Classical and quantum Malus laws

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The classical and the quantum Malus laws for light and spin are discussed. It is shown that for spin $\frac{1}{2}$, the quantum Malus law is equivalent in form to the classical Malus law provided the statistical average involves a quasidistribution function that can become negative. A generalization of Malus's law for arbitrary spin s is obtained in the form of a Feynman path-integral representation for the Malus amplitude. The classical limit of the Malus amplitude for $s \to \infty$ is discussed.

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I. INTRODUCTION

The classical Malus law predicts an attenuation of a polarized light beam through a linear polarizer. This attenuation depends on the relative angle α between the polarization direction $\vec{a}$ of the incoming wave and the orientation $\vec{a}'$ of the polarizer. According to Malus's law the attenuation of the light intensity is just $\cos^2 \alpha$. If the incoming beam consists of a statistical mixture of polarized light, the probability to go through a linear polarizer is

$$p = \int d\Omega P_{cl}(\Omega) \cos^2 \alpha. \tag{1}$$

In this formula the integration is over all possible angles of the random polarization direction $\vec{a}$ described by a solid angle $\Omega = (\theta, \phi)$ and the classical distribution function $P_{cl}(\Omega)$ characterizes the statistical properties of the incident light beam polarization.

It is the purpose of this paper to discuss the quantum Malus law for spin systems. Entangled spin correlations provide examples of such systems. We show that, in general, the quantum Malus law is equivalent in form to the classical Malus law provided the statistical average involves a quasidistribution function that can become negative. A generalization of the Malus law for an arbitrary spin-s system is obtained. Using a Feynman path-integral representation for the Malus amplitude, the relation between Malus's amplitude and the Malus law is obtained in the limit of $s \to \infty$. The classical limit of the Malus law is discussed and classical equations of motion for spin systems are derived. The motivation for the discussion of the Malus law for higher spins comes from the fact that systems involving higher-spin states or many particles exhibit very strong quantum correlations [1,2].

II. QUANTUM MALUS LAW

In quantum mechanics a similar Malus law holds for spin-$\frac{1}{2}$ particles detected by a Stern-Gerlach apparatus oriented in the direction $|\vec{a}'\rangle$. An arbitrary pure state of the spin $\frac{1}{2}$ can be written as a linear superposition of the up $|+\rangle$ and down $|-\rangle$ spin states

$$|\Omega\rangle = e^{i\phi} \sin\frac{\theta}{2}|+\rangle + \cos\frac{\theta}{2}|-\rangle, \tag{2}$$

where the solid angle Ω characterizes the spin orientation on a unit sphere (the Bloch sphere).

The quantum amplitude for the transmission of such a spin state through a Stern-Gerlach apparatus is $\mathcal{A} = \langle\Omega|\vec{a}'\rangle$ and the probability is just the quantum Malus transmission function

$$p = |\mathcal{A}|^2 = \cos^2\frac{\alpha}{2}, \tag{3}$$

where

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{4}$$

is the relative angle between the spherical orientation Ω of the detected state (2) and the spherical direction Ω' of the Stern-Gerlach polarizer. Photons and spins differ in this formulation by a factor $\frac{1}{2}$ in the relative angle involved in the Malus law.

Following the classical Malus law for an unpolarized light beam (1), one can write the following probability for an arbitrary mixed state of the spin-$\frac{1}{2}$ system detected by the Stern-Gerlach apparatus:

$$p = \int d\Omega P(\Omega) \cos^2\frac{\alpha}{2}. \tag{5}$$

The function $P(\Omega)$ plays the role of a statistical distribution for an arbitrary beam of spin-$\frac{1}{2}$ particles. In quantum mechanics one deals with probability amplitudes rather than probabilities and one should sum these amplitudes first, before squaring the result. Malus's ampli-

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tudes can be derived in such a way [3]. Nevertheless, the formula (5) is valid in quantum mechanics if the quantum mechanical distribution $P(\Omega)$ is a quasidistribution. The quantum quasidistribution function is associated with an arbitrary density matrix $\hat{\rho}$ of the spin-$\frac{1}{2}$ system in the following way:

$$\hat{\rho} = \int d\Omega P(\Omega)|\Omega\rangle\langle\Omega|. \tag{6}$$

In this expression the diagonal weight function $P(\Omega)$ is a quantum quasiprobability distribution and accordingly contains all the statistical information about the spin state. This formula is similar in its structure to the diagonal Glauber P representation for a harmonic oscillator if coherent states are used [4]. For spin $\frac{1}{2}$ the corresponding spin coherent states (SCSs) are given by unit directions (2) on the Bloch sphere. From the properties of the SCSs, one concludes that the quasidistribution function is normalized $\int d\Omega P(\Omega) = 1$, but in general it is not positive definite or unique [5]. For example, for the up and the down spin states $|\pm\rangle$, the corresponding quasidistributions are $P_{\pm}(\Omega) = \frac{1}{4\pi}(1 \mp 3\cos\theta)$. These functions can take up negative values that indicate the quantum character of these states. As another example, the incoherent (mixed) state of the spin system described by the density matrix $\hat{\rho} = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$ leads to a distribution that has a purely classical behavior corresponding to a uniform distribution of directions on the Bloch sphere, i.e., $P(\Omega) = \frac{1}{4\pi}$ [6].

The quantum character of the negative quasiprobability is seen best for correlations involving a density operator for an entangled Einstein-Podolsky-Rosen state [7]. For such a correlated system of two spin-$\frac{1}{2}$ particles, labeled by indices a and b, the singlet wave function is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_a \otimes |-\rangle_b - |-\rangle_a \otimes |+\rangle_b\right). \tag{7}$$

The corresponding quasidistribution function has the form [8]

$$P(\Omega_a;\Omega_b) = \frac{1}{(4\pi)^2}[1 + 9\cos\theta_a\cos\theta_b + 9\sin\theta_a\sin\theta_b\cos(\phi_a - \phi_b]. \tag{8}$$

The quantum Malus law (5), if applied to the joint correlations involving two Stern-Gerlach detectors (with directions $\vec{a}$ and $\vec{b}$), provides the following joint probability for detection:

$$p(\vec{a};\vec{b}) = \int d\Omega'_a \int d\Omega'_b P(\Omega'_a;\Omega'_b) \times \cos^2\alpha(\Omega_a,\Omega'_a)\cos^2\alpha(\Omega_b,\Omega'_b). \tag{9}$$

This correlation function evaluated with the expression (8) leads to the well known quantum mechanical result for the joint spin correlation: $p(\vec{a};\vec{b}) = \frac{1}{2}(1 - \vec{a}\cdot\vec{b})$. The distribution function (8) is not unique because the same result is reproduced if one uses the quasidistribution function

$$P(\Omega'_a;\Omega'_b) = \frac{3}{4\pi}\delta^{(2)}(\Omega'_a + \Omega'_b) - \frac{2}{(4\pi)^2}. \tag{10}$$

Formula (9) has the formal structure of a hidden variable theory. In such a theory the joint probability function is calculated from the expression

$$p(\vec{a};\vec{b}) = \int d\lambda_a \int d\lambda_b\ P(\lambda_a;\lambda_b)\ t(\vec{a},\lambda_a)\ t(\vec{b},\lambda_b), \tag{11}$$

where $P(\lambda_a;\lambda_b)$ describes the distribution of some hidden variables λ_a and λ_b and the objective realities of the spin variables are given by the deterministic transmission functions $t(\vec{a},\lambda_a)$ and $t(\vec{b},\lambda_b)$ through the Stern-Gerlach apparatus. It is clear that the quantum mechanical formula (9) has the form of a hidden variable theory with the local spin realities given by $\cos^2\alpha(\Omega_a,\Omega'_a)$ and $\cos^2\alpha(\Omega_b,\Omega'_b)$. In such a theory the hidden parameters are represented by "hidden angles" on a Bloch sphere and are distributed according to Eq. (8) or (10). There the analogy ends because the quantum distribution of these "hidden directions" is given by a nonpositive function that leads to the failure of Bell's inequalities for such an entangled state.

The SCSs and the quantum Malus law (5) can be generalized to an arbitrary spin s. The spin-s coherent states are obtained by a rotation of the maximum down spin state $|s,-s\rangle$ [9]:

$$|\Omega\rangle = \exp(\tau\hat{S}_+ - \tau^*\hat{S}_-)|s,-s\rangle, \tag{12}$$

where $\tau = \frac{1}{2}\theta e^{-i\phi}$ and $\hat{S}_\pm$ are the spin-s ladder operators. The SCSs form an overcomplete set of states on the Bloch sphere

$$\frac{2s+1}{4\pi}\int d\Omega|\Omega\rangle\langle\Omega| = I. \tag{13}$$

Using these formulas, it is easy to calculate Malus's quantum amplitude and the probability for a transmission of such a state through a Stern-Gerlach apparatus. As a result one obtains

$$p = |\langle\Omega|\Omega'\rangle|^2 = \left(\cos\frac{\alpha}{2}\right)^{4s}, \tag{14}$$

with a straightforward generalization involving an arbitrary quasidistribution function $P(\Omega)$ for a density operator of a system with arbitrary spin s. This quantum mechanical expression for the transmission function provides a generalization of the spin-$\frac{1}{2}$ Malus law (3) to the case of an arbitrary spin-s system.

III. PATH-INTEGRAL FORM OF MALUS'S LAW

This quantum Malus law for arbitrary spin is well suited to study the relation between classical and quantum features of the transmission function. In quantum mechanics the primary object is the probability amplitude for the transmission of a SCS $|\Omega\rangle$ through a Stern-

Gerlach apparatus $|\vec{a}\rangle$ characterized by a solid angle Ω'. This probability amplitude is $\mathcal{A} = \langle\Omega|\Omega'\rangle$ and can be cast into a path-integral form exhibiting various quantum paths contributing to the transition. Following the basic idea of path integration [10], one can evaluate the Malus amplitude by dividing the spin trajectories on the Bloch sphere into infinitesimal subintervals $|\Omega_i\rangle$, where $i = 1, \ldots, N$ with $\Omega_0 = \Omega'$ and $\Omega_N = \Omega$. Using the decomposition of unity for the SCS for each subinterval and the infinitesimal form of the Malus amplitude $\langle\Omega_i|\Omega_{i-1}\rangle$, one obtains

$$\mathcal{A} = \int d\Omega_1 \frac{2s+1}{4\pi} \int d\Omega_2 \frac{2s+1}{4\pi} \cdots \times \exp\left(-is\sum_{i=1}^{N}(\phi_i - \phi_{i-1})\cos\theta_{i-1}\right). \quad (15)$$

In the limit of $N \to \infty$, this expression can be written in the form of the spin Feynman path integral

$$\mathcal{A} = \int \mathcal{D}\Omega \frac{2s+1}{4\pi} \exp\left(-is\int d\phi \cos\theta\right), \quad (16)$$

where $\mathcal{D}\Omega$ is the functional path-integration measure over all spin trajectories connecting $|\Omega'\rangle$ with $|\Omega\rangle$ on the Bloch sphere. This path-integral representation of the quantum Malus law can be cast in a more familiar form if the spherical angles are identified with the canonical position and the canonical momentum in the following way: $\phi \Leftrightarrow q$, $\cos\theta \Leftrightarrow p$, and $d\Omega = d\phi d\cos\theta \Leftrightarrow dqdp$. Using these variables we can rewrite the path integral (16) in the form

$$\mathcal{A} = \int \mathcal{D}q\mathcal{D}p \frac{2s+1}{4\pi} \exp\left(-is\int pdq\right), \quad (17)$$

which is the spin analog of the phase-space path integral for the following quantum mechanical amplitude in the configuration space:

$$\langle q|q'\rangle = \int \frac{\mathcal{D}q\mathcal{D}p}{2\pi\hbar} \exp\left(-\frac{i}{\hbar}\int dqp\right). \quad (18)$$

The Malus probability for the spin-s transition of the state $|\Omega\rangle$ through such a Stern-Gerlach apparatus can be expressed as a product of four path integrals

$$|\mathcal{A}|^2 = \int \mathcal{D}q_1\mathcal{D}p_1 \frac{2s+1}{4\pi} \int \mathcal{D}q_2\mathcal{D}p_2 \frac{2s+1}{4\pi} \times \exp\left[i\mathcal{S}(q_1,p_1) - i\mathcal{S}(q_2,p_2)\right], \quad (19)$$

where the classical action is

$$\mathcal{S}(q,p) = s\int pdq + \frac{1}{\hbar}\int \mathcal{H}dt. \quad (20)$$

In this expression a classical Hamiltonian $\mathcal{H}$ has been added in order to describe a possible dynamical time evolution before the particle has reached the Stern-Gerlach apparatus.

In order to see the connection with the quantum Malus law (5) and the classical Malus law (1) one can investigate the properties of Malus's transmission function in the classical limit corresponding to $s \to \infty$. The transition from quantum amplitudes to classical probabilities can be carried out, if the four path integrals can be simplified. In configuration space the classical limit of the path integral can be investigated using a suitable change of variables [11]. In the case of the path integral for Malus's probability this change of variables is

$$q_{1,2} = q \pm \frac{1}{2s}\tilde{q}, \quad p_{1,2} = p \pm \frac{1}{2s}\tilde{p}. \quad (21)$$

In the limit of $s \to \infty$, the path integrals with respect to $\mathcal{D}\tilde{q}$ and $\mathcal{D}\tilde{p}$ can be performed, leading to functional Dirac's functions, and the entire expression for the probability simplifies to

$$\lim_{s\to\infty} |\mathcal{A}|^2 = \int \mathcal{D}q \int \mathcal{D}p\ \delta\left(\dot{q} - \frac{1}{s\hbar}\frac{\partial\mathcal{H}}{\partial p}\right)\ \delta\left(\dot{p} + \frac{1}{s\hbar}\frac{\partial\mathcal{H}}{\partial q}\right). \quad (22)$$

This expression shows that in the classical limit, the spin-s Malus transmission function reduces to a classical dynamics on the Bloch sphere with the following canonical equations of motion:

$$\dot{\phi} = \{\phi, \mathcal{H}\}, \quad \dot{\theta} = \{\theta, \mathcal{H}\}. \quad (23)$$

From the reduced path-integral formula (22) one obtains that the Poisson bracket of the classical dynamics is

$$\{A, B\} = \frac{1}{s\hbar\sin\theta}\left(\frac{\partial A}{\partial\phi}\frac{\partial B}{\partial\theta} - \frac{\partial A}{\partial\theta}\frac{\partial B}{\partial\phi}\right). \quad (24)$$

In these equations one recognizes the classical equations of motion of a particle confined to a sphere. The Poisson bracket in this case has a typical structure for a curved phase space associated with the Bloch sphere [12].

If the Malus law is applied to an arbitrary spin-s system described by a quasidistribution function, in the limit $s \to \infty$ the expression (22) corresponds to a classical statistical mechanics on a unit sphere. These classical trajectories are distributed with a classical distribution function $P_{cl}(\Omega)$ emerging from $P(\Omega)$ in the limit $s \to \infty$.

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