER

In previous sections, we have worked with the quantum model of the micromaser, but it is of course also possible to give a semiclassical description of this device. It is interesting and instructive, because comparison of these two approaches may show which mechanisms are responsible for the lasing without inversion. Like before, we assume that excited two-level atoms are injected into the cavity. The flight time of the atoms across the interaction region τ is much less than the repetition time T, which is constant and equal for all atoms. It means that we consider now the regular pumping case only, but extensions to Poissonian pumping case is immediate. Instead of describing the atom-field interaction by density operator, we use the coupled Maxwell-Bloch equations neglecting atomic relaxations

$$\frac{dv}{dt} = \chi w, \quad \frac{dw}{dt} = -\chi v, \quad \frac{d\chi}{dt} = -\kappa\chi + gv, \tag{42}$$

where g is a coupling constant, v, w are the atomic dipole moment and the population inversion, χ is the Rabi frequency. We assume that τ is much less than the cavity damping time $t_{\rm cav} = \kappa^{-1}$ so we omit a decay term in the third equation while the atom is in the cavity. When atoms are absent the field decays exponentially:

$$\frac{d\chi}{dt} = -\kappa\chi \;\Rightarrow\; \chi(T) = e^{-\kappa T}\chi. \tag{43}$$

Moreover, we eliminate constant g. We define the scaled time variable $t \to t/\sqrt{g}$, and $\chi \to \sqrt{g}\chi$. First, we solve equations (42) approximately. If we assume that the Rabi frequency changes slowly and is constant while the atom is in the cavity, the two Bloch equations have very simple solutions:

$$v(t) = \sin(\chi t), \quad w(t) = \cos(\chi t), \tag{44}$$

which is valid for times $0 \le t \le \tau$. The differential amplification due to the passage of a single atom is given simply by the integral of the equation $\dot\chi = v$ so when we add a losses factor we have a map describing one step in the time evolution of the field

$$\chi_{n+1} = e^{-\kappa T}\left(\tau^2\frac{1-\cos(\chi_n)}{\chi_n} + \chi_n\right). \tag{45}$$

If the field is almost equal to zero we can expand the cosine function and find a threshold condition for the amplification:

$$\chi_{n+1} = \exp(-\kappa T)(\tfrac{1}{2}\tau^2 + 1)\chi_n \;\Rightarrow\; \exp(-\kappa T)(\tfrac{1}{2}\tau^2 + 1) \ge 1, \tag{46}$$

and

$$\exp(-\kappa T) = \frac{2}{2+\tau^2}.$$

It is possible to find a steady-state solution of Eq. (45) and to give stability conditions for this solution but it is not too interesting because we can solve Eqs. (42) exactly. When we substitute

$$v(t) = \sin(\phi(t)), \quad w(t) = \cos(\phi(t)), \tag{47}$$

we have the equation for χ as a equation of a mathematical pendulum: $\frac{d\phi^2}{dt^2} = \sin(\phi(t))$. Solution of this equation is known very well and is expressed by elliptic functions. Instead of giving these solutions in the apparent form, we show results of numerical computations. In Fig. 5 we show the Rabi frequency and the mean inversion as a

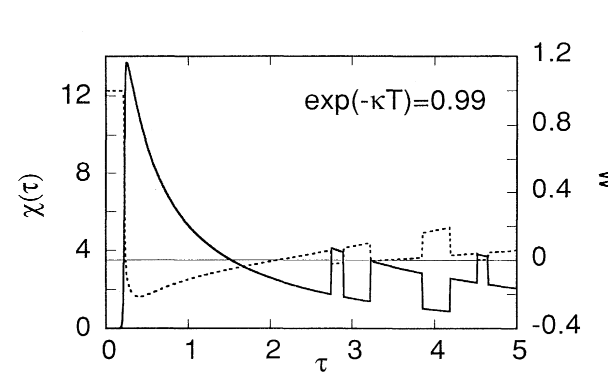

FIG. 5. Semiclassical model. Rabi frequency η and mean inversion $\bar w^\theta$ as a function of time interaction τ for $\exp(-\kappa T) = 0.99$.

function of the interaction time τ for $\exp(-\kappa T) = 0.99$. For small τ we have excellent agreement with results obtained in previous sections. For larger τ the mean inversion becomes positive, that means that our semiclassical model is not valid in this region. The conclusion is that the field quantization, although important is not the major factor in the inversionless action of the micromaser. Much more important may by the neglect of the atomic losses, which can begin to be significative when the active atoms speed is reduced.

VI. CONCLUSIONS

In our paper, we have shown that micromaser is a maser working without mean inversion (defined as time average inversion over the interaction time). We have shown that this effect does not depend on pumping statistics, and of a kind of atomic transition (single or two photon). We have also shown that field quantization although important is not a decisive factor in our analysis. So we think, that the inversionless lasing is a general and interesting property of the micromaser that may be observed in future experiments. Some of the effects presented could be seen in other kinds of lasers where the active atoms cross the interaction region very fast.

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