Interaction between a moving mirror and radiation pressure: A Hamiltonian formulation

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We present a nonrelativistic Hamiltonian of the interaction between a moving mirror and radiation pressure. This Hamiltonian is derived directly from the equation of motion of a moving mirror, and the wave equation with time-varying boundary conditions. We discuss the canonical quantization of both the field and the motion of the mirror.

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I. INTRODUCTION

The mechanical interaction between a moving mirror and a radiation field has been an important topic for the study of very high precision optical interferometers in which radiation pressure effects cannot be ignored [1–4]. The subject is interesting, not only because of practical purposes, but also because a coupled field-mirror system represents a fundamental system in quantum optics. Surprising quantum phenomena, such as photon emission from a nonuniformly moving mirror [5–9], resonance enhancement of Casimir force [10,11], and the vacuum mode locking effect [12] have been predicted according to quantum theory. Recently, there are also experimental proposals indicating that radiation pressure and a movable mirror can be used to generate squeezed light [13,14] and perform quantum nondemolition measurements [15].

One fundamental question regarding the moving-mirror system is the Hamiltonian formalism. Among the previous works on the moving-mirror system there is a lack of a rigorous Hamiltonian approach which can be used to determine the self-consistent coupled dynamics of mirror and field. If a Hamiltonian of the system does exist, it can provide us with a fundamental basis for the mirror-field interaction. More importantly, the Hamiltonian can serve as a starting point for a fully quantized theory in which both the (macroscopic) coordinates of the mirror and the (microscopic) field variables are quantized. It has been a challenging question how to quantize an electromagnetic field when its boundary (e.g., a mirror) moves under the effect of radiation pressure. The situation is complicated by the fact that the boundary condition provided by the movable mirror depends on the field itself. Quantum fluctuations of the field can change the position of the mirror which would in turn affect the field again [16]. Therefore a reasonable approach is to consider the mirror and the field as a self-consistent system and the corresponding Hamiltonian becomes the key to the whole problem.

The main purpose of this paper is to present a nonrelativistic Hamiltonian of a coupled mirror-field system. This Hamiltonian is constructed directly from Newton's equation of the mirror and the wave equation of the field with appropriate boundary conditions. As Fulling and Davies [6] pointed out, the radiation process of a moving mirror is analogous to the radiation by a moving charge, and we find that the mirror-field coupling indeed shares some similarities to the minimal coupling in electrodynamics. For the sake of simplicity, we shall discuss only a one-dimensional configuration. The Hamiltonian is first derived from a classical consideration. We then discuss the canonical quantization of the system.

II. EQUATIONS OF MOTION

We begin by considering a one-dimensional cavity formed by two perfectly reflecting mirrors. One of the mirrors is fixed at the position $x = 0$ and the other moves in a potential well $V(q)$. The motion of this movable mirror is also influenced by the radiation pressure of the cavity fields. We label the mass and the position of the movable mirror by m and $q(t)$, respectively. The mirror and the cavity field constitute an energy conservative system. The Hamiltonian of the system, however, cannot be written down immediately. This is because the explicit form of the mirror-field interaction is not known and there is no obvious way to define the canonical momentum of the mirror, which is not necessarily identical to the kinetic momentum. Our strategy is to examine the equations of motion of the field and the mirror, and to try to identify the Hamiltonian structure of the system. We should mention that when the position of the moving mirror $q(t)$ is treated classically and is a prescribed function of time [i.e., $q(t)$ is not a dynamical degree of freedom], one can find an effective Hamiltonian for the fields [9,17–19]. We shall generalize the effective Hamiltonian in [19] to include the mirror's position and momentum as dynamical variables.

The vector potential $A(x,t)$ of the cavity field is defined in the region $0 \leq x \leq q(t)$ and obeys the wave equation ($c = 1$),

$$\frac{\partial^2 A(x,t)}{\partial x^2} = \frac{\partial^2 A(x,t)}{\partial t^2}. \tag{2.1}$$

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We impose the time-dependent boundary conditions [5]

$$A(0,t) = A(q(t),t) = 0 \tag{2.2}$$

so that the electric fields are always zero in the rest frame of the mirror surface. Notice that it is sufficient to treat $A(x,t)$ as a scalar field in our one-dimensional situation because the two polarizations of the field do not interact with each other.

The nonrelativistic (Newton's) equation of motion of the mirror is given by

$$m\ddot{q} = -\frac{\partial V(q)}{\partial q} + \frac{1}{2}\left(\frac{\partial A(x,t)}{\partial x}\right)^2\Bigg|_{x=q(t)} . \tag{2.3}$$

The radiation pressure force, which is the second term of the right side of Eq. (2.3), can be derived from the radiation pressure force appearing in the rest frame of the movable mirror. This is because in the co-moving frame the radiation pressure force is given by $B'^2/2$, where B' denotes the magnetic field on the mirror's surface in the co-moving frame (the corresponding electric field in the same frame is always zero according to the boundary condition). A straightforward transformation of the force to the laboratory frame in the nonrelativistic limit would yield the force expression given in (2.3). It should be noted that the value of $q(t)$ is strictly positive or zero. The potential well $V(q)$ at $q=0$ acts like an infinite potential wall which forbids the moving mirror from penetrating through the fixed mirror.

The dynamics of the system is completely specified by Eqs. (2.1), (2.2), and (2.3). We now define a set of generalized coordinates $\{Q_k\}$ by

$$Q_k \equiv \sqrt{\frac{2}{q(t)}}\int_0^{q(t)} dx A(x,t) \sin\frac{k\pi x}{q(t)} , \tag{2.4}$$

where k are positive integer. The meaning of Q_j is obvious from its definition, it is basically the mode decomposition of the fields, but unlike the usual situation, the mode basis functions used here are determined by the instantaneous position of the mirror [19]. The completeness of the mode functions enables us to write

$$A(x,t) = \sum_{k=1}^{\infty} Q_k(t)\sqrt{\frac{2}{q(t)}}\sin\frac{k\pi x}{q(t)}. \tag{2.5}$$

Equation (2.5) is a general expression of $A(x,t)$ obeying the time-dependent boundary condition (2.2).

With the help of (2.5) and the orthogonality of the mode functions, we can show that (2.1) and (2.3) are equivalent to

$$\ddot{Q}_k = -\omega_k^2 Q_k + 2\frac{\dot{q}}{q}\sum_j g_{kj}\dot{Q}_j + \frac{\ddot{q}q-\dot{q}^2}{q^2}\sum_j g_{kj}Q_j + \frac{\dot{q}^2}{q^2}\sum_{jl} g_{jk}g_{jl}Q_l , \tag{2.6}$$

$$m\ddot{q} = -\frac{\partial V(q)}{\partial q} + \frac{1}{q}\sum_{k,j}(-1)^{k+j}\omega_k\omega_j Q_k Q_j. \tag{2.7}$$

Here the position-dependent frequencies ω_k are given by

$$\omega_k(q) = \frac{k\pi}{q}, \tag{2.8}$$

and the dimensionless coefficients g_{kj} are given by [20]

$$g_{kj} = \begin{cases} (-1)^{k+j}\dfrac{2kj}{j^2-k^2} , k \neq j & (2.9)\\ 0 , k = j . & (2.10)\end{cases}$$

Therefore the dynamics of the coupled field-mirror system is defined by Eqs. (2.6) and (2.7).

III. THE HAMILTONIAN AND QUANTIZATION

Now the basic question is whether the dynamical equations (2.6) and (2.7) can be considered as a consequence of a set of Euler-Lagrange equations with respect to a Lagrangian L. By examining Eqs. (2.6) and (2.7), we find that one can indeed construct a Lagrangian L,

$$L(q,\dot{q},Q_k,\dot{Q}_k) = \frac{1}{2}\sum_k\left[\dot{Q}_k^2 - \omega_k^2(q)Q_k^2\right] + \frac{1}{2}m\dot{q}^2 - V(q) - \frac{\dot{q}}{q}\sum_{j,k} g_{kj}\dot{Q}_k Q_j + \frac{\dot{q}^2}{2q^2}\sum_{j,k,l} g_{kj}g_{kl}Q_l Q_j, \tag{3.1}$$

so that the corresponding Euler-Lagrangian equations are equivalent to (2.6) and (2.7). The Hamiltonian associated with this L is defined by

$$H(P_k,Q_j,p,q) \equiv p\dot{q} + \sum_k P_k\dot{Q}_k - L(q,\dot{q},Q_k,\dot{Q}_k) , \tag{3.2}$$

where P_k and p are canonical momenta conjugate to Q_k and q, respectively,

$$P_k = \dot{Q}_k - \frac{\dot{q}}{q}\sum_j g_{kj}Q_j, \tag{3.3}$$

$$p = m\dot{q} - \frac{1}{q}\sum_{j,k} g_{kj}P_k Q_j. \tag{3.4}$$

We see that the mirror's canonical momentum p is not equal to the kinetic momentum $m\dot{q}$ for nonzero fields. The explicit expression of the Hamiltonian (3.2) now reads

$$H = \frac{1}{2m}\left(p + \frac{1}{q}\sum_{j,k} g_{kj}P_k Q_j\right)^2 + V(q) + \frac{1}{2}\sum_k\left[P_k^2 + \omega_k^2 Q_k^2\right]. \tag{3.5}$$

It is not difficult to check that H itself represents the total energy of the system, i.e.,

$$H = H_{field} + \frac{1}{2}m\dot{q}^2 + V(q) , \tag{3.6}$$

where H_{field} is the field energy defined by

$$H_{field} = \frac{1}{2}\int_0^{q(t)} dx \left[\left(\frac{\partial A(x,t)}{\partial t}\right)^2 + \left(\frac{\partial A(x,t)}{\partial x}\right)^2\right]. \tag{3.7}$$

As a remark, when $q(t)$ is treated as a prescribed time-dependent parameter (i.e., not as a dynamical variable), one can recover the same effective Hamiltonian obtained in Refs. [18,19] from our Lagrangian (3.1). The Hamiltonian (3.5) is a more general expression because the mirror is included as a dynamical degree of freedom, and so allows us to consider the effects of radiation pressure.

The classical Hamiltonian (3.5) provides us with a basis for the quantization of the system. Following the canonical quantization procedure, we let the variables p, q, P_k, Q_k be operators, which obey the commutation relations

$$[\hat{q}, \hat{Q}_j] = [\hat{q}, \hat{P}_k] = [\hat{p}, \hat{Q}_j] = [\hat{p}, \hat{P}_k] = 0, \tag{3.8}$$

$$[\hat{q}, \hat{p}] = i\hbar, \quad [\hat{Q}_j, \hat{P}_k] = i\delta_{jk}\hbar. \tag{3.9}$$

In order to specify the quantum state of the field in the Fock space, we define the cavity-length-dependent creation and annihilation operators for each cavity mode by

$$a_k(\hat{q}) = \sqrt{\frac{1}{2\hbar\omega_k(\hat{q})}}\left[\omega_k(\hat{q})\hat{Q}_k + i\hat{P}_k\right], \tag{3.10}$$

$$a_k^\dagger(\hat{q}) = \sqrt{\frac{1}{2\hbar\omega_k(\hat{q})}}\left[\omega_k(\hat{q})\hat{Q}_k - i\hat{P}_k\right]. \tag{3.11}$$

The dependence on the operator $\hat{q}$ indicates that for each position of the mirror we have a set of Fock states associated with that position. We label such a set of Fock states by $|\{n_l\}, q\rangle$ where $\{n_l\} = \{n_1, n_2, n_3, ...\}$ denotes the set of occupation numbers for different cavity modes. The state vector $|\{n_l\}, q\rangle$ is the simultaneous eigenvectors of the number operator $a_k^\dagger(\hat{q})a_k(\hat{q})$ and the position operator $\hat{q}$, i.e.,

$$a_k^\dagger(\hat{q})a_k(\hat{q})|\{n_l\}, q\rangle = n_k|\{n_l\}, q\rangle, \tag{3.12}$$

$$\hat{q}|\{n_l\}, q\rangle = q|\{n_l\}, q\rangle. \tag{3.13}$$

Such a set of states is orthogonal and is assumed complete, so any quantum state $|\Psi\rangle$ of the whole system can be expressed as superposition of these states, i.e.,

$$|\Psi\rangle = \sum_{\{n_l\}}\int_0^\infty dq \;\; C(\{n_l\}, q)|\{n_l\}, q\rangle, \tag{3.14}$$

where $C(\{n_l\}, q)$ is the probability-amplitude density. It should be noted that the creation and annihilation operators are not defined at the point $q = 0$. This is the situation when the cavity has zero length. We eliminate this problem by imposing a boundary condition of the system wave function such that the wave function is identically zero at $q = 0$. This is analogous to the infinite potential well problem in quantum mechanics.

The quantal Hamiltonian now reads

$$H = \frac{(p+\Gamma)^2}{2m} + V(q) + \hbar\sum_k \omega_k(q)\left[a_k^\dagger a_k + \frac{1}{2}\right], \tag{3.15}$$

where

$$\Gamma \equiv \frac{i\hbar}{2q}\sum_{k,j} g_{kj}\left[\frac{k}{j}\right]^{1/2}\left[a_k^\dagger a_j^\dagger - a_k a_j + a_k^\dagger a_j - a_j^\dagger a_k\right]. \tag{3.16}$$

We have used a short notation $a_k = a_k(q)$ for convenience. It is interesting to recognize the similarity between (3.15) and the minimal coupling Hamiltonian in electrodynamics. The major difference here is that the operator Γ is quadratic, which contributes to two-photon emission and absorption processes [19].

The vacuum field energy appearing in (3.15) is divergent and is the origin of the Casimir force. We follow the usual procedure [21] to obtain the Casimir energy $-\hbar\pi/24q$ for one-dimensional space,

$$H = \frac{(p+\Gamma)^2}{2m} + V(q) + \hbar\sum_k \omega_k(q)a_k^\dagger a_k - \frac{\hbar\pi}{24q}. \tag{3.17}$$

However, we must point out that in replacing the vacuum energy terms by the Casimir energy, we actually "borrow" the field energy from the *outside* in order to compensate the infinite change of the vacuum energy inside the cavity as the mirror changes its position. The fact that the Casimir force is finite is because of the cancellation of the divergent parts of the vacuum pressure from both sides of the mirror [22]. Therefore one cannot single out the Casimir energy without taking the outside field into account, that is, the field at $x > q$. To maintain the consistency of the theory, one should include the outside field as additional dynamical degrees of freedom. Hamiltonian (3.17) therefore is an approximation, since it only counts the static part (the Casimir effect) of the interaction between the mirror and the outside field. The dynamic part, which describes the changing of the field outside the cavity, is ignored. Nevertheless, in most physical situations where the cavity field is dominant, Hamiltonian (3.17) is a good approximation. A simple case is when the cavity initially contains an appreciable number of photons, so the dynamic effects of the field outside the cavity can be neglected. We, however, remark that if the movable mirror is also perfectly reflective for the field outside the cavity, we can generalize our method to obtain a full Hamiltonian which includes the outside field.

IV. LINEAR APPROXIMATION

The Hamiltonian (3.17) exhibits the nonlinear nature of the coupling between a field and a moving mirror. In most situations the mirror is bounded by a potential $V(q)$ which keeps the mirror moving around a certain equilibrium position l_0, and the radiation pressure force acts as a small perturbation. We now present a linearized form of the Hamiltonian when the displacement $x_m \equiv q - l_0$ is small compared with l_0. In this case, one can write

$$\Gamma \approx \Gamma_0 \ , \tag{4.1}$$

where $\Gamma_0 = \Gamma|_{q=l_0}$ is the operator evaluated at the equilibrium position. Furthermore, the small parameter x_m/l_0 allows us to make the following expansions:

$$a_k(q) \approx a_{k0} - \frac{x_m}{2l_0} a_{k0}^\dagger, \tag{4.2}$$

$$\omega_k(q) \approx \omega_{k0} \left(1 - \frac{x_m}{l_0} \right) , \tag{4.3}$$

where a_{k0} and ω_{k0} denote the annihilation operator and the frequency associated with the equilibrium position, respectively.

We substitute Eqs. (4.1)–(4.3) into the Hamiltonian (3.17), and make a unitary transformation $H' = T^\dagger H T$ with the transformation operator

$$T = \exp \{ i x_m \Gamma_0 / \hbar \} . \tag{4.4}$$

It then follows that

$$H' \approx \frac{p^2}{2m} + u(x_m) + \hbar \sum_k \omega_{k0} a_{k0}^\dagger a_{k0} - x_m F_0. \tag{4.5}$$

Here $u(x_m) = V(q) - \hbar\pi/24q$ is the potential which includes the Casimir energy. The symbol F_0 denotes the normally ordered radiation pressure force,

$$F_0 = \frac{\hbar}{2l_0} \sum_{k,j} (-1)^{k+j} \sqrt{\omega_{k0}\omega_{j0}} \times (a_{k0}a_{j0} + a_{k0}^\dagger a_{j0}^\dagger + a_{k0}^\dagger a_{j0} + a_{j0}^\dagger a_{k0}). \tag{4.6}$$

The correction terms to Eq. (4.5) involve higher power of x_m which can be shown to have negligible contributions when x_m is much smaller than the wavelengths of the field λ. The physical meaning of H' is quite apparent. Because of the unitary transformation, the canonical momentum p now becomes the same as the kinetic momentum. In this picture, we can see that the mirror-field interaction is linearized to a familiar form $x_m F_0$, which is analogous to xE in dipole interaction.

It is worth mentioning that in the special case where the cavity field is contributed dominantly from a single cavity mode k_0, the interaction term $x_m F_0$ can be reduced to the one given in Refs. [4,13], i.e.,

$$x_m F_0 \approx x_m \frac{\hbar \omega_{k0}}{l_0} a_{k0}^\dagger a_{k0}. \tag{4.7}$$

However, this single-mode expression requires the motion of the mirror to be adiabatically slow so that the scattering of photons from the k_0 mode to other cavity modes can be ignored [19]. More precisely, a necessary condition for Eq. (4.7) to be valid is that the frequencies of x_m are much smaller than the frequency spacing of neighboring cavity modes. Our general Hamiltonians (3.17) and (4.5) do not require such restriction.

V. CONCLUSION

Finally, we address a fundamental limitation of our model. Hamiltonian (3.17) is valid only in the nonrelativistic domain. As in the nonrelativistic quantum electrodynamics, Hamiltonian (3.17) fails to describe physical phenomena which involve fields of arbitrarily high frequencies. The mirror self-energy problem, for example, cannot be properly handled without a relativistic consideration. For the sake of consistency we introduce a cutoff frequency ω_c to distinguish the nonrelativistic domain. We require that $\hbar\omega_c$ is much smaller than the rest energy of the mirror, but ω_c has to be sufficiently large so that the Hamiltonian can cover a wide spectral interval. For those physical processes which happen at the frequencies well below ω_c, we should expect the Hamiltonian (3.17) to be a correct description of the system. From a realistic point of view, since real mirrors do have a plasma cutoff frequency above which they become transparent, the introduction of a cutoff would make our model closer to practical situations. However, we emphasize that a study of a real mirror system should require a more sophisticated investigation which includes the dephasing of the field produced by imperfect reflections [16,23].

In conclusion, we have presented a nonrelativistic Hamiltonian of a one-dimensional mirror-field coupled system in a cavity configuration. Both the cavity field and the position of the mirror are treated as dynamical variables. The Hamiltonian itself is the energy of the system and the Hamilton equations obtained from it agree with the presumed equations of motion. We have also discussed the canonical quantization of the system. Although we have considered only the closed cavity situation, our method of finding the Hamiltonian can be generalized to a partially open cavity system. This can possibly be achieved by replacing the stationary mirror with a thin slab of dielectric at $x = 0$, and putting an additional stationary mirror at $x = -L$ ($\to -\infty$). Such a cavity model is the same as the one studied by Lang, Scully, and Lamb [24] in laser physics, except that one of the cavity mirrors $x = q$ (which is perfectly reflective) is movable. A detailed analysis of this model should be useful when cavity loss becomes important and input-output problems are involved. We hope to discuss this more difficult system in the future.

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[20] The coefficient g_{kj} is obtained from the integral

$$g_{kj} = q \int_0^q \varphi_j \frac{\partial \varphi_k}{\partial q} dx, \quad \text{where } \varphi_k = \sqrt{\frac{2}{q}} \sin \frac{k\pi x}{q}.$$

In deriving Eq. (2.6), we have used the relation

$$\sum_k g_{jk} g_{lk} = q^2 \int_0^q \frac{\partial \varphi_j}{\partial q} \frac{\partial \varphi_l}{\partial q} dx.$$

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