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Generation of sub-Poissonian and squeezed fields in the thermal superposition Jaynes-Cummings model

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Approximate analytical expressions for the time evolution of the photon-number statistics and squeezing in the thermal Jaynes-Cummings model with a two-level atom being initially in a coherent superposition of its ground and excited states are presented.

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I. INTRODUCTION

The Jaynes-Cummings model (JCM), describing the essential physics of the interaction of a single-mode radiation field with a two-level atom, with respect to its relative simplicity, is one of the most intensively studied models in quantum optics. Despite this simplicity its behavior is far from simple. Many nonclassical effects such as vacuum-field Rabi oscillations, sub-Poissonian photon-number statistics, antibunching, and squeezing of the radiation field have been predicted (for the latest reviews see [1,2]). Recent experiments with Rydberg atoms in high-Q microwave cavities have allowed experimental observations of the main dynamical features of the model [3–7].

The thermal JCM with an initially excited [8–10] or unexcited atom [11] can exhibit a sub-Poissonian field (but not squeezing) for sufficiently small initial photon numbers. Quite recently some approximate analytical formulas for this effect have been presented [12,13]. The bound on sub-Poissonian photon number statistics is much more restricted for an initially unexcited atom.

If the atom is initially prepared in a coherent superposition of its upper and lower levels, an interesting dynamics emerges; the mean values of the atomic-dipole operators, and in consequence the mean values of the operators of an initially thermal field, do not vanish, which may lead to the possibility of the appearance of squeezing. For an initially coherent field squeezing may be revealed in the JCM even for an initially unexcited or excited atom [14]. The coherent JCM is also able to exhibit higher-order [15] and higher-power squeezing [16] in the sense of Hong and Mandel's [17] and Hillery's [18] definitions, respectively. The coherent JCM with multiphoton transitions between the atomic levels may produce not only higher-order squeezing, but also higher-order intrinsic squeezing [17] for the photon multiplicities of the atomic transition equal or greater than 4 [15]. Knight [19] and Wódkiewicz *et al.* [20] have studied squeezing produced by the interaction of a suitably prepared two- or three-level atom with a cavity mode initially in the vacuum state.

One of our aims here is to show that the standard JCM with a two-level atom initially prepared in a coherent superposition of its excited and ground levels and coupled to an initially thermal field can manifest squeezing for small numbers of thermal photons. We also show how the photon-number statistics varies with the atomic excitation angle θ and the phenomenon of incoherent trapping is discussed.

II. MODEL

The effective rotating-wave approximation Hamiltonian for the model under discussion for exact resonance reads

$$\begin{aligned} H &= H_{\text{free}} + H_{\text{int}} , \\ H_{\text{free}} &= \hbar\omega S_z + \hbar\omega a^\dagger a , \\ H_{\text{int}} &= \hbar g[a^\dagger S_- + aS_+] . \end{aligned} \tag{1}$$

ω denotes the frequency of the field mode and g is atom-field coupling. S_-, S_+, and S_z are atomic pseudospin lowering, raising, and inversion operators, respectively, and

$$[S_+, S_-] = 2S_z. \tag{2}$$

The exact operator solution for the model was presented by Ackerhalt and Rzążewski [21]. We shall work here in the interaction picture.

The density operator ρ of a thermal field

$$\rho = \sum_{n=0}^{\infty} P_n |n\rangle\langle n| , \tag{3}$$

where the photon-number distribution function P_n has the form

$$P_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \tag{4}$$

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and $\bar{n}$ is the initial mean photon number, has only diagonal matrix elements in the photon-number representation $|n\rangle$. Hence it is convenient to perform calculations in the photon-number representation and then to make the final summation over n with the geometric distribution function P_n.

If the field is initially in a number state $|n\rangle$ and the atom is prepared in a coherent superposition of its ground $|-\rangle$ and excited $|+\rangle$ states, the initial state of the atom-field system reads

$$\begin{aligned}|\Psi(0)\rangle &= \cos\theta|-,n\rangle + e^{i\varphi}\sin\theta|+,n\rangle \\ &= C_-^{(n)}(0)|-,n\rangle + C_+^{(n)}(0)|+,n\rangle\,. \end{aligned} \tag{5}$$

The interaction-picture state vector of the system at any time $t > 0$ has the form

$$\begin{aligned}|\Psi(t)\rangle &= C_-^{(n)}(t)|-,n\rangle + C_{-+}^{(n)}(t)|+,n-1\rangle \\ &\quad + C_+^{(n)}(t)|+,n\rangle + C_{+-}^{(n)}(t)|-,n+1\rangle\,. \end{aligned} \tag{6}$$

The state $|-,n\rangle$ is coupled with the state $|+,n-1\rangle$ and the state $|+,n\rangle$ is coupled with the state $|-,n+1\rangle$. The time-dependent Schrödinger equation with the initial condition (5) gives the following solutions for the probability amplitudes $C_k^{(n)}(t)$, $(k = -,+,-+,+-)$:

$$\begin{aligned}C_-^{(n)}(t) &= \cos\theta\cos\Omega_n t\,, \\ C_{-+}^{(n)}(t) &= -i\cos\theta\sin\Omega_n t\,, \end{aligned} \tag{7}$$

and

$$\begin{aligned}C_+^{(n)}(t) &= e^{i\varphi}\sin\theta\cos\Omega_{n+1} t\,, \\ C_{+-}^{(n)}(t) &= -ie^{i\varphi}\sin\theta\sin\Omega_{n+1} t\,. \end{aligned} \tag{8}$$

Ω_n is the quantum Rabi frequency of the oscillations of the model

$$\Omega_n = g\sqrt{n}\,. \tag{9}$$

Due to the special initial preparation of the atom, the density matrix of the whole atom-field system has nonzero off-diagonal elements carrying a phase information, important from the point of view of squeezing.

III. PHOTON-NUMBER STATISTICS

Owing to Eqs. (6)–(8), the time evolution of the mean photon number (the first-order field correlation function) ${}^nG^{(1)}(t)$ is given by the sum of the mean photon numbers for an initially unexcited and excited atom multiplied by $\cos^2\theta$ and $\sin^2\theta$, respectively,

$$\begin{aligned}{}^nG^{(1)}(t) &= n|C_-^{(n)}(t)|^2 + (n-1)|C_{-+}^{(n)}(t)|^2 \\ &\quad + n|C_+^{(n)}(t)|^2 + (n+1)|C_{+-}^{(n)}(t)|^2 \\ &= {}^nG_{\text{un}}^{(1)}(t)\cos^2\theta + {}^nG_{\text{ex}}^{(1)}(t)\sin^2\theta\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned}{}^nG_{\text{un}}^{(1)}(t) &= n - \sin^2\Omega_n t\,, \\ {}^nG_{\text{ex}}^{(1)}(t) &= n + \sin^2\Omega_{n+1} t\,. \end{aligned} \tag{11}$$

The superscript n denotes that the field is initially in a Fock state. Since the system evolves with two incommensurate Rabi frequencies, the phenomenon of quantum beats will lead to aperiodic time behavior. In particular, incomplete collapses of the oscillations, more evident for greater photon numbers and $\theta = \pi/4$, will occur.

For a thermal field, after summation over n one gets

$$G^{(1)}(t) = G_{\text{un}}^{(1)}(t)\cos^2\theta + G_{\text{ex}}^{(1)}(t)\sin^2\theta\,, \tag{12}$$

and

$$\begin{aligned}G_{\text{un}}^{(1)}(t) &= \bar{n} - \sum_{n=1}^{\infty} P_n \sin^2(g\sqrt{n}t)\,, \\ G_{\text{ex}}^{(1)}(t) &= \bar{n} + \sum_{n=0}^{\infty} P_n \sin^2(g\sqrt{n+1}t)\,. \end{aligned} \tag{13}$$

Using the obvious property of the geometrical distribution $P_{n+1} = qP_n$, where

$$q = \frac{\bar{n}}{1+\bar{n}}, \tag{14}$$

we find that

$$\sum_{n=1}^{\infty} P_n \sin^2(g\sqrt{n}t) = q\sum_{n=0}^{\infty} P_n \sin^2(g\sqrt{n+1}t)\,. \tag{15}$$

In consequence, the thermal JCM with an initially excited and unexcited atom oscillates with the same spread of the Rabi frequencies, but with different amplitudes. As arises from Eqs. (13), the oscillations are shifted in phase by π.

Due to the relation (15), Eq. (12) transforms into

$$G^{(1)}(t) = \bar{n} + (\sin^2\theta - q\cos^2\theta)\sum_{n=0}^{\infty} P_n \sin^2(g\sqrt{n+1}t)\,. \tag{16}$$

For an arbitrary $\bar{n}$, if only the following equality is satisfied:

$$q = \tan^2\theta, \tag{17}$$

the photon number in the thermal JCM becomes a constant of motion and the phenomenon of radiation trapping takes place [22,23]. Since $0 < q < 1$, this effect occurs for $0 < \theta < \pi/4$. The excitation angle θ asymptotically approaches $\pi/4$ for very strong thermal fields. This particular case of trapping for strong fields has recently been discussed [24]. In other words, for an initially unexcited or excited atom there is no trapping, as it must be, since this phenomenon is reached for mixed states, which are better studied in the dressed-state approach. The term trapping is employed here to refer to a persistent probability for occupying a given state despite the existence of both the field and atomic transitions. It is simply a result of competition between the absorption and the spontaneous and stimulated emission processes. Trapping in this model does not depend on the phase and the coherence between the atom and radiation. So, we deal with incoherent trapping as distinct from the coherent JCM, where restricted trapping depending on both the atomic and the field phase occurs [25]. For every θ below and above θ_{trap}, the oscillations of the system are in phase. However, the oscillations above θ_{trap} are shifted by π in comparison with those below θ_{trap}. This is because the signs of the expression $\sin^2\theta - q\cos^2\theta$ are different in both intervals of θ (Fig. 1).

The excitation number operator

$$N = a^\dagger a + S_z + \frac{1}{2} \tag{18}$$

commutes with the Hamiltonian (1), i.e., is an integral of motion. Therefore, the time evolution of the expectation value of the atomic inversion is simply given as

$$\begin{aligned}\langle S_z(t)\rangle &= -\frac{1}{2}\cos 2\theta + \bar{n} - G^{(1)}(t)\\ &= -\frac{1}{2}\cos 2\theta - (\sin^2\theta - q\cos^2\theta)\\ &\quad\times\sum_{n=0}^{\infty} P_n \sin^2(g\sqrt{n+1}t)\,,\end{aligned} \tag{19}$$

where $-\cos 2\theta/2$ is the initial inversion. It is evident that for $\theta_{\text{trap}} < \theta \leq \pi/4$ the expectation value of the atomic inversion remains negative for any time, while for $\theta < \theta_{\text{trap}}$ it may also take only negative values during the whole time evolution of the system for small initial photon numbers.

For trapping, due to Eq. (17) the initial inversion may be rewritten in the form

$$\langle S_z(0)\rangle_{\text{trap}} = -\frac{1}{2(1+2\bar{n})}. \tag{20}$$

Depending on $\bar{n}$ it takes the values from the interval $-1/2 < \langle S_z(0)\rangle_{\text{trap}} < 0$ and, obviously, remains constant in the course of the atom-field interaction.

Gea-Banacloche [26] has pointed out the usefulness of the Bloch vector in the description of the fully quantized JCM. Arancibia-Bulnes *et al.* [24] have studied for the thermal superposition JCM the Bloch vector trajectory for $\theta = \pi/4$. In our notation, the expectation values of the components $S_x = (S_- + S_+)/2$ and $S_y = i(S_- - S_+)/2$ of the atomic dipole operator read

$$\begin{aligned}\langle S_x\rangle &= \frac{1}{2}\sin 2\theta\cos\varphi\sum_{n=0}^{\infty} P_n \cos\Omega_n t\cos\Omega_{n+1}t\,,\\ \langle S_y\rangle &= -\langle S_x\rangle\tan\varphi\,.\end{aligned} \tag{21}$$

Both dipole components vanish for an initially unexcited or inverted atom. For $\varphi = 0$ the Bloch vector moves in the x-z plane. It was shown [24] that the trajectory $\langle S_z(t)\rangle$ against $\langle S_x(t)\rangle$ is almost regular for small initial photon numbers, while for stronger fields ($\bar{n} = 10$) it exhibits quasistochastic fluctuations. From the present considerations it additionally arises that in the case of trapping ($\langle S_z\rangle = \text{const}$) this trajectory reduces for any $\bar{n}$ to the segment of the straight line parallel to the x axis and distant from it by the absolute value of (20).

The second-order normally ordered field correlation function for an initially Fock field reads

$$\begin{aligned}{}^nG^{(2)}(t) &= \langle\Psi(t)|a^\dagger a^\dagger aa|\Psi(t)\rangle\\ &= {}^nG^{(2)}_{\text{un}}(t)\cos^2\theta + {}^nG^{(2)}_{\text{ex}}(t)\sin^2\theta\,,\end{aligned} \tag{22}$$

where

$$\begin{aligned}{}^nG^{(2)}_{\text{un}}(t) &= n^2 - n - 2(n-1)\sin^2(gt\sqrt{n}),\\ {}^nG^{(2)}_{\text{ex}}(t) &= n^2 - n + 2n\sin^2(gt\sqrt{n+1}).\end{aligned} \tag{23}$$

For a thermal field one gets

$$G^{(2)}(t) = G^{(2)}_{\text{un}}(t)\cos^2\theta + G^{(2)}_{\text{ex}}(t)\sin^2\theta\,, \tag{24}$$

where

$$\begin{aligned}G^{(2)}_{\text{un}}(t) &= 2\bar{n}^2 - 2\sum_{n=2}^{\infty} P_n(n-1)\sin^2(g\sqrt{n}t),\\ G^{(2)}_{\text{ex}}(t) &= 2\bar{n}^2 + 2\sum_{n=1}^{\infty} P_n n\sin^2(g\sqrt{n+1}t).\end{aligned} \tag{25}$$

Since

$$\sum_{n=2}^{\infty}(n-1)P_n\sin^2(g\sqrt{n}t) = q\sum_{n=1}^{\infty} nP_n\sin^2(g\sqrt{n+1}t)\,, \tag{26}$$

we have

$$\begin{aligned}G^{(2)}(t) &= 2\bar{n}^2 + 2(\sin^2\theta - q\cos^2\theta)\\ &\quad\times\sum_{n=1}^{\infty} nP_n\sin^2(g\sqrt{n+1}t)\,.\end{aligned} \tag{27}$$

Obviously, in the case of trapping the photon number statistics remains thermal.

In what follows, we shall use the normally ordered variance defined as

$$V(t) = G^{(2)}(t) - \left[G^{(1)}(t)\right]^2. \tag{28}$$

All the conclusions regarding the phase relations of the oscillations of the correlation function $G^{(2)}(t)$ [but not $V(t)$; see Fig. 2] remain the same as for the function $G^{(1)}(t)$.

IV. FIELD QUADRATURES AND SQUEEZING

In order to study the squeezing of the field we introduce the two Hermitian slowly varying in-phase and out-of-phase quadrature components Q and P:

$$Q = a_s + a_s^\dagger, \quad P = (a_s - a_s^\dagger)/i, \tag{29}$$

where a_s and $a_s^\dagger$ are slowly varying parts of the photon annihilation and creation operators

$$a_s = a e^{i\omega t}, \quad a_s^\dagger = a^\dagger e^{-i\omega t}. \tag{30}$$

In general, the variances of the quadrature components are

$$\begin{aligned} \langle(\Delta Q)^2\rangle &= 1 + 2G^{(1)} + 2\mathrm{Re}\langle a_s^2\rangle - 4\left(\mathrm{Re}\langle a_s\rangle\right)^2, \\ \langle(\Delta P)^2\rangle &= 1 + 2G^{(1)} - 2\mathrm{Re}\langle a_s^2\rangle - 4\left(\mathrm{Im}\langle a_s\rangle\right)^2. \end{aligned} \tag{31}$$

The field is squeezed to the second order if one of the above variances satisfies the condition

$$\langle(\Delta Q)^2\rangle < 1 \qquad \text{or} \qquad \langle(\Delta P)^2\rangle < 1. \tag{32}$$

Fluctuations in one field quadrature are then reduced below vacuum fluctuations. For minimum uncertainty states the reduction of fluctuations in one quadrature can only take place at the expense of enhanced fluctuations in the other quadrature. For states that are not minimum uncertainty states, the above condition has not to be fulfilled.

For the thermal JCM the expectation values of the square of the photon annihilation $\langle a_s^2\rangle$ and creation $\langle a_s^{\dagger 2}\rangle$ operators are equal to zero and the variances (31) reduce to

$$\begin{aligned} \langle(\Delta Q)^2\rangle &= 1 + 2G^{(1)} - 4\left(\mathrm{Re}\langle a_s\rangle\right)^2, \\ \langle(\Delta P)^2\rangle &= 1 + 2G^{(1)} - 4\left(\mathrm{Im}\langle a_s\rangle\right)^2. \end{aligned} \tag{33}$$

It is obvious that the thermal JCM cannot exhibit amplitude-squared squeezing [18]; however, it may produce higher-order squeezing as defined by Hong and Mandel [17].

For the time evolution of the annihilation operator we have

$$\begin{aligned} 2\mathrm{Re}\left[{}^n\langle a_s(t)\rangle\right] &= \sqrt{n+1}\left[C_-^{(n)*}(t)C_{+-}^{(n)}(t) + C_-^{(n)}(t)C_{+-}^{(n)*}(t)\right] \\ &\quad + \sqrt{n}\left[C_{-+}^{(n)*}(t)C_+^{(n)}(t) + C_{-+}^{(n)}(t)C_+^{(n)*}(t)\right], \\ 2\mathrm{Im}\left[{}^n\langle a_s(t)\rangle\right] &= -i\Big\{\sqrt{n+1}\Big[C_-^{(n)*}(t)C_{+-}^{(n)}(t) - C_-^{(n)}(t)C_{+-}^{(n)*}(t)\Big] \\ &\quad + \sqrt{n}\Big[C_{-+}^{(n)*}(t)C_+^{(n)}(t) - C_{-+}^{(n)}(t)C_+^{(n)*}(t)\Big]\Big\}. \end{aligned} \tag{34}$$

Hence, using (7) and (8), we get

$$\begin{aligned} 2\mathrm{Re}^n\langle a_s(t)\rangle &= \sin 2\theta \sin\varphi\left[\sqrt{n+1}\cos\Omega_n t\sin\Omega_{n+1}t - \sqrt{n}\sin\Omega_n t\cos\Omega_{n+1}t\right], \\ 2\mathrm{Im}^n\langle a_s(t)\rangle &= -2\mathrm{Re}^n\langle a_s(t)\rangle\cot\varphi \end{aligned} \tag{35}$$

and for an initially thermal field arrive at

$$\begin{aligned} 2\mathrm{Re}\langle a_s(t)\rangle &= \sin 2\theta\sin\varphi \\ &\quad\times\left(\sum_{n=0}^{\infty} P_n\sqrt{n+1}\cos\Omega_n t\sin\Omega_{n+1}t - \sum_{n=1}^{\infty} P_n\sqrt{n}\sin\Omega_n t\cos\Omega_{n+1}t\right), \\ 2\mathrm{Im}\langle a_s(t)\rangle &= -2\mathrm{Re}\langle a_s(t)\rangle\cot\varphi. \end{aligned} \tag{36}$$

The optimal choice of the atomic phase φ for squeezing in the quadrature Q is $\varphi = \pi/2$, while for squeezing in the quadrature P is $\varphi = 0$. Then, if additionally the trapping condition (17) is satisfied, fluctuations in one quadrature remain constant and equal to their initial value $1 + 2\bar{n}$. The other quadrature may then reveal squeezing since the squeezed states produced in the model are not minimum uncertainty states. For an initially unexcited ($\theta = 0$) and purely excited ($\theta = \pi/2$) atom the quantities (36) vanish and there is no atomic phase present in (33). In such a case, quantum fluctuations in the quadratures become equal. At equal fluctuations the appearance of squeezing is impossible (the variances are then at least equal to one for minimum uncertainty states and greater than one for other states).

Field squeezing is connected here with squeezing in the atomic dipole operators [20,27–29]. The expectation value of the Q quadrature has the same φ dependence as $\langle S_y\rangle$, while the out-of-phase field quadrature P is related to S_x (21). The origin of such a relationship is concerned with the form of the interaction Hamiltonian (1) [20]. It was shown [20] that in order to observe field squeezing, the initial atomic superposition state has to satisfy two conditions: the atomic-dipole operator S_x or S_y should be squeezed and the expectation value of the commutator $[S_+, S_-]$ should be negative. This commutator is simply related to the atomic inversion (2) and its initial value $-\cos 2\theta/2$ may be negative for $0 \le \theta < \pi/4$. For an initially unexcited atom the phase information is lost. Therefore, in principle, field squeezing in the model in question is possible in the interval $0 < \theta < \pi/4$, even in the case of trapping. The other bound is connected with the initial mean number of thermal photons.

V. ANALYTICAL SOLUTION FOR WEAK THERMAL FIELDS

The JCM dynamics is exactly solvable in the rotating-wave approximation when the field is in a number state, showing the quantum Rabi oscillations. When the field mode is prepared in another state, the dynamics is not so simple and usually one of the main problems is the appearance of infinite sums over n. Some attempts have been made to overcome these summations. For the coherent JCM, the analytical solution has been found with the help of saddle-point techniques [30,31]. By definition, this approximation should hold rather for large mean photon numbers. However, agreement with the numerical results is fairly good already for $\bar{n} \simeq 4$. In turn, if the atom interacts with a classical thermal field, the final analytical result for the time evolution of the mean photon number may be presented in terms of the Dawson integral [32]. Then the collapsed system does not revive.

In what follows we are interested in weak thermal fields. Only for such fields is it possible to observe sub-Poissonian photon-number statistics and squeezing in the JCM. Obviously, in this case one can always perform direct numerical evaluations that are not so time consuming, but it is always interesting to find analytical results. Only recently an analytical solution, based on a linear approximation for the Rabi frequency, has been presented for an initially excited and unexcited atom. Namely, for small $\bar{n}$ the sums over n may be analytically calculated if we make the following substitution for the square roots of n and $n+1$ [12,13,24]:

$$\sqrt{n} = 1 + (n-1)(\sqrt{2}-1)\,, \qquad n = 1,2,3,\ldots$$
$$\sqrt{n+1} = 1 + n(\sqrt{2}-1)\,, \qquad n = 0,1,2,\ldots\,. \tag{37}$$

Inserting the second relation (37) into Eq. (16), after some algebra one finds [13]

$$\sum_{n=0}^{\infty} P_n \sin^2(g\sqrt{n+1}t) = \frac{1}{2}\left\{1 - (1-q)\frac{\cos[T+\Phi(\tau)]}{\sqrt{D(\tau)}}\right\}, \tag{38}$$

where

$$D(\tau) = 1 + q^2 - 2q\cos\tau\,,$$

$$\cos\Phi(\tau) = \frac{1-q\cos\tau}{\sqrt{D(\tau)}}\,, \qquad \sin\Phi(\tau) = \frac{q\sin\tau}{\sqrt{D(\tau)}}\,, \tag{39}$$
$$T = 2gt\,, \qquad \tau = 2g(\sqrt{2}-1)t\,.$$

Hence, from Eq. (16) we finally get

$$G^{(1)}(t) = \bar{n} + \frac{1}{2}\left\{1 - (1-q)\frac{\cos[T+\Phi(\tau)]}{\sqrt{D(\tau)}}\right\} \times(\sin^2\theta - q\cos^2\theta)\,. \tag{40}$$

The second-order field correlation function is found to be

$$G^{(2)}(t) = 2\bar{n}^2 + \left(\bar{n} - (1-q)\left\{\frac{\cos[T+2\Phi(\tau)]}{D(\tau)} - \frac{\cos[T+\Phi(\tau)]}{\sqrt{D(\tau)}}\right\}\right)(\sin^2\theta - q\cos^2\theta)\,. \tag{41}$$

Therefore, the normally ordered variance (28) reads

$$V(t) = \bar{n}^2 + \left\{(1+\bar{n})\frac{\cos[T+\Phi(\tau)]}{\sqrt{D(\tau)}} - \frac{\cos[T+2\Phi(\tau)]}{D(\tau)}\right\} \times(1-q)(\sin^2\theta - q\cos^2\theta) - \frac{1}{4}\left\{1-(1-q)\frac{\cos[T+\Phi(\tau)]}{\sqrt{D(\tau)}}\right\}^2 \times(\sin^2\theta - q\cos^2\theta)^2\,. \tag{42}$$

In order to calculate $\langle a_s(t)\rangle$, one has to approximate additionally oscillation amplitudes, also using (37). From (36) we have

$$2\mathrm{Re}\langle a_s(t)\rangle = \sin 2\theta\sin\varphi(1-q)\left\{\sin\frac{T}{2} + \bar{n}(\sqrt{2}-1) \times\left(\bar{n}+\frac{3}{2}+\sqrt{2}\right)\sin\frac{\tau}{2} + \frac{\sqrt{2}-1}{2}\left[\frac{1}{\sqrt{D(\tau)}}\sin\left(T-\frac{\tau}{2}+\Phi(\tau)\right) - \sin\left(T-\frac{\tau}{2}\right)\right]\right\}. \tag{43}$$

VI. DISCUSSION

Figure 1 shows the time evolution of the mean photon number obtained from the exact formula (16). In fact, the differences between the plots of the exact and the approximate $G^{(1)}$ would not be noticeable on this scale of the graph. For $\theta = 0$ the system oscillates with the amplitudes determined by $\bar{n}$. The amplitudes diminish with growing θ and the oscillations are completely suppressed for θ_{trap}. Further on, the system starts to oscillate again and the oscillations amplitudes progressively increase, reaching their maximal values for the purely excited atom.

Let us study now very roughly the signs of V and, in this manner, the field statistics. Obviously, in the case of trapping the photon-number statistics remains thermal. Figure 2 presents a general view of the time evolution of

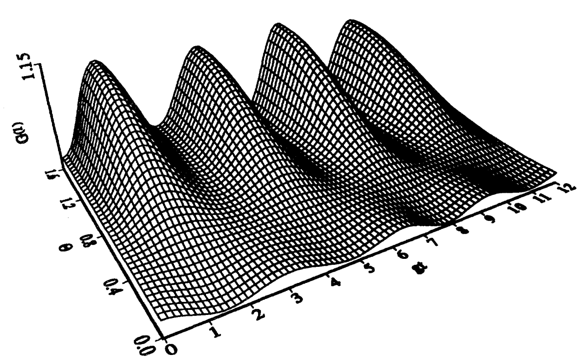

FIG. 1. Time and θ dependence of the mean photon number for $\bar{n} = 0.15$ (exact solution).

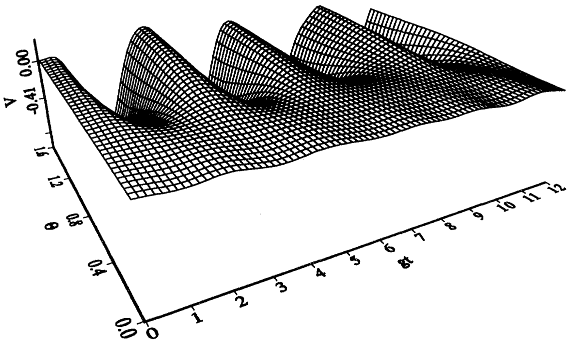

FIG. 2. Time and θ dependence of the normally ordered photon-number variance for $\bar{n} = 0.15$ (exact solution).

V. From Figs. 1 and 2 it is seen that for $\theta > \theta_{\text{trap}}$ the variance $V(t)$ reaches its minima almost at the points at which the mean photon number reaches its maxima [$gt = (2k-1)\pi/2$, $k = 1, 2, 3, \ldots$]. This relation is better for $\theta > \pi/4$ and becomes more and more accurate as θ increases. As an example, the graphs of these functions are presented for $\theta = \pi/2$ in Fig. 3. The exact results (solid lines) are compared with the approximate ones (40) and (42) (dashed lines). The maximum of $G^{(1)}$, most interesting for us, is the third one (k=3) when the value of this function is very close to its higher possible value $1+\bar{n}$ ($\simeq 1.2$ in this case): $\cos T = -1, \cos\tau \approx 1, D \approx (1-q)^2$, and $\cos\Phi \approx \cos 2\Phi \approx 1$. Then, in general, for $\theta > \theta_{\text{trap}}$ we get

$$G^{(1)} \approx \bar{n} + \sin^2\theta - q\cos^2\theta\,, \tag{44}$$

$$V \approx \bar{n}^2 - (\sin^2\theta - q\cos^2\theta)^2\,. \tag{45}$$

From the above equation it is evident that for a given $\bar{n}$ the possibility of emergence of sub-Poissonian field statistics ($V < 0$) enhances with θ. In turn, for a given θ this possibility grows as $\bar{n}$ diminishes. Sub-Poissonian statistics may be revealed for $\theta > \theta_{\text{trap}}$ if

$$\cos\theta < \sqrt{\frac{1-\bar{n}^2}{1+2\bar{n}}}\,. \tag{46}$$

To conclude, the bound on sub-Poissonian photon statistics grows with θ and reaches its maximum for a purely excited atom [13]. This limit on $\bar{n}$ is readily seen, namely, $\bar{n} < 1$. It remains in good agreement with the numerically found limit $\bar{n} \leq 1.16$ [11].

For $\theta < \theta_{\text{trap}}$ the normally ordered variance V takes its first and deepest negative minimum for $gt \approx \pi$. Its graph for an initially unexcited atom is presented in Fig. 4. Then, $\cos T \approx 1$ and $\tau \approx 2\pi(\sqrt{2}-1)gt$ and, in general, for $\theta < \theta_{\text{trap}}$, we find approximately

$$V \approx \bar{n}^2 + q(\sin^2\theta - q\cos^2\theta)(1 - \cos 2\sqrt{2}\pi)\,. \tag{47}$$

As seen from Fig. 2 and analytically obvious from the above formula, for a given $\bar{n}$, the possibility of emergence of the sub-Poissonian field decreases as θ increases to θ_{trap}. In particular, for $\theta = 0$ our previous result for the bound for an initially unexcited atom is recovered: $\bar{n} < 0.36$ [12]. In the times considered $gt \leq 12$, the numerically found limit amounts to $\bar{n} < 0.31$ [11].

In the discussion of squeezing we restrict our attention to the quadrature Q. Putting $\bar{n} = 0$ in Eqs. (16) and (27) or in Eqs. (40) and (43), from Eq. (33) we recover the result for spontaneous squeezing [19,20]:

$$\left\langle [\Delta Q(t)]^2 \right\rangle = 1 + 2\sin^2 gt \sin^2\theta \left(1 - 2\cos^2\theta \sin^2\varphi\right)\,. \tag{48}$$

This variance reaches periodically its minima at $gt = (2k-1)\pi/2$, $k = 1, 2, 3, \ldots$. For $\varphi = \pi/2$ we have

$$\left\langle [\Delta Q(t)]^2 \right\rangle = 1 - 2\sin^2 gt \sin^2\theta \cos 2\theta\,. \tag{49}$$

For $\theta = \pi/6$, maximal squeezing, amounting to 25%, occurs [19,20]. A three-dimensional view of spontaneous squeezing is presented in Fig. 5. In fact, for the clarity of the graph, the function $1 - \langle(\Delta Q)^2\rangle$ is plotted versus gt and θ. Hence the positive values along the Z axis point to squeezing. As evident from (49) and seen from the graph,

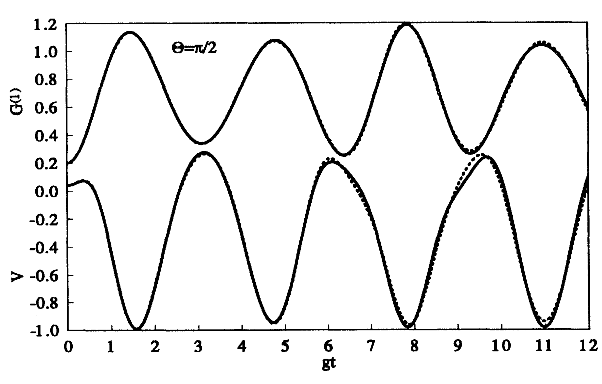

FIG. 3. Comparison of the exact solutions (solid lines) and the approximate ones (dahed lines) for the mean photon number and the normally ordered variance for $\bar{n} = 0.2$ and $\theta = \pi/2$ (excited atom).

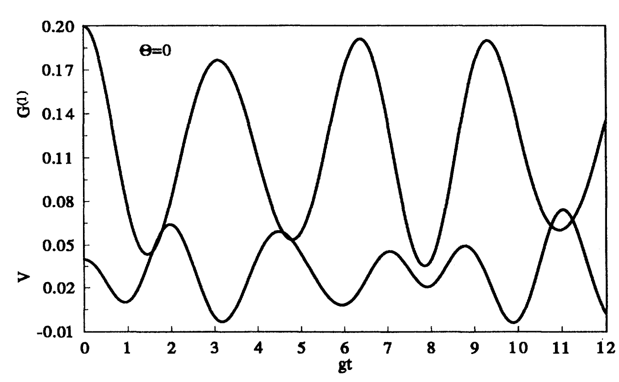

FIG. 4. Time evolution of the exact mean photon number and the normally ordered variance for $\bar{n} = 0.2$ and $\theta = 0$ (unexcited atom).

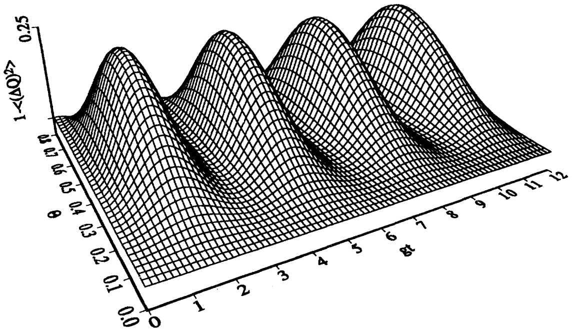

FIG. 5. Time and θ dependence of squeezing in the quadrature Q for $\varphi = \pi/2$ in the spontaneous case. For the clarity of the graph the quantity $1 - \langle(\Delta Q)^2\rangle$ is plotted.

spontaneous squeezing may occur for $0 < \theta < \pi/4$.

Figure 6 shows how spontaneous squeezing changes in the presence of thermal photons ($\bar{n} = 0.15$). As before, the quantity $1 - \langle(\Delta Q)^2\rangle$ is plotted. Some details are more evident from Fig. 7. Here only non-negative values of the quantity $1 - \langle(\Delta Q)^2\rangle$ are presented. This function has its maxima approximately at $gt = (2k - 1)\pi/2$ and the positions of the peaks change only slightly with $\bar{n}$. However, their heights strongly depend on $\bar{n}$. Already for $\bar{n} = 0.15$ not all of them have positive values; therefore only three hummocks are seen on the graph in comparison with four in the spontaneous case (the one corresponding to k=2 has already vanished). The width of the θ interval in which squeezing appears varies in time. In the times gt assumed, the maximal magnitude of squeezing, amounting to 18.5% for the above value of $\bar{n}$, is reached at $gt \approx 7\pi/2$ and $\theta = \pi/6$. A slightly greater amount of squeezing occurs at later times ($gt \approx 20.5\pi$), less practical from the experimental point of view.

In Fig. 8 the exact formula (28) (solid line) and the approximate one obtained from (40) and (43) (dashed line) have been compared for $\bar{n} = 0.1$ and $\theta = \pi/6$. Agreement between these two curves is satisfactory, although now not only the Rabi frequencies have been linearly approximated in n as in the case of photon statistics [12,13],

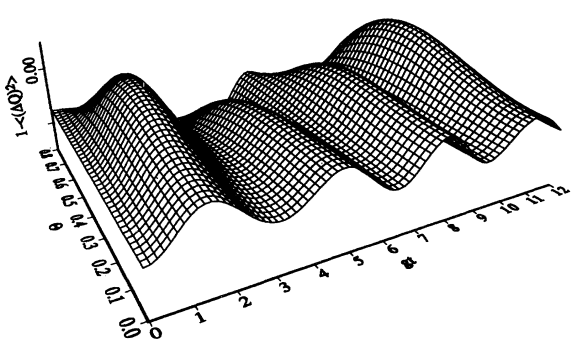

FIG. 6. Same as in Fig. 5, but for the thermal model with $\bar{n} = 0.15$.

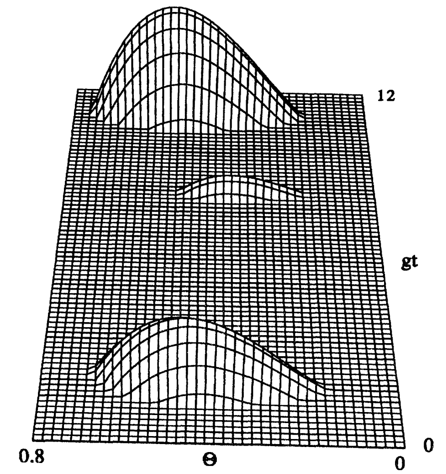

FIG. 7. Non-negative values of $1 - \langle(\Delta Q)^2\rangle$ versus gt and θ for $\bar{n} = 0.15$ and $\varphi = \pi/2$.

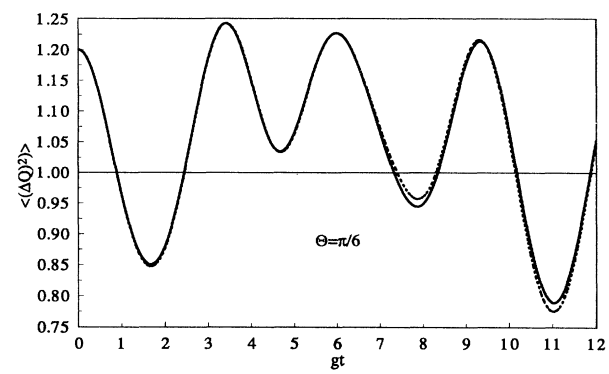

FIG. 8. Comparison of the exact $\langle(\Delta Q)^2\rangle$ (solid line) and the approximate one (dashed line) for $\bar{n} = 0.1$, $\theta = \pi/6$, $\varphi = \pi/2$.

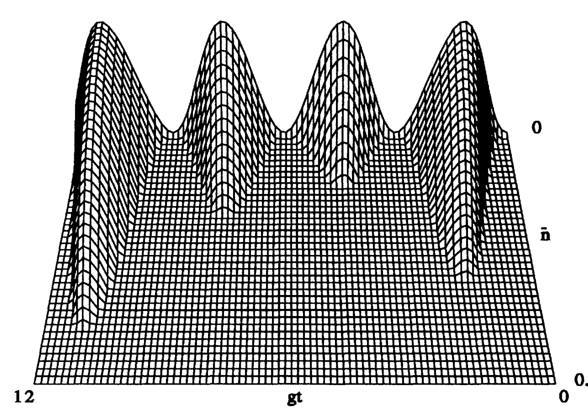

FIG. 9. Non-negative values of $1 - \langle(\Delta Q)^2\rangle$ versus gt and $\bar{n}$ for $\theta = \pi/6$ and $\varphi = \pi/2$.

but the oscillation amplitudes as well. Only subtle differences start to appear at $gt \approx 7.5$.

In the time interval assumed in this paper, squeezing is maximal in the vicinity of $gt \approx 7\pi/2$. This value of gt is taken in further discussion on the bounds of squeezing in the thermal JCM. From Eqs. (33), (40), and (43), for the value of gt mentioned above, we get approximately

$$\left\langle \left[\Delta Q \left(gt = \frac{7\pi}{2} \right) \right]^2 \right\rangle \approx 1 + 4\sin^4\theta - 2\sin^2\theta + 4\bar{n}^2. \tag{50}$$

Quantum fluctuations in this quadrature are reduced below vacuum fluctuations if

$$2\sin^4\theta - \sin^2\theta + 2\bar{n}^2 < 0. \tag{51}$$

For a given $\bar{n}$, squeezing is possible in the following interval of θ:

$$\arcsin\left(\frac{\sqrt{1-\sqrt{1-16\bar{n}^2}}}{2}\right) < \theta < \arcsin\left(\frac{\sqrt{1+\sqrt{1-16\bar{n}^2}}}{2}\right). \tag{52}$$

Since we deal here with the trigonometric function, the expression under the square root has to be real and we immediately get the bound on $\bar{n}$ for which squeezing can still be present, namely, $\bar{n} \leq 0.25$. This value of $\bar{n}$ remains in good agreement with the numerically found limit $\bar{n} < 0.29$. From Eq. (52) it is evident that the interval of θ with squeezing becomes narrower as $\bar{n}$ increases and for $\bar{n} = 0.25$ it turns to the point $\theta = \pi/6$, which is the point of minimum of the left-hand side of the inequality (51). The quantity $1 - \langle(\Delta Q)^2\rangle$ in function of gt and $\bar{n}$ for $\theta = \pi/6$ is plotted in Fig. 9.

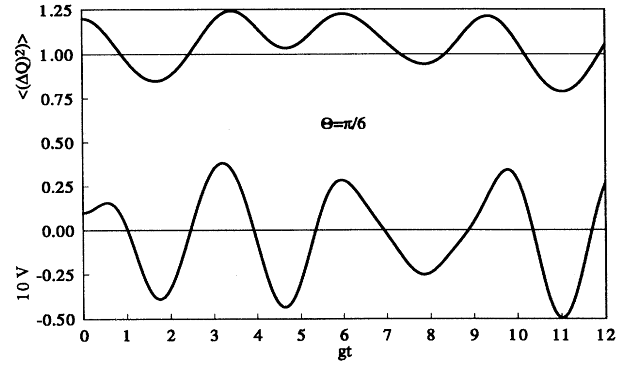

FIG. 10. Time dependence of V and $\langle(\Delta Q)^2\rangle$ for $\bar{n} = 0.1$ and $\theta = \pi/6$ ($\varphi = \pi/2$).

In Fig. 10 the time evolution of $\langle(\Delta Q)^2\rangle$ and V is compared for $\bar{n} = 0.1$ and $\theta = \pi/6$. In this case squeezing in its maximal points is always accompanied by sub-Poissonian photon-number statistics (although at $gt \approx 10.2$ squeezing starts to appear a little earlier than sub-Poissonian statistics). However, the opposite statement is not true, e.g., in the time interval around $gt \approx 3\pi/2$. In turn, in the vicinity of the trapping condition (17), the model discussed can manifest squeezing solely. For $\bar{n} > 0.29$, sub-Poissonian photon number statistics may be still present, while the appearance of squeezing is already impossible, even for the optimal excitation angles for this effect $\theta \approx \pi/6$. To conclude, there is no general relation between the emergence of squeezing and the sub-Poissonian field for $0 < \theta < \pi/4$; they may be encountered together or may appear alone.

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