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Number and phase uncertainties of the q-analog quantized field

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The q-analog coherent states $|z\rangle_q$ are used to identify some of the canonical physical properties of the single-mode q-analog quantized radiation field in the $|z\rangle_q$ "classical limit" where $|z|$ is large. In this quantum-optics-like limit, the fractional uncertainties of most physical quantities (momentum, position, amplitude, phase) which characterize the quantum field are shown to be $O(1)$, and only vanish as $O(1/|z|)$ when $q=1$. In contrast to this more-quantum-like behavior for $q\neq 1$, the fractional uncertainties do still approach zero for the usual number operator, N, and the N-Hamiltonian $H_N \equiv \hbar\omega(N+\frac{1}{2})$ which describes a free q-boson gas. An empirical signature for q-boson counting statistics is that $(\Delta N)^2/\langle N\rangle \to 0$ as $|z|\to\infty$. Properties of the q-analog generalizations of the phase operators of Susskind and Glogower (SG) and of the phase operator $\hat{\phi}_q$ of Pegg and Barnett are investigated. In contrast to the manifest q deformed properties of SG operators for moderate $|z|^2$, the "Hermitian" phase operator $\hat{\phi}_q$ still exhibits almost normal classical behavior in the $|z\rangle_q$ basis. In particular, the conventional (approximate) number-phase uncertainty relation $\Delta N \Delta\hat{\phi}_q \geq 1/2$ and approximate commutation relation $[N,\hat{\phi}_q]=i$ are found to follow for the single-mode q-analog quantized field. So N and $\hat{\phi}_q$ are *almost canonically conjugate* operators in the $|z\rangle_q$ classical limit. The $|z\rangle_q$ coherent states minimize this uncertainty relation for moderate $|z|^2$. q-analog generalizations of the P, Q, and W phase-space representations are treated in the Appendix.

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I. INTRODUCTION AND MOTIVATION

We assume that if q oscillators [1,2] exist in nature which realize the remarkable symmetries of the new quantum algebras, then there will also exist a q-analog quantum field which has such q oscillators as its normal modes [3]. In the Heisenberg representation, we consider [4,5] a specific mode of the generic q-analog radiation field having a specific polarization $\hat{\epsilon}$, where for q real, $O<q<1$,

$$aa^\dagger - q^{\pm 1/2}a^\dagger a = q^{\mp N/2} , \tag{1.1}$$

with $[N,a^\dagger]=a^\dagger$, $[N,a]=-a$, and $[a,a]=0$. As $q\to 1$, these reduce to the usual boson commutation relations. We suppress both the subscript $\mathbf{k}$ and the polarization vectors $\hat{\boldsymbol{\epsilon}}$ for the q-analog electric and magnetic fields, etc.

In order to recognize the presence of such an underlying q-boson quantum field, we need to know its canonical physical properties. In particular, what are its number and phase signatures? Since the usual quasiclassical coherent states (CS's) approximately characterize many types of cooperative behavior in the $q=1$ case, it is reasonable to use the q-analog coherent states $|z\rangle_q$ to investigate and identify the experimental signatures of a generic q-analog quantized field for cooperative phenomena.

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The q-analog CS's satisfy

$$a|z\rangle_q = z|z\rangle_q , \tag{1.2}$$

where z is a complex number. Up to a phase choice, they are

$$|z\rangle_q = N(z)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n\rangle_q , \tag{1.3}$$

where $N(z)=e_q(|z|^2)^{-1/2}$. Here $e_q(z)$ is the q-exponential function defined by the power series

$$e_q(z)\equiv\sum_{n=0}^{\infty}\frac{z^n}{[n]!} , \tag{1.4}$$

where $[n]!=[n][n-1]\cdots[1],[0]!=1$. Note that the "bracket no." is defined by ($s=\ln q$)

$$[x]_q \equiv [x] = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}} \tag{1.5a}$$

$$= \frac{\sinh(sx/2)}{\sinh(s/2)} . \tag{1.5b}$$

$[x]$ is called the q deformation of x. It is invariant under $q\leftrightarrow 1/q$, so we can often consider $0<q<1$ without loss of generality.

In the $|n\rangle_q$ occupation number basis, $\langle m|n\rangle=\delta_{mn}$ and

$$a^\dagger|n\rangle=\sqrt{[n+1]}|n+1\rangle,\quad a|n\rangle=\sqrt{[n]}|n-1\rangle , \tag{1.6}$$

with q-boson vacuum $|0\rangle_q$ such that $a|0\rangle_q=0$. We will often suppress the q subscript on the number basis states, etc. Notice also that besides the orthodox number operator

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$$N|n\rangle_q = n|n\rangle_q \tag{1.7}$$

there are two q-deformed numberlike operators $[N]$ and $[N+1]$ with

$$a^\dagger a|n\rangle_q = [N]|n\rangle_q = [n]|n\rangle_q \ , \tag{1.8a}$$

$$aa^\dagger|n\rangle_q = [N+1]|n\rangle_q = [n+1]|n\rangle_q \ . \tag{1.8b}$$

The normalization [see Eq. (1.3)] and resolution of unity for the q-analog CS's involve the q-exponential function $e_q(z)$, Eq. (1.4). Since $|e_q(z)| \le e_q(|z|) \le \exp(|z|)$, the series representation for $e_q(z)$ converges uniformly and absolutely for all finite z independent of the value of q. For $0 \le q < 1$, $e_q(z)$ is an order zero entire function [6]. For $x > 0$, $e_q(x)$ is positive, but for $x < 0$ and $q < (q_1^* \approx 0.14)$ there is a universal behavior independent of the value of q, consisting of an infinite number of increasing amplitude oscillations of decreasing frequency as $x \to (-\infty)$. For q small and/or n large, the asymptotic formula for the infinite number of real zeros of $e_q(x)$ is $\tilde{\mu}_n^e = -q^{1/2(1-n)}/(1-q)$, $n = 1, 2, \ldots$. As q increases above the first collision point at $q_1^* \approx 0.14$, these zeros collide in pairs and then move off the negative real axis into the complex z plane. They move off as (and remain) a complex conjugate pair. Thus, $e_q(x) \to \exp(x)$ as $q \to 1$.

It is not yet known whether nature makes use of q bosons as some type of nonlinear quasiparticle excitation of the ordinary electromagnetic field (or other known field) or as the quanta for a more novel type of cooperative phenomena. Possibly the physical occurrence of q bosons requires a background lattice, as in the case of phonons, or requires some other type of material medium to break the Lorentz invariance.

In this paper we use the q-analog CS's to investigate the $|z\rangle_q$ classical limit, where the modulus of z is large, for various quantities characterizing the single-mode q-analog quantized radiation field.

In Sec. II, we show analytically that the fractional uncertainties of most physical quantities (momentum, position, amplitude, phase) are of order 1 and only vanish as $O(1/|z|)$ when $q = 1$. In this respect, for $q \ne 1$ the $|z\rangle_q$ classical limit exhibits a more-quantum-like behavior than occurs in the $q = 1$ case. The fractional uncertainty still approaches zero, however, for both the usual N operator and for the N-Hamiltonian $H_N \equiv \hbar\omega(N + \frac{1}{2})$ which describes a free q-boson gas. At the end of Sec. II, it is emphasized that the N-Hamiltonian H_N does possess conventional physical properties but that the quadratic $\hat{P}$, $\hat{Q}$ Hamiltonian $H_{P,Q} \equiv \frac{1}{2}(\hat{P}^2 + \hat{Q}^2)$ probably does not permit a consistent physical interpretation [3] based on a smooth limit to a conventional free quantized field.

In Sec. III, we review the explicit reasoning [7] which shows that the usual Bose-Einstein energy distribution holds for a *free* q-boson gas. We analyze the q-boson number distribution $P_n^q(z)$ for $|z\rangle_q$ CS's, which differs distinctively from a Poisson distribution. Specifically, for $q \ne 1$ there is the important signature for q-boson counting statistics that

$$\lim_{|z| \to \infty} \frac{(\Delta N)^2}{\langle N \rangle} \to O \ . \tag{1.9}$$

Section IV reviews the q-analog generalizations [3] of the phase operators of Susskind and Glogower (SG) [8], $\widehat{\cos}(\phi)$, $\widehat{\sin}(\phi)$, and of the phase operator of Pegg and Barnett (PB) [9,10], $\hat{\phi}_q$. In Sec. V properties of the q-analog SG phase operators are studied analytically [6,11] in the $|z\rangle_q$ classical limit (with $|z|$ large). In general, the q deformation of the polar decomposition of the creation and annihilation operators, $a \equiv ([N+1])^{1/2}\widehat{\exp}(i\phi)$, affects both the angular and radial parts of functions of the SG phase operators. We find, however, that some operators (class II) which are functions of the SG phase operators possess the property that only their radial parts are q deformed in the $|z\rangle_q$ basis in the large-$|z|$ limit. So if the relevant physical observables for a q-boson superfluid, or superconductor, are class II, then, in analogy with the $q = 1$ description [12], for such observables the true state would be expected to have expectation values of the order parameter of the form

$$\langle \psi(\mathbf{x},t) \rangle \approx \sqrt{\rho_q(\mathbf{x},t)} \exp\{ i\phi(\mathbf{x},t) \} \ , \tag{1.10}$$

with only a q deformation of the amplitude, $\sqrt{\rho_q(\mathbf{x},t)}$, and no q deformation of the phase $\phi(\mathbf{x},t)$. So in spontaneous symmetry breaking, by Eq. (1.10) the q deformation would only directly affect the Higgs modes and not the Goldstone modes.

Also studied are the $\widehat{\cos}(\phi)$, $\widehat{\sin}(\phi)$, and N uncertainty relations. Unlike the usual $q = 1$ classical behavior, for $q \ne 1$ we find for moderate $|z|^2 (\sim 10 < |z|^2 <$ few 100) that the SG phase operators for describing the q-analog radiation field remain correlated, and therefore nonclassical. This result prompts the following question: What mechanism causes the SG phase operators to be q deformed and to behave nonclassically in the $|z\rangle_q$ classical limit when $q \ne 1$, but to behave classically when $q = 1$? While additional work is required to fully answer this important question, note that the q-deformed commutation relations also give a finite ΔN and a nonzero $\Delta\phi_q$ for the PB phase width.

For $q \ne 1$, the q generalization of the SG phase operators continues to assign a nonrandom phase behavior to the q-analog vacuum state $|0\rangle_q$. This property of the $\widehat{\sin}(\phi)$ and $\widehat{\cos}(\phi)$ operators in the $q = 1$ case has been criticized in the literature [9]. Note, however, that now when $0 < q < 1$, the vacuum component of the q-analog CS's, $|z\rangle_q$, has a larger relative amplitude a_0/a_n for $n > 1$ in the expansion $|z\rangle_q = \sum a_n |n\rangle_q$ than in the more familiar $q = 1$ case. In Eq. (1.3) for $0 < q < 1$, the bracket factorial $[n]!$ increases more rapidly [11] with n than does the usual $n!$ for $q = 1$. In addition, from the statistical viewpoint, the q-boson number distribution $P_n^q(z)$ for the $|z\rangle_q$ coherent states is non-Poisson for $q \ne 1$. Thus for $q \ne 1$ the intrinsic coherence and interference in the quantum field in the $|z\rangle_q$ basis are not that of statistically independent sources; in particular the contribution of the vacuum state is more significant.

In contrast, in Sec. VI it is shown that the q-analog PB Hermitian phase operator $\hat{\phi}_q$, does exhibit almost normal classical behavior in the $|z\rangle_q$ basis. The $\hat{\phi}_q$ phase operator gives a random phase to the q-analog vacuum state $|0\rangle_q$. The q-boson PB phase distribution $\bar{P}_q(\theta_m)$ for $|z\rangle_q$

CS's is introduced. It is used to show that $\langle\hat{\phi}_q\rangle=\theta=\arg(z)$, to investigate the variance $(\Delta\hat{\phi}_q)^2$, and to show that $\lim_{|z|\to\infty}\bar{P}_q(\theta_m)|_{q\neq 1}\to 2\pi\delta_q(\theta-\theta_m)$, $q\neq 1$. The fiducial quantity $\delta_q(\theta-\theta_m)$ is a bell-shaped function which is centered on $\theta=\arg(z)$ with finite width and finite height. For $q=1$, it is a Dirac δ distribution [9]. A q-dependent constant η_q characterizes the q deformation with $\Delta N\to\eta_q$ and $\Delta\hat{\phi}_q\to(2\eta_q)^{-1}$ for moderate $|z|^2$.

In Sec. VII it is shown that the q-boson CS's $|z\rangle_q$ give the approximate number-phase commutation relation [12] $[N,\hat{\phi}_q]\approx i$. So in contrast to the more-quantum behavior for $q\neq 1$ of the standard $\hat{Q}$ and $\hat{P}$ operators, the number and phase operators N and $\hat{\phi}_q$ do turn out to be almost canonically conjugate in the $|z\rangle_q$ classical limit. Thus the conventional (approximate) number-phase uncertainty relation of Dirac [12] still does follow for the generic single-mode q-analog quantum field:

$$\Delta N\Delta\hat{\phi}_q\geq\tfrac{1}{2}\ . \tag{1.11}$$

The $|z\rangle_q$ CS's are also found to minimize this uncertainty relation for moderate $|z|^2$.

Further applications of PB formalism to $q\neq 1$ are discussed in Appendix A. In Appendix B, we use $|z\rangle_q$ CS's to generalize to $q\neq 1$ the phase operator treatment of Paul. Appendix C contains a brief discussion of q-analog generalizations [3] of the standard P, Q, and W phase-space representations.

II. FRACTIONAL UNCERTAINTIES IN $|z\rangle_q$ CLASSICAL LIMIT

Using q-analog CS's, we can obtain analytically the fractional uncertainties in the classical limit for various quantities characterizing the q-analog quantized radiation field. Some of these are tabulated in Table I.

A. q-boson resolution function $\lambda(z)$

This analysis exploits the relationship between the q-boson resolution function $\lambda(z)$ and the q-exponential function $e_q(z)$:

$$\begin{aligned}\lambda(z)&\equiv\langle z|\hat{\Lambda}|z\rangle\ ,\\ &\equiv\langle z|[N+1]|z\rangle-\langle z|[N]|z\rangle \qquad (2.1a)\\ &=(q^{1/2}-1)|z|^2+\{e_q(q^{-1/2}|z|^2)/e_q(|z|^2)\} \qquad (2.1b)\\ &=(q^{-1/2}-1)|z|^2+\{e_q(q^{+1/2}|z|^2)/e_q(|z|^2)\}\ . \qquad (2.1c)\end{aligned}$$

Note that $\lambda(0)=1$ for the usual vacuum state, and that as $q\to 1$, $\lambda(z)\to 1$. The last line follows by the $q\leftrightarrow 1/q$ symmetry. Equivalently [3],

$$\lambda(z)=N(z)^2\sum_{n=0}^{\infty}\frac{|z|^{2n}\cosh\{s(2n+1)/4\}}{[n]!\cosh\{s/4\}}\ , \tag{2.1d}$$

where $q=\exp(s)$, and $N(z)$ is the CS normalization factor; see Eq. (1.3).

Notice that the associated resolution operator $\hat{\Lambda}$ is simply the basic commutator

$$\hat{\Lambda}\equiv[a,a^\dagger]\ . \tag{2.2}$$

On the other hand, the anticommutator,

$$\{a,a^\dagger\}_+=[N+1]+[N] \tag{2.3}$$

is proportional to the quadratic $\hat{P}$, $\hat{Q}$ Hamiltonian

$$\begin{aligned}H_{P,Q}&\equiv\tfrac{1}{2}\hbar\omega(a^\dagger a+aa^\dagger)\\ &=\tfrac{1}{2}(\hat{P}^2+\hat{Q}^2)\ . \qquad (2.4)\end{aligned}$$

TABLE I. Fractional uncertainties for the single-mode q-analog quantized radiation field. Except for N and H_N, quantum effects are pervasive in the $|z\rangle_q$ classical limit for $q\neq 1$.

Quantity	$\hat{O}$	$\frac{\Delta\hat{O}}{\lvert\langle\hat{O}\rangle\rvert}$ as $\lvert z\rvert\to\infty$	Case $q=1$
Position	$\hat{Q}$	$\frac{1}{2\cos\theta}\sqrt{(q^{-1/2}-1)+\epsilon_1/\lvert z\rvert^2}$	$\frac{1}{2\cos\theta\lvert z\rvert}$
Momentum	$\hat{P}$	$\frac{1}{2\sin\theta}\sqrt{(q^{-1/2}-1)+\epsilon_1/\lvert z\rvert^2}$	$\frac{1}{2\sin\theta\lvert z\rvert}$
E,**B** fields' amplitudes	$\hat{E},\hat{B}$	$\frac{1}{2}\sqrt{(q^{-1/2}-1)+\epsilon_1/\lvert z\rvert^2}$	$\frac{1}{2\lvert z\rvert}$
Deformed number operator	$[N]$	$\sqrt{(q^{-1/2}-1)+\epsilon_1/\lvert z\rvert^2}$	$\frac{1}{\lvert z\rvert}$
Deformed $(N+1)$ operator	$[N+1]$	$\sqrt{(q^{-1/2}-1)+(-1+2q)\epsilon_1/\lvert z\rvert^2+\cdots}$	$\frac{1}{\lvert z\rvert+\frac{1}{\lvert z\rvert}}$
Quadratic $\hat{P},\hat{Q}$ Hamiltonian	$H_{P,Q}$	$\sqrt{(q^{-1/2}-1)+(-1+2q^{1/2})\epsilon_1/\lvert z\rvert^2+\cdots}$	$\frac{1}{\lvert z\rvert+\frac{1}{2\lvert z\rvert}}$
Number operator and N Hamiltonian	$H_{N,N}$	approaches zero[a]	$\frac{1}{\lvert z\rvert}$

[a]For $|z|^2>$ few 100.

This resolution function characterizes the $\hat{Q}$ and $\hat{P}$ uncertainty relation

$$\langle z|[\hat{Q},\hat{P}]|z\rangle = i\hbar\langle z|\hat{\Lambda}|z\rangle = i\hbar\lambda(z) \geq i\hbar \tag{2.5}$$

in the $|z\rangle_q$ basis. For

$$z = |z|e^{i\theta} \tag{2.6}$$

just as for $q=1$, the mean values [13],

$$\langle z|\hat{P}|z\rangle = (2\hbar\omega)^{1/2}|z|\sin\theta \ , \tag{2.7a}$$

$$\langle z|\hat{Q}|z\rangle = (2\hbar/\omega)^{1/2}|z|\cos\theta \ , \tag{2.7b}$$

where

$$\hat{P} \equiv i(\hbar\omega/2)^{1/2}(a^\dagger - a) \ , \tag{2.8a}$$

$$\hat{Q} \equiv (\hbar/2\omega)^{1/2}(a^\dagger + a) \ . \tag{2.8b}$$

Equation (2.8) assumes the canonical relation between $\hat{Q}$ and $\hat{P}$, and a and $a^\dagger$. While in principle N-dependent functions which became unity in the limit $q \to 1$ could be introduced and investigated, Eq. (2.8) is both simpler and preserves free-quanta additivity in the field-theoretic momentum for significantly dispersed quanta.

$|z\rangle_q$'s are minimum uncertainty states, for they still do minimize the fundamental commutation relation $[\hat{Q},\hat{P}]$. For example, if we define the correlation parameter

$$U_{Q,P} \equiv \frac{2\Delta Q \Delta P - |\langle [Q,P]\rangle|}{|\langle [Q,P]\rangle|} \geq 0 \ , \tag{2.9}$$

we find for the $|n\rangle_q$ states that

$$U|_{|n\rangle \neq |0\rangle} = \frac{3[n]+[n+1]}{[n+1]-[n]} > 0 \ . \tag{2.10}$$

But, for the $|z\rangle_q$ coherent states, and the usual vacuum state $|0\rangle_q$, $U_{Q,P}=0$. It is interesting that this minimization occurs for both the q-coherent states and the q-vacuum state because, for $q=1$, several of the important properties of the usual CS's can be understood simply by considering them as being either displaced, or projected [14], vacuum states.

B. Fractional uncertainties

This resolution function also determines [3] the variances of the generic **E** and **B** fields (and of their associated potentials)

$$(\Delta\hat{E})^2|_{|z\rangle} = (\Delta\hat{B})^2|_{|z\rangle} = \frac{\hbar\omega}{2\epsilon_0 V}\lambda(z) > \frac{\hbar\omega}{2\epsilon_0 V} \ , \tag{2.11}$$

where

$$\hat{E} = i(\hbar\omega/2\epsilon_0 V)^{1/2}\{a \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) - a^\dagger \exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)\} \tag{2.12}$$

has the expectation value

$$\langle z|\hat{E}|z\rangle = -2(\hbar\omega/2\epsilon_0 V)^{1/2}|z|\sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta) \ , \tag{2.13}$$

since

$$\langle z|a|z\rangle = z = |z|\exp(i\theta) \ .$$

Expression (2.13) has the formal structure of a classical field, and the amplitude could provide an operational definition [15] for the modulus $|z|$, see Eq. (2.6). However, the physical situation is unclear since it follows from Eq. (2.12) that the fractional uncertainty in the $\hat{E}$ amplitude is of order 1 for $|z|$ large.

By $q \leftrightarrow 1/q$ symmetry, that is from Eq. (2.1c),

$$\lambda(z) = (q^{-1/2}-1)|z|^2 + \{e_q(q^{1/2}|z|^2)/e_q(|z|^2)\} \ . \tag{2.14}$$

So, because $e_q(|z|^2)$ increases monotonically in $|z|$, $\lambda(z)$ diverges linearly in $|z|^2$,

$$\lim_{|z|\to\infty} \lambda(z) \to (q^{-1/2}-1)|z|^2 + \epsilon_1 \ , \tag{2.15}$$

for $0 < q \leq 1$. It is useful to define two non-negative, q-dependent functions $\epsilon_i(|z|^2)$:

$$\epsilon_1 \equiv \lim_{|z|\to\infty} \{e_q(q^{1/2}|z|^2)/e_q(|z|^2)\} = \lim_{|z|\to\infty} \left\{ \prod_{n=1}^{\infty} \frac{\left[1+q^{1/2}\frac{z}{z_n}\right]}{\left[1+\frac{z}{z_n}\right]} \right\} = \prod_{n=1}^{\infty}(q^{1/2}) = 0, \quad q<1 \ , \tag{2.16}$$

by the product representation of $e_q(z)$ [see Eq. (37) of Ref. [6]]. Note z_n is the value of the nth zero of $e_q(z)$.

$$\epsilon_2 \equiv \lim_{|z|\to\infty} \{e_q(q|z|^2)/e_q(|z|^2)\} = \lim_{|z|\to\infty} \epsilon_2(|z|^2) = 0, \quad q<1 \ . \tag{2.17}$$

Note that $\epsilon_{i+1}(|z|^2) < \epsilon_i(|z|^2) < 1$ for $q \neq 1$, but $\epsilon_i(|z|^2) \to 1$ as $q \to 1$. For arbitrary $|z|$ values, $\lambda(z)$ is bounded:

$$\{1,(q^{-1/2}-1)|z|^2\} \leq \lambda(z) \leq \{(q^{1/2}-1)|z|^2+1\} \ , \tag{2.18}$$

where the more restrictive lower bound depends on q and $|z|$. The ϵ_i's characterize the "q-exponential falloff" of various mean values and uncertainties for the q-analog quantized radiation field in the classical limit ($x=|z|^2$):

$$\langle H_{P,Q}\rangle = (1+q^{-1/2})x + \epsilon_1(x) \ , \tag{2.19}$$

$$(\Delta H_{P,Q})^2 = (q^{-1/2}+1)^2(q^{-1/2}-1)x^2 + (q^{-1}+1+2q^{1/2})x\epsilon_1(x) + \epsilon_2(x) - (\epsilon_1(x))^2 \ , \tag{2.20}$$

in units of $(\hbar\omega/2)$, and

$$\langle [N]\rangle = x \ , \tag{2.21}$$

$$(\Delta[N])^2 = (q^{-1/2}-1)x^2 + x\epsilon_1(x) \ , \tag{2.22}$$

$$\langle [N+1]\rangle = q^{-1/2}x + \epsilon_1(x) \ , \tag{2.23}$$

$$(\Delta[N+1])^2 = (q^{-3/2}-q^{-1})x^2 + (q^{-1}-2q^{-1/2}+2)x\epsilon_1(x) + \epsilon_2(x) - (\epsilon_1(x))^2 \ . \tag{2.24}$$

It is straightforward to work out upper and lower bounds for these variances analogous to those for $\lambda(z)$ in Eq. (2.18).

Unlike for the other quantities, the last row of Table I shows that the fractional uncertainty is still zero for both the usual N operator and for the elementary N-Hamiltonian operator

$$H_N \equiv \hbar\omega(N+\tfrac{1}{2}) . \tag{2.25}$$

This is a numerical result, see Sec. III. In contrast with the situation for the $H_{P,Q}$ Hamiltonian operator, the N-Hamiltonian operator H_N does possess the conventional properties of (i) free-quanta additivity in energy for widely separated quanta, (ii) an orthodox free-field limit including operator diagonalization in the Fock $|n\rangle_q$ basis and zero fractional uncertainty, and (iii) mathematical independence from the basic commutation relation $[a,a^\dagger]=\hat{\Lambda}$ when $q\neq 1$. All of these properties are absent for $H_{P,Q}$. In particular, due to the hyperbolic nature of the definition of $[x]$ [see Eq. (1.5b)], there is the operator identity [3] for $q\neq 1$ which relates $H_{P,Q}$ and $[\hat{Q},\hat{P}]=i\hbar[a,a^\dagger]=i\hbar\hat{\Lambda}$:

$$\{\hat{\Lambda}\cosh(\tfrac{1}{4}s)\}^2-\left\{\frac{2}{\hbar\omega}H_{P,Q}\sinh(\tfrac{1}{4}s)\right\}^2=1 . \tag{2.26}$$

Because $H_{P,Q}$ lacks (i)–(iii), it is doubtful that the Hamiltonian operator $H_{P,Q}$ permits a consistent physical interpretation [3] based on a smooth limit to a conventional free quantized field. In Sec. VI we also find that while the commutator $[H_N,\hat{\phi}_q]$ has canonical $q=1$-type behavior in the $|z\rangle_q$ basis in the $|z\rangle_q$ classical limit, this is not the case for the $[H_{P,Q},\hat{\phi}_q]$ commutator. Here $\hat{\phi}_q$ is the Hermitian Pegg-Barnett phase operator, see Secs. IV, VI, and VII below.

III. q-BOSON COUNTING STATISTICS

For the sake of completeness, we begin this section by reviewing some known material on properties of the free q-boson gas. Knowledgeable readers should skip to about Eq. (3.9).

The physically important but mathematically trivial $[a,a]=0$ implies that the usual Bose-Einstein energy distribution still follows for a free q-boson gas (with a nondegenerate equally spaced spectrum). Several authors [7] have explicitly confirmed this for the N-Hamiltonian $H_N=\hbar\omega(N+\frac{1}{2})$. The nontrivial thermal averages for these free excitations give an equilibrium distribution of cavity excitations

$$\langle [N]\rangle_T=\langle [N+1]\rangle_T e^{-\hbar\omega/kT} . \tag{3.1}$$

By Eq. (1.6), this means that, for q bosons in a blackbody cavity,

$$\text{(absorption rate)}=\text{(spontaneous and induced emission rate)}$$
$$\times\text{(Boltzmann ratio of emitting to absorbing wall molecules)}$$

Note that Eq. (3.1) follows quite simply since ($\beta\equiv\hbar/kT$)

$$\begin{aligned}\langle [N+1]\rangle_T &\equiv \frac{1}{Z}\sum e^{-n\beta\omega}[n+1]\\ &=\frac{1}{Z}e^{\beta\omega}\sum e^{-(n+1)\beta\omega}[n+1]\\ &=e^{\beta\omega}\langle [N]\rangle_T .\end{aligned} \tag{3.2}$$

The q-boson absorption and emission rates themselves are q dependent, with

$$\langle [N]\rangle_T=\frac{\langle N\rangle_T}{\mathcal{D}} , \tag{3.3}$$

$$\langle [N+1]\rangle_T=\frac{\langle N\rangle_T+1}{\mathcal{D}} , \tag{3.4}$$

where

$$\mathcal{D}=\{1+(1-q^{1/2})\langle N\rangle_T\}\{1+(1-q^{-1/2})\langle N\rangle_T\}\langle N\rangle_T .$$

However, in (3.4) there is the usual, q-independent Bose-Einstein energy distribution

$$\langle N\rangle_T=\left[\frac{1}{e^{\hbar\omega/kT}-1}\right] . \tag{3.5}$$

An important corollary is that ratios of thermal averages [16] for a free q-boson gas satisfy the equalities

$$\frac{\langle [N]\rangle_T}{\langle [N+1]\rangle_T}=\frac{\langle N\rangle_T}{\langle N\rangle_T+1}=e^{-\hbar\omega/kT} \tag{3.6}$$

independent of the q value.

On the other hand, the q-analog CS's $|z\rangle_q$ do not give a Poisson number distribution for $q\neq 1$ since [1,17] the q-boson number distribution

$$\begin{aligned}P_n^q(z) &\equiv |{}_q\langle n|z\rangle_q|^2\\ &=\frac{|z|^{2n}}{[n]!e_q(|z|^2)}\end{aligned} \tag{3.7}$$

with $\sum_{n=0}^{\infty}P_n^q(z)=1$. Note that, for $q\neq 1$, $|z|^2$ is the eigenvalue of the deformed number operator in the $|z\rangle_q$ basis

$${}_q\langle z|[N]|z\rangle_q=|z|^2 . \tag{3.8}$$

Of course, in the limit $q\to 1$, the Poisson distribution (for statistically independent sources) occurs.

Since for $q\neq 1$, it is the expectation value of the deformed number operator $[N]$ which is both q independent and analytically simple, we have chosen to show the figures in this paper versus $|z|^2$ instead of versus the mean value $\langle z|N|z\rangle$ which is q dependent.

For fixed $|z|^2=100$, the peak of $P_n^q(z)$ narrows and shifts to smaller n as q decreases; see Fig. 1. However, the behavior of the fractional uncertainty, $(\Delta N)/\langle N\rangle$, of

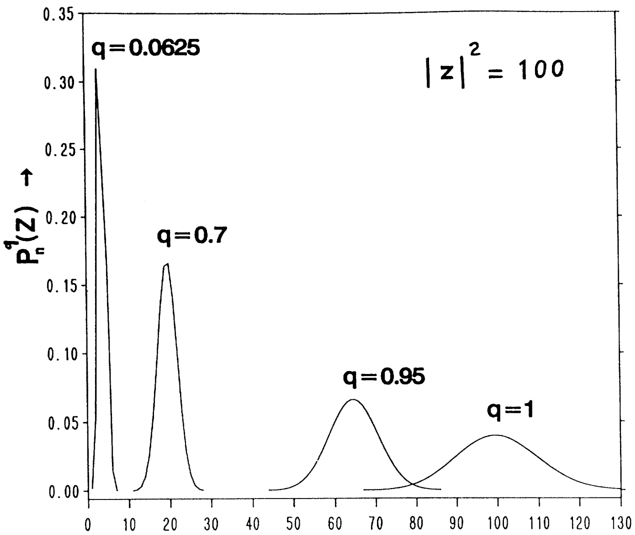

FIG. 1. The q-boson number distribution in the q-analog coherent state for fixed $|z|^2=100$. Note that $P_n^q(z)\equiv|_q\langle n|z\rangle_q|^2$ and that $|z|^2={}_q\langle z|a^\dagger a|z\rangle_q={}_q\langle z|[N]|z\rangle_q$. Thus $|z|^2$ is the expectation value of the deformed number operator in the q-analog CS basis. In the limit $q\to1$ (i.e., normal boson statistics), $P_n^q(z)$ is the Poisson distribution and $|z|^2={}_q\langle z|[N]|z\rangle_q\to\langle N\rangle$, the expectation value of the conventional number operator in the conventional CS basis. In this paper, *all* expectation values denote q-analog coherent states.

the number operator N versus $|z|$ is not very q dependent; see Fig. 2. Nevertheless, as shown by Table II, there is a simple, though unusual, property of q-boson statistics that, for $q\neq1$,

$$\lim_{|z|\to\infty}\frac{(\Delta N)^2}{\langle N\rangle}\to0\ . \tag{3.9}$$

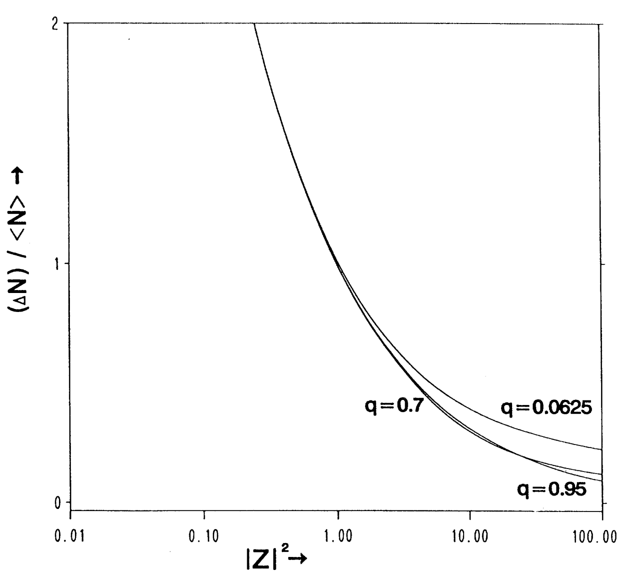

FIG. 2. Behavior of the fractional uncertainty in N, that is of $(\Delta N)/\langle N\rangle$, as $|z|^2$ varies for fixed $q\neq1$. For $q=1$, $(\Delta N)/\langle N\rangle=1/|z|$.

TABLE II. Comparison of q-boson counting statistics with known photon statistics.

Bose-Einstein statistics	$\frac{(\Delta N)^2}{\langle N\rangle}=\langle N+1\rangle$	Thermal source
Poisson statistics	$\frac{(\Delta N)^2}{\langle N\rangle}=1$	Laser light ($q=1$ CS's)
q-boson statistics	$\frac{(\Delta N)^2}{\langle N\rangle}\xrightarrow[\lvert z\rvert\to\infty]{}0$	$q\neq1$ CS's

That is, the ratio of the variance to the mean value for q-boson counting statistics is radically different from that of ordinary photon counting statistics. This simple signature Eq. (3.9) should prove to be an important test for identifying q bosons versus other field quanta.

Table III summarizes the number operator and phase operator uncertainties in the $|z\rangle_q$ classical limit. We will discuss the Hermitian Pegg-Barnett phase operator $\hat\phi_q$ in Secs. IV, VI, and VII. Figure 3 shows that the mean value $\langle N\rangle=\langle z|N|z\rangle$ still increases, although with a smaller effective slope, as q decreases ($s=\ln q$):

$$\langle N\rangle=\frac{2}{s}\left\langle\sinh^{-1}\left\{[N]\sinh\left[\frac{s}{2}\right]\right\}\right\rangle$$
$$\to2\alpha_q\ln|z|+\beta_q,\quad\text{for }\sim1<|z|^2\lesssim\text{ few }100\ . \tag{3.10}$$

The second line is a crude numerical estimate with α_q and β_q some q-dependent constants. For large $|z|$, the second line with $\alpha_q=(-2/s)$ follows from the first line upon replacing $[N]\to|z|^2$, which is a diagonal CS approximation. We find numerically that the number uncertainty

$$\lim_{|z|\to\infty}\Delta N\to\eta_q\ , \tag{3.11}$$

where η_q is another q-dependent constant. This is shown in Fig. 4, with $\eta_{0.7}\approx2.4$ and $\eta_{0.06}\approx0.9$.

In summary, the number of q bosons appears to be a meaningful quantity in the $|z\rangle_q$ classical limit since the fractional uncertainty $\Delta N/\langle N\rangle\to$zero. Thus the addi-

TABLE III. N and $\hat\phi_q$ uncertainties in the $|z\rangle_q$ classical limit. Note how η_q characterizes the q deformation for both N and $\hat\phi_q$.

Uncertainty	Behavior when $q\neq1$	Case $q=1$
ΔN	$\eta_q=\begin{cases}q\text{ dependent}\\\text{const}\end{cases}$	$\lvert z\rvert$
$\Delta\hat\phi_q$	$\frac{1}{2\eta_q}$	$\frac{1}{2\lvert z\rvert}$
$\Delta N\Delta\hat\phi_q$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\Delta N}{\langle N\rangle}$	$\left[\frac{\eta_q}{2\alpha_q\ln\lvert z\rvert+\beta_q}\right]^{\text{a}}$	$\frac{1}{\lvert z\rvert}$
$\frac{\Delta\hat\phi_q}{\lvert\langle\hat\phi_q\rangle\rvert}$	$\frac{1}{2\eta_q\lvert\langle\phi_q\rangle\rvert}$	$\frac{1}{2\lvert z\rvert\lvert\langle\phi\rangle\rvert}$

[a]For $\sim1<|z|^2\lesssim$ few 100.

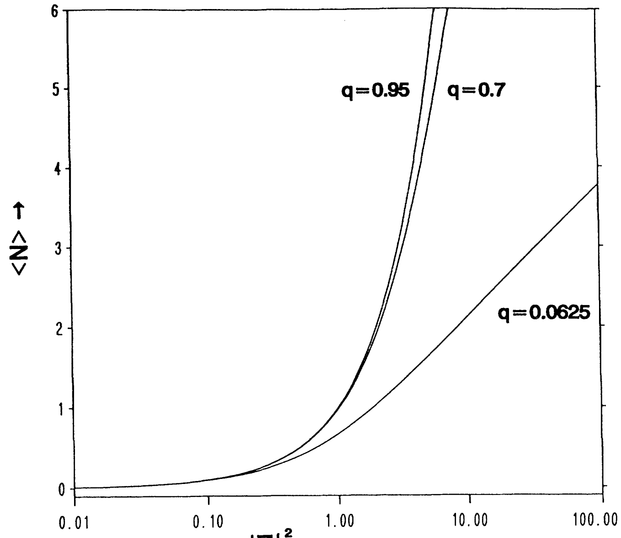

FIG. 3. Behavior of the mean value $\langle N\rangle \equiv {}_q\langle z|N|z\rangle_q$ as $|z|^2$ varies for fixed $q\neq 1$. For $q=1$, $\langle N\rangle = |z|^2$.

tive energy of the free q-boson gas is also meaningful in this limit,

$$\lim_{|z|\to\infty} \frac{\Delta H_N}{\langle H_N\rangle} \to \frac{\eta_q}{2\alpha_q \ln|z| + \beta_q} \quad \text{for } \sim 1 < |z|^2 \lesssim \text{ few } 100 \tag{3.12a}$$

$$\to \text{zero}, \tag{3.12b}$$

in contrast to the significant quantum corrections for other quantities which we found in Sec. II. The various q-boson uncertainty ratios involving the number operator are both simple and distinctive when $q\neq 1$, so in principle it is possible by q-boson counting experiments to empirically identify a q-boson gas in this limit in spite of the ordinary Bose-Einstein frequency distribution Eq. (3.5). Such ratios include Eq. (3.9) and the various abnormal (normal) fractional uncertainty ratios shown in Table I.

IV. q-ANALOG PHASE OPERATORS

Different phase definitions for the quantum field have been, and are being, investigated in the theoretical and experimental literature in quantum optics [8–10,18–20]. Mathematically [8] a Hermitian phase operator conjugate to N or to $[N]=a^\dagger a$ does not exist. Nevertheless, in physical analyses in a variety of physical systems it has been fruitful to investigate and apply phase operators and number-phase commutation relations.

A. Susskind-Glogower phase operators $\widehat{\cos}\phi$ and $\widehat{\sin}\phi$

An $\widehat{\exp}(i\phi)_q$ generalization [8,3] of the Susskind-Glogower phase operator can be defined by

$$a \equiv ([N+1])^{1/2}\widehat{\exp}(i\phi), \tag{4.1a}$$

$$a^\dagger \equiv \widehat{\exp}(-i\phi)([N+1])^{1/2} \tag{4.1b}$$

with $\widehat{\exp}(-i\phi) \equiv \{\widehat{\exp}(i\phi)\}^\dagger$. So, there are the Hermitian operators

$$\widehat{\cos}(\phi) \equiv \tfrac{1}{2}\{\widehat{\exp}(i\phi)+\widehat{\exp}(-i\phi)\}, \tag{4.2a}$$

$$\widehat{\sin}(\phi) \equiv (-\tfrac{1}{2}i)\{\widehat{\exp}(i\phi)-\widehat{\exp}(-i\phi)\}. \tag{4.2b}$$

In the $|n\rangle_q$ basis these definitions correspond to

$$\widehat{\exp}(i\phi)_q \equiv \sum_{n=0}^{\infty} |n\rangle_{q\,q}\langle n+1|. \tag{4.3}$$

This form of the definition $\widehat{\exp}(i\phi)_q$ is manifestly q independent, nonunitary, and obviously a simple q analog of the SG operator.

B. Pegg-Barnett phase operator $\hat{\phi}_q$

Similarly, a q generalization [9,3] of the Pegg-Barnett phase operator $\hat{\phi}_q$ is obtained by introducing a complete,

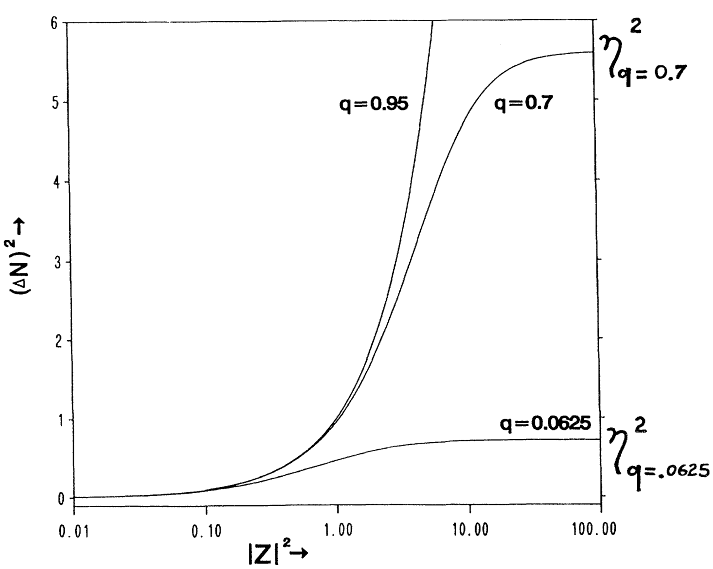

FIG. 4. Behavior of the variance $(\Delta N)^2$ as $|z|^2$ varies for fixed $q\neq 1$. While for $q=1$, $(\Delta N)^2=\langle N\rangle=|z|^2$, for $q\neq 1$ the figure indicates that $(\Delta N)^2 \to (\eta_q)^2$ where η_q is a q-dependent constant.

orthonormal basis of $(s+1)$ phase states:

$$|\theta_m\rangle=(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle_q\ , \tag{4.4a}$$

with

$$\theta_m=\theta_0+\{2m\pi/(s+1)\},\quad m=0,1,\ldots,s\ , \tag{4.4b}$$

where θ_0 is a reference, or indicial phase. Note that these states are introduced in an $(s+1)$-dimensional finite subspace $\Psi_{\rm PB}^{(q)}$ and that in calculations the $s\rightarrow\infty$ limit is only to be taken after matrix elements, $\langle a|\hat{O}(\hat{\phi}_q,\ldots)|b\rangle$, of the phase-dependent operators are calculated. These $|\theta_m\rangle$'s are eigenstates of the Hermitian phase operator

$$\hat{\phi}_q\equiv\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m| \tag{4.5}$$

and of the unitary

$$\exp(i\hat{\phi}_q)\equiv|0\rangle_{q\,q}\langle 1|+|1\rangle_{q\,q}\langle 2|+\cdots+|s-1\rangle_{q\,q}\langle s| +\exp\{i(s+1)\theta_0\}|s\rangle_{q\,q}\langle 0|\ . \tag{4.6}$$

Equation (4.6) only differs in the last term versus the SG expression, Eq. (4.3). These operators are also manifestly q independent in the $|n\rangle_q$ basis.

A polar decomposition operator h has been used in the analysis of the $\mathrm{SU}(2)_q$ algebra by Chaichian and Ellinas [3]. h is the same as $\exp(i\hat{\phi}_q)$ if the reference phase of h is taken to be s dependent, $\phi_R=(s+1)\theta_0$. Note that the term "classical limit" in their work means the $q\rightarrow 1$ limit, and not the large $|z|$ limit of matrix elements in the $|z\rangle_q$ basis.

Since Eqs. (4.3), (4.4), and (4.6) are all manifestly q independent, the nontrivial q-analog phase effects we obtain in this paper in the $|z\rangle_q$ classical limit for both the SG and PB phase operators arise due to the q dependence of the mapping $|n\rangle_q\rightarrow|z\rangle_q$.

V. PROPERTIES OF q-ANALOG SG PHASE OPERATORS

A. Properties of $\widehat{\sin}(\phi)$ and $\widehat{\cos}(\phi)$ in $|z\rangle_q$ basis

The $\widehat{\sin}(\phi)$ and $\widehat{\cos}(\phi)$ operators in the $|z\rangle_q$ basis do exhibit some correspondence-principle-like behavior for arbitrary q. For example, with $z=|z|e^{i\theta}$,

$$\frac{\langle z|\widehat{\sin}(\phi)|z\rangle}{\langle z|\widehat{\cos}(\phi)|z\rangle}=\frac{\sin\theta}{\cos\theta}\ , \tag{5.1a}$$

$$\langle z|\widehat{\cos}(\phi)^2+\widehat{\sin}(\phi)^2|z\rangle=1-\tfrac{1}{2}e_q(|z|^2)^{-1} \rightarrow 1\ \text{ for } |z|^2\rightarrow\infty\ , \tag{5.1b}$$

$$\langle z|[\widehat{\cos}(\phi),\widehat{\sin}(\phi)]|z\rangle=\tfrac{1}{2}ie_q(|z|^2)^{-1} \rightarrow 0\ \text{ for } |z|^2\rightarrow\infty\ , \tag{5.1c}$$

and so, for $q\neq 1$, the SG phase operators $\widehat{\cos}(\phi)$ and $\widehat{\sin}(\phi)$ ultimately do commute for $|z|$ sufficiently large. However, for moderate $|z|^2$ we do find a significant q dependence for other expectation values and find that the uncertainty product $\{\Delta\widehat{\cos}(\phi)\Delta\widehat{\sin}\phi\}$ is nonvanishing.

Note that acting on $|z\rangle_q$ the $\widehat{\cos}(\phi)$ and $\widehat{\sin}(\phi)$ operators give

$$\widehat{\cos}(\phi)|z\rangle=\tfrac{1}{2}N(z)\sum_{n=0}^{\infty}\left\{\frac{z^{n+1}}{([n+1]!)^{1/2}}|n\rangle+\frac{z^n}{([n]!)^{1/2}}|n+1\rangle\right\}, \tag{5.2a}$$

$$\widehat{\sin}(\phi)|z\rangle=(-\tfrac{1}{2}i)N(z)\sum_{n=0}^{\infty}\left\{\frac{z^{n+1}}{([n+1]!)^{1/2}}|n\rangle-\frac{z^n}{([n]!)^{1/2}}|n+1\rangle\right\}. \tag{5.2b}$$

So, besides Eqs. (5.1), we find

$$\begin{aligned}
&\langle z|\widehat{\cos}(\phi)|z\rangle=\cos\theta I_1(|z|)\ ,\\
&\langle z|\widehat{\sin}(\phi)|z\rangle=\sin\theta I_1(|z|)\ ,\\
&\langle z|\widehat{\cos}(\phi)^2-\widehat{\sin}(\phi)^2|z\rangle=\cos 2\theta I_2(|z|)\ ,\\
&\langle z|\{\widehat{\cos}(\phi),\widehat{\sin}(\phi)\}_+|z\rangle=\sin 2\theta I_2(|z|)\ ,
\end{aligned} \tag{5.3}$$

where the functions $I_i(|z|)$ are defined by the power series

$$I_1(|z|)=|z|e_q(|z|^2)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{[n]!([n+1])^{1/2}}\ , \tag{5.4a}$$

$$I_2(|z|)=|z|^2e_q(|z|^2)^{-1}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{[n]!([n+2][n+1])^{1/2}}\ . \tag{5.4b}$$

For $|z|^2\ll 1$, the asymptotic limits of these functions are

$$I_1(|z|)\sim|z|e_q(|z|^2)^{-1}\left\{1+\frac{|z|^2}{\sqrt{2}}+\frac{|z|^4}{[2]!\sqrt{[3]}}+\ldots\right\},$$

$$I_2(|z|)\sim|z|^2e_q(|z|^2)^{-1}\left\{\frac{1}{\sqrt{[2]}}+\frac{|z|^2}{\sqrt{[3][2]}}+\frac{|z|^4}{[2]!\sqrt{[4][3]}}+\ldots\right\},$$

so

$$\begin{aligned}
&\langle z|\widehat{\sin}(\phi)|z\rangle\sim|z|\sin\theta,\quad \langle z|\widehat{\cos}(\phi)|z\rangle\sim|z|\cos\theta\ ,\\
&\langle z|\widehat{\sin}(\phi)^2|z\rangle\sim\langle z|\widehat{\cos}(\phi)^2|z\rangle\sim\tfrac{1}{4}\ ,\\
&\langle z|[\widehat{\cos}(\phi),\widehat{\sin}(\phi)]|z\rangle\sim-\tfrac{1}{2}i\ ,\\
&\langle z|[\widehat{\cos}(\phi),\widehat{\sin}(\phi)]_+|z\rangle\sim\text{ zero}\ ,
\end{aligned}$$

Note that there is no (leading order) q dependence in these expectation values for $|z|\ll 1$. In the $q=1$ case, Ref. [19] calls this $|z|\ll 1$ limit the quantum limit since the CS states are the prototype quantum-mechanical states for the radiation emitted by a classical current source.

For moderate $|z|^2$, the behavior of the functions $I_i(|z|)$ is shown in Figs. 5 and 6. Note that for $|z|^2>(\sim 10)$

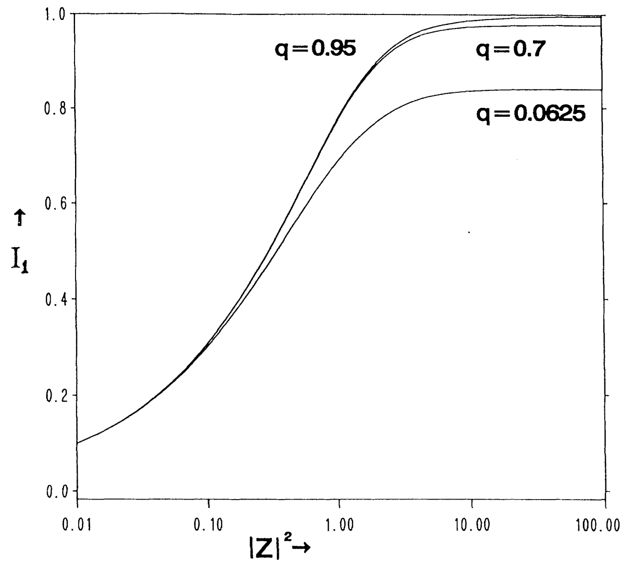

FIG. 5. As $|z|^2$ increases, the behavior of the function $I_1(|z|)$. It characterizes expectation values of the $\hat{\cos}(\phi)$ and $\hat{\sin}(\phi)$ operators, see Eq. (5.3).

these functions appear to level off, but not at the value of one which occurs in the $q \rightarrow 1$ limit:

$$I_1(|z|)|_{q=1} = |z| e^{-|z|^2} \Psi_1(|z|^2) \underset{|z|\rightarrow\infty}{\sim} 1-(8|z|^2)^{-1}+\cdots , \tag{5.5a}$$

$$I_2(|z|)|_{q=1} = |z|^2 e^{-|z|^2} \Psi_2(|z|^2) \underset{|z|\rightarrow\infty}{\sim} 1-(2|z|^2)^{-1}-(8|z|^4)^{-1}+\cdots , \tag{5.5b}$$

where the power-series definitions [19] for Ψ_i are evident

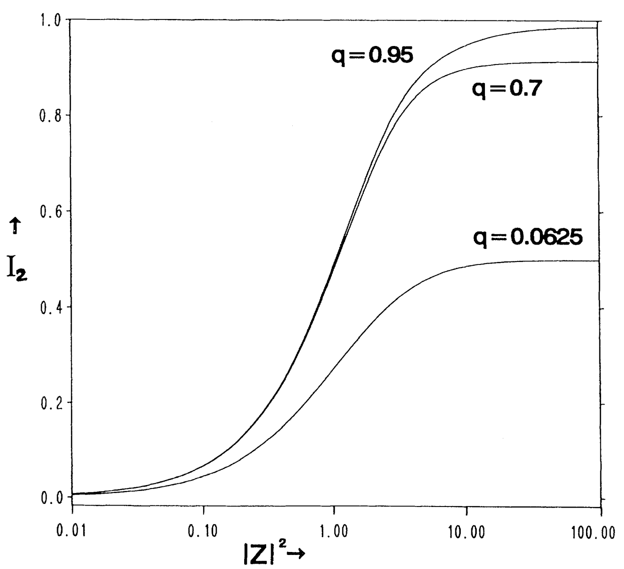

FIG. 6. As $|z|^2$ increases, the behavior of the function $I_2(|z|)$. It characterizes expectation values involving some squares of $\hat{\cos}(\phi)$ and $\hat{\sin}(\phi)$, see Eq. (5.3). By Eq. (5.6), there is the analytic result that $I_2 \rightarrow (p=q^{1/4})$ as $|z| \rightarrow \infty$.

from Eqs. (5.4). As we discuss in Sec. V B, there is a similar plateau behavior for the associated number-phase uncertainty relations which involve the SG phase operators.

For $|z| \rightarrow \infty$, the function

$$\begin{aligned} I_2(|z|)|_{q\neq 1} &= |z|^2 e_q(|z|^2)^{-1} \psi_2(|z|^2) \leq p\{1-e_q(|z|^2)^{-1}\} \\ &\rightarrow p\{1-\tfrac{1}{2}(1-q)\epsilon_2(|z|^2) \\ &\qquad -\tfrac{1}{8}(1-q)(1+3q)\epsilon_4(|z|^2)+\cdots\} \\ &\rightarrow p , \end{aligned} \tag{5.6}$$

where $p=q^{1/4}$ (positive fourth root). Therefore, the plateau shown in Fig. 6 does not rise for larger $|z|$ values. In fact, its asymptotic value has almost been reached by $|z|^2 \sim 20$ for $q<(\sim 0.7)$ since $p=0.915$ for $q=0.7$ and $p=0.5$ for $q=0.0625$.

Equation (5.6) follows from (5.4b) by using

$$[n+2][n+1]=[n+\tfrac{3}{2}]^2-[\tfrac{1}{2}]^2$$

and

$$(c^2-x^2)^{-1/2}=c^{-1}\left\{1+\frac{1}{2}\left[\frac{x}{c}\right]^2+\frac{3}{8}\left[\frac{x}{c}\right]^4+\cdots\right\},$$

so

$$\psi_2(|z|^2) \rightarrow \sum_{n=0}^{\infty} \frac{|z|^{2n}}{[n+1]!}\left[\frac{[n+1]}{[n+\frac{3}{2}]}\right] \rightarrow q^{1/4} e_q(|z|^2)|z|^{-2}+\cdots$$

since for q small or n large,

$$[n]=q^{(1-n)/2}\left\{\frac{1-q^n}{1-q}\right\} \approx \frac{p^{2(1-n)}}{1-q} .$$

In I_2 the subasymptotic terms, which occur due to the power-series expansion and the approximation to $[n]$, fall off q exponentially, i.e., as $e_q(q^{m/2}|z|^2)/e_q(|z|^2) = \epsilon_m(|z|^2)$ where $m>0$; compare Eq. (2.16).

As a consequence of Eq. (5.6), the $|z\rangle_q$ CS expectation values of the SG phase operators are class I,

$$\langle z|\hat{\cos}(\phi)^2|z\rangle \rightarrow (1+p\cos 2\theta)/2 , \tag{5.7a}$$

$$\langle z|\hat{\sin}(\phi)^2|z\rangle \rightarrow (1-p\cos 2\theta)/2 , \tag{5.7b}$$

and class II,

$$\langle z|\{\hat{\cos}(\phi)^2-\hat{\sin}(\phi)^2\}|z\rangle \rightarrow p\cos 2\theta , \tag{5.7c}$$

$$\langle z|\{\hat{\cos}(\phi),\hat{\sin}(\phi)\}_+|z\rangle \rightarrow p\sin 2\theta . \tag{5.7d}$$

Hence the q deformation of the polar decomposition of the creation and annihilation operators,

$$a \equiv ([N+1])^{1/2}\hat{\exp}(\iota\phi) ,$$

affects, in general, both the angular and radial parts of the functions of the SG phase operators. So in describing a physical q-boson system undergoing a coherent cooperative behavior (e.g., spontaneous symmetry breaking as would occur in a q-boson superfluid or superconductor), the usefulness of a field-theoretic ansatz for the order parameter (cf. Anderson [12] and Carruthers and Nieto [19])

$$\langle \psi(\mathbf{x},t)\rangle \approx \sqrt{\rho_q(\mathbf{x},t)}\exp\{i\phi(\mathbf{x},t)\}\ , \tag{5.8}$$

where only $\sqrt{\rho_q(\mathbf{x},t)}$ undergoes a q deformation, depends crucially on whether the relevant physical observables are class II or not. Notice that the operator combinations in Eqs. (5.1b) and (5.1c) are class II and that from Fig. 5, $I_1 \to (\sim p^{1/4})$ as $|z|^2 \to (>100)$, so

$$\langle z|\hat{\cos}(\phi)|z\rangle \to \sim p^{1/4}\cos\theta\ , \tag{5.9a}$$

$$\langle z|\hat{\sin}(\phi)|z\rangle \to \sim p^{1/4}\sin\theta\ . \tag{5.9b}$$

So an ansatz of the form of Eq. (5.8) for the condensate wave function has the additional constraint of yielding the multiplicative p-scaling variation of Eqs. (5.1b), (5.1c), (5.7c), (5.7d), (5.9a), and (5.9b).

Being an analytic result, Eq. (5.6) shows that at least for I_2 the generic plateau behavior shown in the figures in this paper is not due to a very gradual approach to 1 as $|z| \to \infty$. Instead, the nonclassical plateau value for $q \neq 1$ for moderate $|z|$ is indeed due to the q deformation of the limiting value.

B. q-analog $\hat{\sin}(\phi)$, $\hat{\cos}(\phi)$, and N uncertainty relations

For $\hat{\sin}(\phi)$ and $\hat{\cos}(\phi)$ the usual number-phase uncertainty relations hold for arbitrary q:

$$\Delta N \Delta\, \hat{\sin}(\phi) \geq \tfrac{1}{2}|\langle \hat{\cos}(\phi)\rangle|\ , \tag{5.10a}$$

$$\Delta N \Delta\, \hat{\cos}(\phi) \geq \tfrac{1}{2}|\langle \hat{\sin}(\phi)\rangle|\ . \tag{5.10b}$$

Therefore, the θ-independent associated Carruthers-Nieto uncertainty relation [19,20] follows:

$$U(|z|) \equiv (\Delta N)^2 \frac{[(\Delta S)^2 + (\Delta C)^2]}{[\langle S\rangle^2 + \langle C\rangle^2]} \geq \tfrac{1}{4}\ , \tag{5.10c}$$

where $S \equiv \hat{\sin}(\phi)$ and $C = \hat{\cos}(\phi)$.

However, for moderate values of $|z|^2$, we find numerically that the $|z\rangle_q$ CS no longer minimize these relations when $q \neq 1$. [In the $q \to 0$ limit, for $|z| \to \infty$ this lack of minimization for Eqs. (5.10a) and (5.10b) was shown analytically in Ref. [3].] We begin with the left-hand side of Eq. (5.10a). For $q = 0.7$ and 0.0625, respectively, Figs. 7 and 8 display

$$S(\theta) \equiv (\Delta N)^2 (\Delta\, \hat{\sin}(\phi))^2 \tag{5.10d}$$

as a function of $|z|^2$ for various values of the phase θ of $z = |z|\exp(i\theta)$. Note that Fig. 8 shows that for $\theta = \pi/2$ the function $S(\theta = \pi/2)$ is not zero for $|z|^2 \gtrsim 10$, but instead S plateaus at a nonzero value.

Indeed, as shown in Figs. 9 and 10, for $q \neq 1$ the CS's do *not* minimize the simplest uncertainty relations [21] of Eqs. (5.10) for moderate $|z|^2$ values (unless $\theta = 0$ or $\pi/2$). Note that

$$Q(\theta) \equiv \frac{(\Delta N)^2 (\Delta\, \hat{\sin}(\phi))^2}{\langle \hat{\cos}(\phi)\rangle^2}\ . \tag{5.11}$$

Figures 11 and 12 show the behavior of the Carruthers and Nieto [19] minimization function $U(|z|)$ [see Eq. (5.10c)] for $q = 0.7$ and 0.0625, respectively. Note that $1/2 \geq U \geq 1/4$.

From the q-independent commutator

$$[\hat{\cos}(\phi), \hat{\sin}(\phi)] = \frac{1}{2i}[a^\dagger([N+1])^{-1}a - 1]\ , \tag{5.12}$$

it follows that, in the $|z\rangle_q$ basis,

$$\Delta\, \hat{\sin}(\phi) \Delta\, \hat{\cos}(\phi) \geq \tfrac{1}{4} e_q(|z|^2)^{-1}\ . \tag{5.13}$$

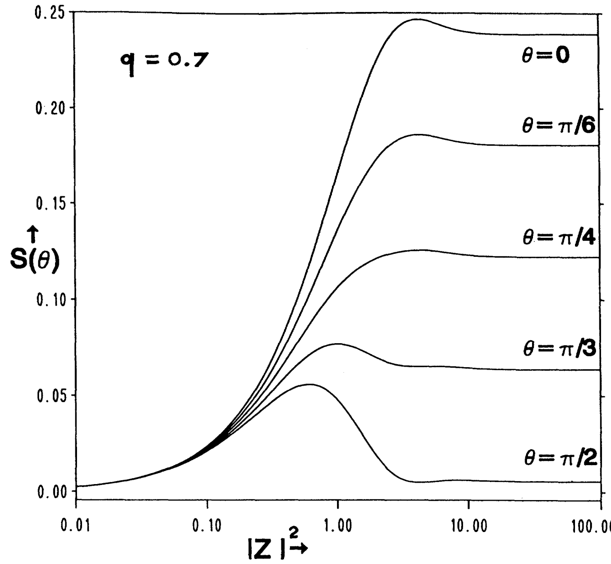

FIG. 7. For $q = 0.7$, the behavior of the uncertainty product $S(\theta) \equiv (\Delta N)^2(\Delta\, \hat{\sin}(\phi))^2$ as a function of $|z|^2$ for various values of the phase θ of the complex number $z = |z|e^{i\theta}$. $\hat{\sin}(\phi)$ is the q analog of the Susskind-Glogower Hermitian sine operator.

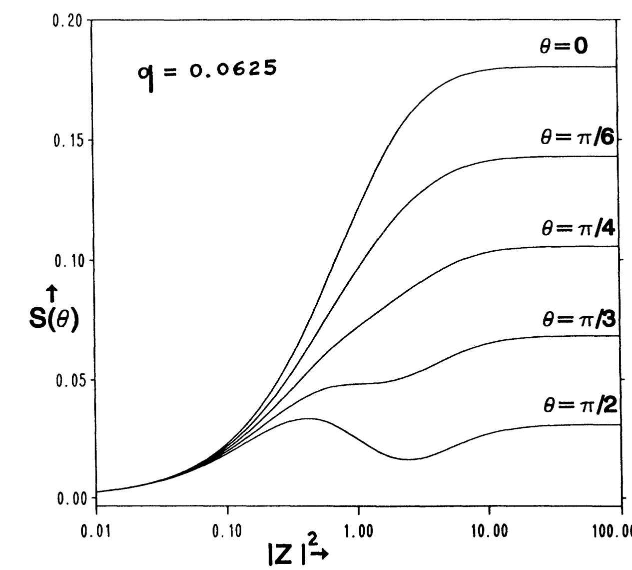

FIG. 8. Same as for Fig. 7, except now $q = 0.0625$. Unlike here, for $q = 1$ the function $S(\theta = \pi/2) =$ zero for $|z|^2 \gtrsim 10$.

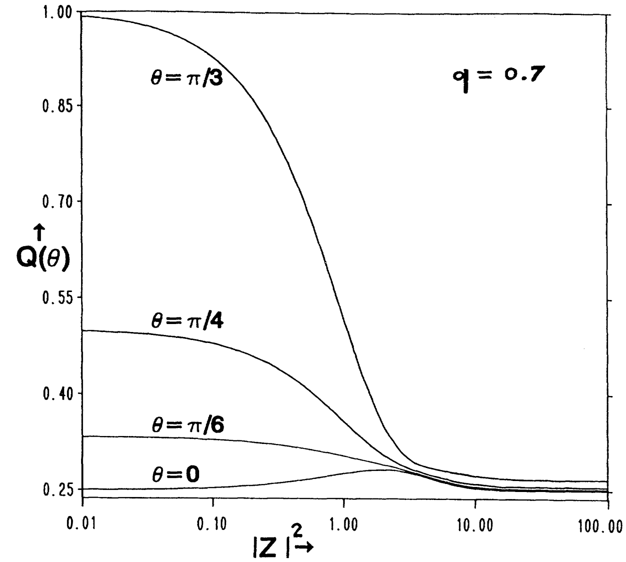

FIG. 9. For $q=0.7$, behavior of the minimization function $Q(|z|,\theta)\equiv(\Delta N)^2(\Delta\widehat{\sin}(\phi))^2/\langle\widehat{\cos}(\phi)\rangle^2$ as a function of $|z|^2$ for various values of the phase θ. By Eq. (5.10a), $Q(|z|,\theta)\geq\frac{1}{4}$ just as for $q=1$. However, here unlike for $q=1$ the q-analog CS's do not minimize the function Q for $|z|^2\gtrsim 80$; see Fig. 10.

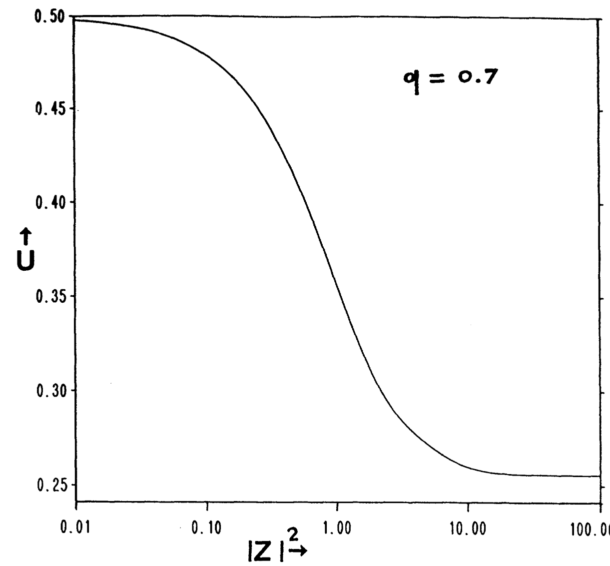

FIG. 11. For $q=0.7$, behavior of the Carruthers-Nieto minimization function $U(|z|)$, defined in Eq. (5.10c), as $|z|^2$ increases. It is independent of θ and greater than $\frac{1}{4}$. Here again, unlike for $q=1$ the q-analog CS's do not minimize U for $|z|^2\gtrsim 80$; see Fig. 12.

So for $|z|^2\ll 1$ (the quantum limit), the equality is satisfied for any q value. However, for moderate $|z|$ values, the left-hand side of Eq. (5.12) is very q dependent, as shown in Fig. 13. Therefore, the product

$$\Delta\,\widehat{\sin}(\phi)\Delta\,\widehat{\cos}(\phi)\not\rightarrow\text{zero},\quad \sim 10<|z|^2<\text{few }100 \qquad (5.14)$$

for $q\neq 1$, which is unlike the usual $q=1$ classical behavior.

The leveling off for moderate $|z|^2$ values of the various uncertainty-relation minimization functions [Eqs. (10d), (5.11), and (5.14)] means, of course, that the associated SG phase operators for describing the q-analog radiation field remain correlated, in this region of $|z|^2$. Hence for the SG operators the $|z\rangle_q$ classical limit is a q-deformed analog of the $q=1$ case, and it is not a limit for moderate $|z|^2$, which physically exhibits classically uncorrelated $\widehat{\sin}(\phi)$, $\widehat{\cos}(\phi)$, and N operators. For much larger $|z|^2$, $|z|^2\gg 100$, Eq. (5.6) shows that such plateaus do not always approach their $q=1$ classical values.

In Sec. VI, we show that the q-analog generalization of

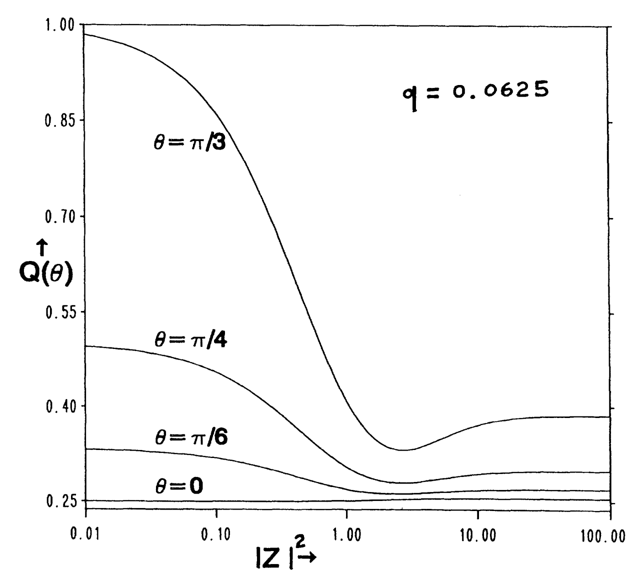

FIG. 10. Same as Fig. 9, except $q=0.0625$.

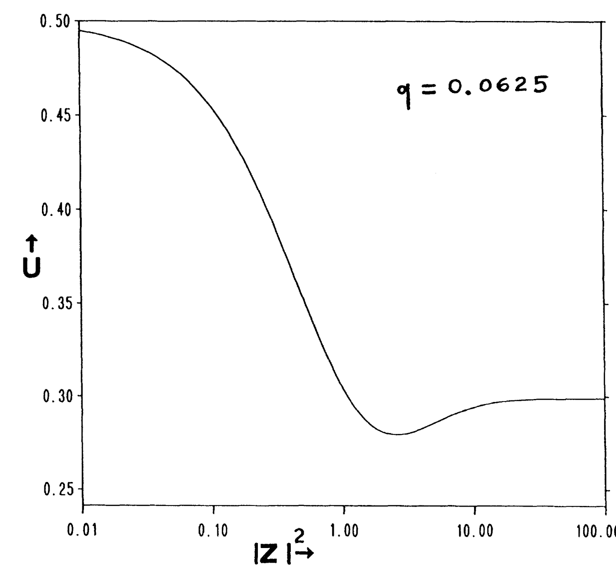

FIG. 12. Same as Fig. 9, except $q=0.0625$.

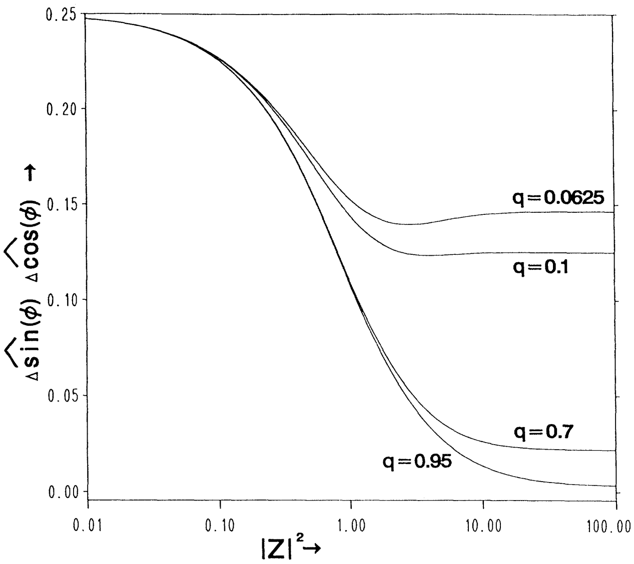

FIG. 13. For various fixed values of q, the behavior of the product of the uncertainties of the q-analog Susskind-Glogower sine and cosine operators $\Delta\,\widehat{\sin}(\phi)\Delta\,\widehat{\cos}(\phi)$. Again, only for $q=1$ does this product vanish for $|z|^2 \gtrsim 80$.

the PB phase operator $\hat{\phi}_q$ for $q \neq 1$ does not exhibit such manifest non-($q=1$), nonclassical behavior in the $|z\rangle_q$ basis in the large-$|z|$ limit. However, the associated phase width for a $|z\rangle_q$ coherent state is q deformed since $\Delta\hat{\phi}_q \rightarrow (2\eta_q)^{-1}$ for moderate $|z|^2$.

VI. PROPERTIES OF q-ANALOG PB PHASE OPERATORS

A. Properties of $\hat{\phi}_q$

Unlike for the SG phase operators, the classical behavior for the phase operator (as expected from the correspondence principle) does indeed still hold for $q \neq 1$ for the PB phase operator. In particular, since

$$\cos\hat{\phi}_q = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (\hat{\phi}_q)^{2n} ,$$

$$\sin\hat{\phi}_q = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} (\hat{\phi}_q)^{2n+1} \tag{6.1}$$

for arbitrary q value, it still follows that

$$[\cos\hat{\phi}_q, \sin\hat{\phi}_q] = 0 , \tag{6.2a}$$

$$\cos^2\hat{\phi}_q + \widehat{\sin}^2\phi_q = 1 , \tag{6.2b}$$

and using

$$|n\rangle_q = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{-in\theta_m} |\theta_m\rangle ,$$

$$\frac{1}{s+1} \sum_{m=0}^{s} \rightarrow \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+2\pi} d\theta_m ,$$

that

$$_q\langle n|\sin^2\hat{\phi}_q|n\rangle_q = {}_q\langle n|\cos^2\hat{\phi}_q|n\rangle_q = \tfrac{1}{2} \tag{6.2c}$$

for $n=0,1,2,\ldots$. In particular, for arbitrary q, the q-boson vacuum state $|0\rangle_q$ remains a state with random phase. In contrast, for arbitrary q the SG phase operators satisfy [3,19,9] Eq. (6.2c) for $n \neq 0$, but for $n=0$ the $\frac{1}{2} \rightarrow \frac{1}{4}$ in Eq. (6.2c).

B. The q-boson phase distribution $\bar{P}_q(\theta_m)$

We consider a phase distribution analog to the q-boson number distribution $P_n^q(z)$ in the CS basis. That is, as opposed to $P_n^q(z)$ of Sec. III, we study its conjugate-variable analog using the PB operator $\hat{\phi}_q$. We consider the q-boson phase distribution

$$\bar{P}_q(\theta_m) \equiv \lim_{s\rightarrow\infty} (s+1)|\langle\theta_m|z\rangle_q|^2 , \tag{6.3}$$

with the normalization

$$\frac{1}{2\pi} \int_0^{2\pi} \bar{P}_q(\theta_m) d\theta_m = 1 . \tag{6.4}$$

The bar denotes our insertion of the $(s+1)$ factor in the definition in Eq. (6.3).

The qualitative behavior of $\bar{P}_q(\theta_m)$ for $q \neq 1$ can be obtained analytically, as was done previously in the literature [22] for the case $q=1$. For $z=|z|\exp(i\theta)$, we find

$$\bar{P}_q(\theta_m) = 1 + 2(N(z))^2 \times \sum_{n>l}^{\infty} \frac{|z|^{n+l}}{\sqrt{[n]![l]!}} \cos\{(n-l)(\theta-\theta_m)\} . \tag{6.5}$$

The mean value of $\hat{\phi}_q$ in the $|z\rangle_q$ basis is

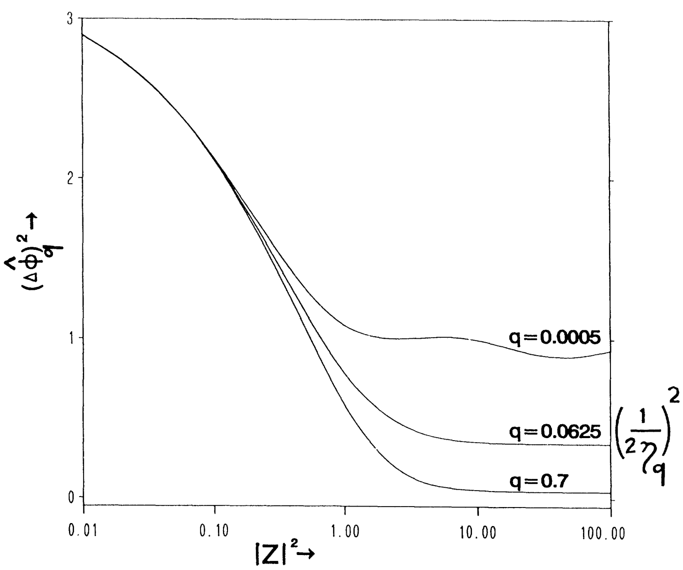

FIG. 14. Behavior of the variance $(\Delta\hat{\phi}_q)^2$ of the q-analog Pegg-Barnett phase operator as $|z|^2$ varies for fixed $q \neq 1$. While for $q=1$, $(\Delta\hat{\phi}_q)^2 \rightarrow 1/(4|z|^2)$ for $|z|^2 \gtrsim 10$, for $q \neq 1$ this figure shows the q deformation $(\Delta\hat{\phi}_q)^2 \rightarrow (1/2\eta_q)^2$. Recall that this q-dependent constant also appeared in Fig. 4; see its caption.

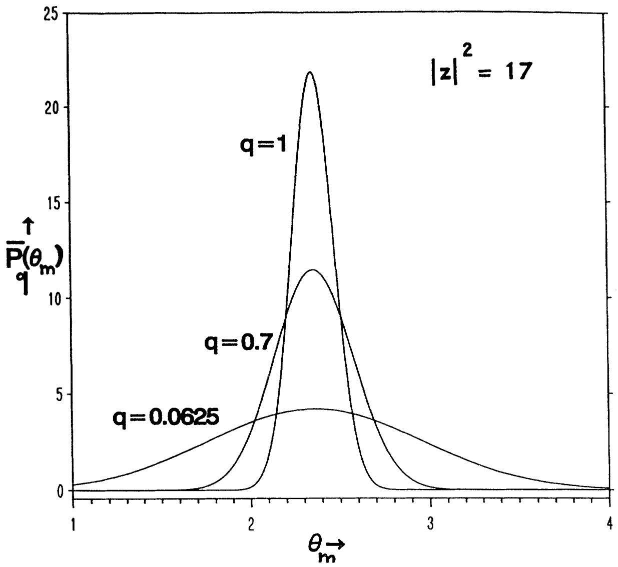

FIG. 15. For fixed $|z|^2=17$, the q-boson Pegg-Barnett normalized phase distribution function $\bar{P}_q(\theta_m)$; see Eq. (6.3). We set $\arg(z)=3\pi/4$ in this and the following plots for display purposes.

$$\langle \hat{\phi}_q \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \theta_m \bar{P}_q(\theta_m) d\theta_m \tag{6.6a}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\theta+y)\bar{P}_q(\theta+y) dy \ . \tag{6.6b}$$

To obtain Eq. (6.6b), we have chosen the reference phase [see Eq. (4.4b)], to be $\theta_0 \equiv \theta - \pi s/(s+1)$, and then changed the variables by $\theta_m=\theta+y$, with $y=2\pi(m-s/2)/(s+1)$. Then by Eq. (6.4) the mean value

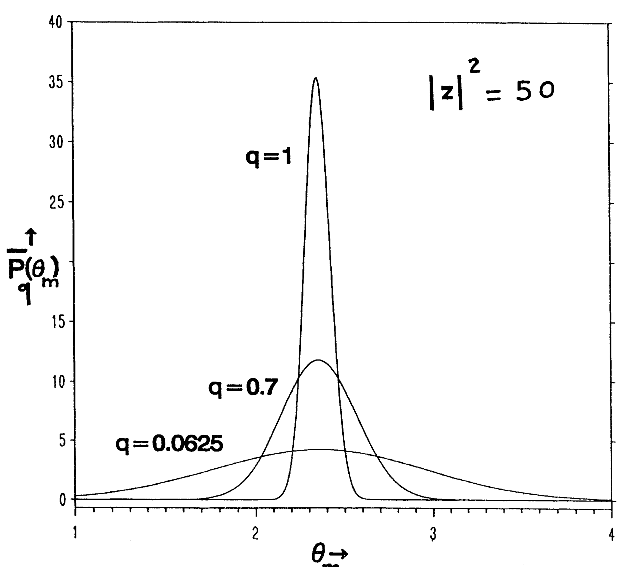

FIG. 16. Same as Fig. 15, except for increased $|z|^2=50$. As opposed to the previous plot, note that the vertical scale has been changed, and so only for the $q=1$ case do these plots display a δ-function limiting behavior.

$$\langle \hat{\phi}_q \rangle = \theta \tag{6.7a}$$

$$= \arg(z) \ , \tag{6.7b}$$

because the y term in the integrand of Eq. (6.6b) is odd from Eq. (6.5). Similarly, we find that the variance of the phase operator is

$$(\Delta\hat{\phi}_q)^2 = \frac{\pi^2}{3} + 4(N(z))^2 \sum_{n>l}^{\infty} \frac{(-1)^{n+l}}{(n-l)^2} \left\{ \frac{|z|^{n+l}}{\sqrt{[n]![l]!}} \right\} . \tag{6.8}$$

The behavior of $(\Delta\hat{\phi}_q)^2$ for moderate values of $|z|^2$ is shown in Fig. 14; see its caption.

Figures 15 and 16 show how the phase distribution $\bar{P}_q(\theta_m)$ changes as $|z|^2=17$ is increased to $|z|^2=50$. In these figures, we set $(\arg z)=\theta=3\pi/4$ for display purposes. At this point, it is also useful to reconsider Table III. When $q=1$, the limiting fiducial distribution [9]

$$\lim_{|z|\to\infty} \bar{P}(\theta_m)|_{q=1} \to 2\pi\delta(\theta-\theta_m), \quad q=1 \tag{6.9}$$

is a Dirac δ distribution. However, for $q\neq 1$, since the variance $(\Delta\hat{\phi}_q)^2 \to (2\eta_q)^{-2}$ for moderate $|z|^2$ values, the normalization requirement Eq. (6.4) implies that a Dirac δ distribution no longer occurs. Instead

$$\lim_{|z|\to\infty} \bar{P}(\theta_m)|_{q\neq 1} \to 2\pi\delta_q(\theta-\theta_m), \quad q\neq 1 \ , \tag{6.10}$$

where the fiducial $\delta_q(\theta-\theta_m)$ is a bell-shaped function (see Figs. 15 and 16) which is centered on $\theta=(\arg z)$, and has finite width $(\Delta\hat{\phi}_q \to 1/2\eta_q)$ and finite height.

In the case of the $|n\rangle_q$ basis, the amplitude for the $|z\rangle_q$ CS to be in the $|n\rangle_q$th state is, indeed, a complex amplitude

$$\langle n|z\rangle = \frac{|z|^n}{\sqrt{[n]!}}(\cos n\theta + i \sin n\theta) \ . \tag{6.11}$$

However, the real and imaginary parts of $\langle n|z\rangle$ are not actually centered on the mean value $\langle z|N|z\rangle$. Such a centering at $\langle z|N|z\rangle$ is only seen at the $P_n^q(z)=|\langle n|z\rangle|^2=\{\mathrm{Re}(\langle n|z\rangle)\}^2+\{\mathrm{Im}(\langle n|z\rangle)\}^2$ probability distribution level.

In contrast to Eq. (6.11), in the case of the PB eigenket $|\theta_m\rangle$ the real and imaginary parts of the $\langle\theta_m|z\rangle$ amplitude do individually display centering about $\theta=\arg(z)$ for any q. If we consider the amplitude for $|z\rangle_q$ to be in the PB phase eigenket $|\theta_m\rangle$, then

$$\lim_{s\to\infty} \{(s+1)^{1/2}\langle\theta_m|z\rangle_q\} \equiv \mathcal{R}(\theta_m,z)+i\mathcal{I}(\theta_m,z) \ , \tag{6.12}$$

with

$$\mathcal{R}(\theta_m,z)=N(z)\sum_{n=0}^{\infty}\frac{|z|^n}{\sqrt{[n]!}}\cos\{(\theta-\theta_m)n\} \ , \tag{6.13a}$$

$$\mathcal{I}(\theta_m,z)=N(z)\sum_{n=0}^{\infty}\frac{|z|^n}{\sqrt{[n]!}}\sin\{(\theta-\theta_m)n\} \ . \tag{6.13b}$$

For $q=1$, one has

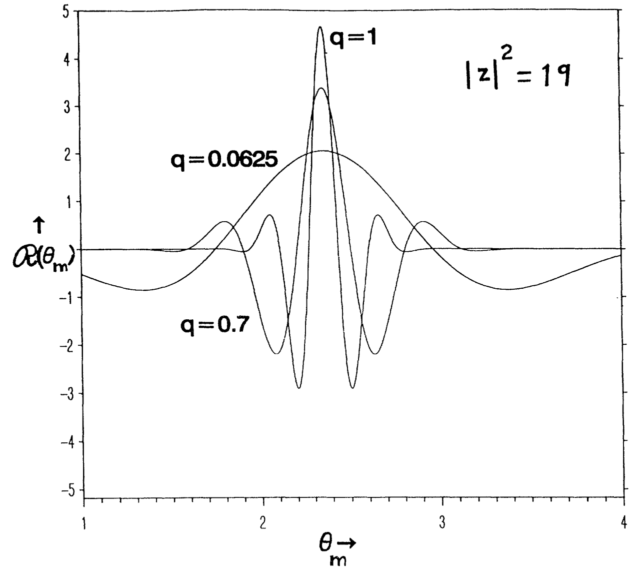

FIG. 17. For $|z|^2=19$, behavior of the real part $\mathcal{R}(\theta_m)\sim\text{Re}\{{}_q\langle\theta_m|z\rangle_q\}$ for $q=1$, 0.7, and 0.0625; see Eq. (6.12).

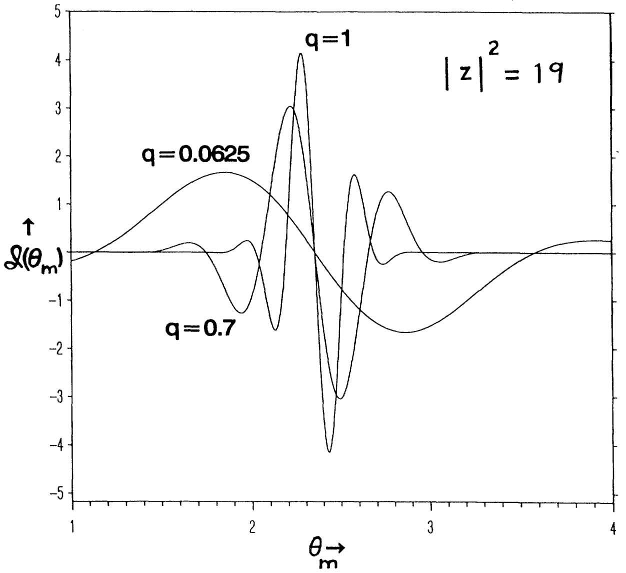

FIG. 18. Same as Fig. 17, except for the imaginary part $\mathcal{I}(\theta_m)\sim\text{Im}\{{}_q\langle\theta_m|z\rangle_q\}$.

$$\mathcal{R}(\theta_m,z)|_{q=1}=(8\pi|z|^2)^{1/4}\exp\{-|z|^2(\theta-\theta_m)^2\}\times\cos\{|z|^2(\theta-\theta_m)\}\ , \tag{6.14a}$$

$$\mathcal{I}(\theta_m,z)|_{q=1}=(8\pi|z|^2)^{1/4}\exp\{-|z|^2(\theta-\theta_m)^2\}\times\sin\{|z|^2(\theta-\theta_m)\}\ . \tag{6.14b}$$

However, even for $q\neq1$ there is damping in $|\theta-\theta_m|$, as shown in Figs. 17 and 18. Note that as q decreases from 1, there is a decrease in the frequency of the oscillations in $|\theta-\theta_m|$. This frequency, per Eqs. (6.14), also increases for $q\neq1$ as $|z|$ as increased.

Note also the $|z\rangle_q$ expectation value of $(\hat{\phi}_q)^r$, for r a positive integer, is given by a finite series in θ with θ^r the largest power:

$$\langle(\hat{\phi}_q)^r\rangle\equiv{}_q\langle z|(\hat{\phi}_q)^r|z\rangle_q=\sum_{l=0}^{[r/2]}\binom{r}{2l}\theta^{r-2l}c_l\ , \tag{6.15a}$$

which is in terms of the usual binomial coefficient, and

$$c_l=\frac{\pi^{2l}}{(2l+1)}+2(N(z))^2\sum_{m>n}^{\infty}\frac{|z|^{m+n}}{\sqrt{[m]![n]!}}\sum_{k=0}^{[(2l-1)/2]}\frac{(-1)^{k+m+n}\pi^{2(l-k-1)}(2l)!}{(m-n)^{2(k+1)}(2l-2k-1)!}\ , \tag{6.15b}$$

where $[r/2]=$[greatest integer $\leq(r/2)$] in the summations' upper limits.

VII. APPROXIMATE NUMBER-PHASE COMMUTATION RELATION $[N,\hat{\phi}_q]$ IN $|z\rangle_q$ CLASSICAL LIMIT

In the $|z\rangle_q$ basis, the q-boson phase distribution $\bar{P}_q(\theta_m)$ function considered in Sec. VI also appears in Dirac's approximate number-phase commutation relation

$${}_q\langle z|[N,\hat{\phi}_q]|z\rangle_q=i-i\bar{P}_q(\theta_0)\ . \tag{7.1}$$

This result simply follows, cf. Pegg and Barnett by inserting the $|n\rangle_q$ basis completeness Pegg and Barnett relation $\sum|n\rangle_q\,{}_q\langle n|=1$ in the left-hand side to obtain

$${}_q\langle z|[N,\hat{\phi}_q]|z\rangle_q=\sum_{m,n}\langle z|m\rangle\langle m|[N,\hat{\phi}_q]|n\rangle\langle n|z\rangle\ . \tag{7.2a}$$

But in the $|n\rangle_q$ basis, there is the q-independent relation [9]

$$\langle m|[N,\hat{\phi}_q]|n\rangle=-i(1-\delta_{mn})\exp\{i(m-n)\theta_0\}\ , \tag{7.2b}$$

so

$${}_q\langle z|[N,\hat{\phi}_q]|z\rangle_q=i-i\left|\sum_{n=0}^{\infty}{}_q\langle n|z\rangle_q\exp(-in\theta_0)\right|^2=i-i\bar{P}_q(\theta_0) \tag{7.2c}$$

by Eq. (6.3) since ${}_q\langle n|z\rangle_q=N(z)z^n/\sqrt{[n]!}$.

So for large $|z|$, for $q\neq1$,

$$\lim_{|z|\to\infty}{}_q\langle z|[N,\hat{\phi}_q]|z\rangle_q=i-i2\pi\delta_q(\theta-\theta_0)\quad(q\neq1) \tag{7.3}$$

for $\hat{\phi}_q$ eigenvalues from the indicial θ_0 to $(\theta_0+2\pi)$. Recall [see Sec. VI] that this extra δ_q term in Eq. (7.3) is a bell-shaped fiducial function as displayed in Figs. 15 and

16. This extra term serves a physical role analogous to that of a smeared magnetic monopole string in that it appears in the classical limit to uniquely specify the classical phase angle. For $q=1$, the smearing is absent as δ_q is replaced by a δ-function distribution [9]. Note that the appearance of the smeared $\delta_q(\theta-\theta_0)$ instead of a sharp $\delta(\theta-\theta_0)$ indicial referencing of θ versus θ_0 in Eq. (7.3) is in physical agreement with greater fractional $\Delta\hat{\phi}_q/|\langle\hat{\phi}_q\rangle|$ for $q\neq 1$ shown in the last line of Table III.

Hence for the q-boson quantized radiation field, in contrast to the $\lambda(z)$ factor in the commutation relation for the position and momentum operators [see Eq. (2.5)], the operators N and $\hat{\phi}_q$ do turn out to be almost canonically conjugate in the $|z\rangle_q$ classical limit.

So, neglecting the indicial-referencing term, we find that the usual (though approximate) number-phase uncertainty relation follows from Eq. (7.3) for the q-boson quantized field

$$\Delta N \Delta\hat{\phi}_q \geq \tfrac{1}{2} \quad (7.4)$$

[see Eqs. (A13) and (A15) for extension to PB-type physical states]. Equivalently, for free q bosons in the $|z\rangle_q$ classical limit, there is the usual approximate, energy-phase uncertainty relation

$$\Delta H_N \Delta\hat{\phi}_q \geq \hbar\omega/2 \ . \quad (7.5)$$

From Eq. (7.3), the classical-quantum canonical correspondence for a single mode of the q-boson field is

$$\left\{ \frac{d\phi_{\text{clas}}}{dt} \equiv [\phi_{\text{clas}}, H] \right\} \rightleftharpoons \left\{ \frac{d\hat{\phi}_q}{dt} = \frac{[\hat{\phi}_q, H_N]}{i\hbar} \right\} , \quad (7.6a)$$

where

$$\frac{[\hat{\phi}_q, H_N]}{i\hbar} \sim \{ -\omega + 2\pi\omega\delta_q(\theta-\theta_0) \} \quad (7.6b)$$

in the $|z\rangle_q$ classical limit. Numerically (see Fig. 19) we find that the $|z\rangle_q$ coherent states do indeed minimize this approximate number-phase uncertainty relation for the q-boson quantized field.

When $q\neq 1$, the generalization of the ordinary number operator N is not unique, as emphasized at the end of Sec.

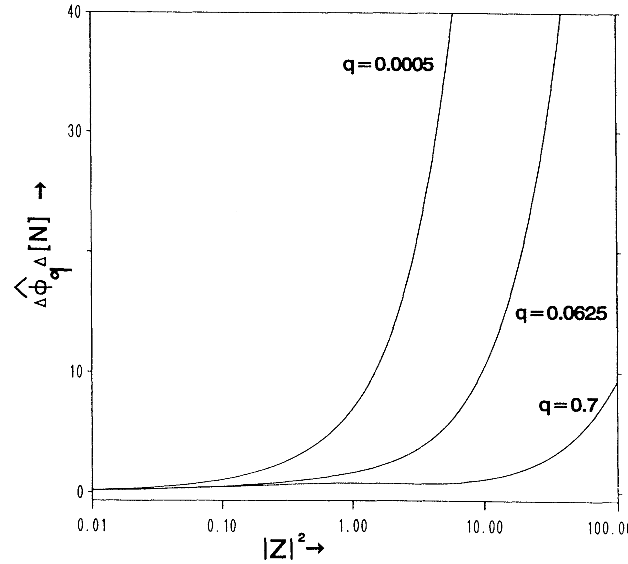

FIG. 20. For contrast to the behavior shown in Fig. 19, this plot shows the behavior of the uncertainty product $(\Delta\hat{\phi}_q)(\Delta[N])$ as $|z|^2$ varies.

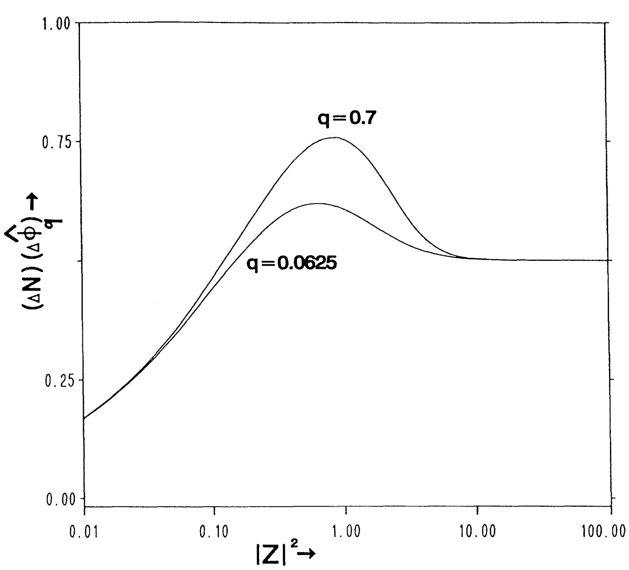

FIG. 19. For $q=0.7$ and 0.0625, this plot shows the behavior of the number-phase uncertainty product $(\Delta N)(\Delta\hat{\phi}_q)$ as $|z|^2$ varies. Here $\hat{\phi}_q$ is the Hermitian Pegg-Barnett phase operator. Note that the figure shows that this product is minimized for $|z|^2 \gtrsim 10$ independent of the q value {at least for $1\geq q \gtrsim 0.06$}.

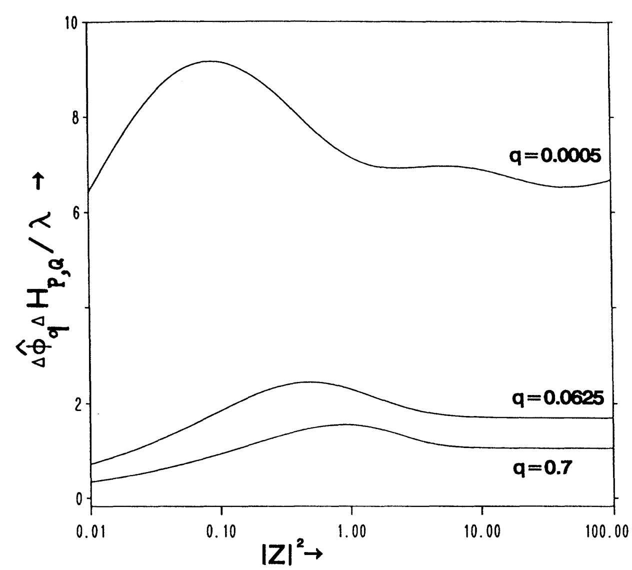

FIG. 21. Also for contrast to Fig. 19, this plot shows the behavior of the uncertainty product $\Delta\hat{\phi}_q \Delta H_{P,Q}$ divided by the $\lambda(z)$ resolution function in units of $(\hbar\omega/2)$. When $q=1$, this function $\lambda(z)=1$. But for $q\neq 1$, $\lambda(z)$ diverges linearly in $|z|^2$ as $|z|\to\infty$; see Eq. (2.15).

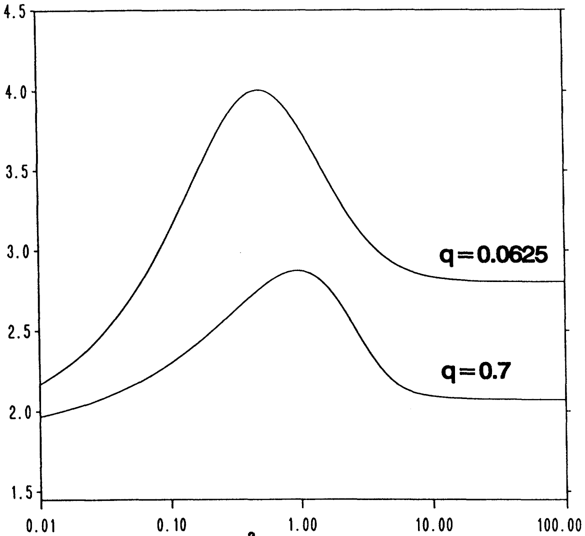

FIG. 22. Also for contrast to Fig. 19, this plot shows the behavior of the function $M(|z|)\equiv 2\Delta\hat{\phi}_q \Delta H_{P,Q}/\langle [H_{P,Q},\hat{\phi}_q]\rangle$ as $|z|^2$ increases. Unlike for Fig. 19, there is no apparent minimization for $q\neq 1$.

II. So we have also investigated numerically the properties of the uncertainty products

$$\Delta\hat{\phi}_q \Delta[N] \ , \tag{7.7a}$$

$$(\Delta\hat{\phi}_q \Delta H_{P,Q})/\lambda(z) \ , \tag{7.7b}$$

and

$$M(|z|)=2\Delta\hat{\phi}_q \Delta H_{P,Q}/\langle [H_{P,Q},\hat{\phi}_q]\rangle \tag{7.7c}$$

as $|z|^2$ varies from 0.1 to ~ 100.0. Figures 20–22 show that unlike for N and $\hat{\phi}_q$, these products do not exhibit canonical ($q=1$)-type behavior in the $|z\rangle_q$ basis in the $|z\rangle_q$ classical limit.

ACKNOWLEDGMENTS

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APPENDIX A: FURTHER APPLICATIONS OF THE PB FORMALISM

(i) Since the additional term in the $[\hat{\phi}_q,N]$ commutator has direct physical consequences, it is important also to check in the $q\neq 1$ case whether there are possibly other physically observable consequences.

Following the work of Pegg and Barnett [9], we define a physically accessible, or preparable state in the $(s+1)$-dimensional $\Psi_{\rm PB}^{(q)}\equiv\{|0\rangle_q,|1\rangle_q,\ldots,|s\rangle_q\}$ subspace,

$$|p_{\rm PB}\rangle\equiv\sum_{n=0}^{s} c_n|n\rangle_q \ , \tag{A1}$$

where the c_n coefficients are such that the expectation value

$$\langle p_{\rm PB}|\{[N]\}^I|f_{\rm PB}\rangle \ , \tag{A2}$$

converges as $s\rightarrow\infty$ for I any non-negative integer. This implies that

$$\lim_{s\rightarrow\infty}(|c_s|^2\{[s]\}^M)\rightarrow O \tag{A3}$$

for $M=0,1,2,\ldots$.

In $\Psi_{\rm PB}$ we define creation and annihilation operators

$$\begin{aligned}\bar{a}&\equiv\exp(i\hat{\phi}_q)\sqrt{[N]}\\&=|0\rangle_{q\,q}\langle 1|+\sqrt{[2]}|1\rangle_{q\,q}\langle 2|+\cdots\\&\quad+\sqrt{[s]}|s-1\rangle_{q\,q}\langle s| \ .\end{aligned} \tag{A4}$$

The second line follows from Eq. (4.6). Note that $\bar{a}|0\rangle_q=0$. We use a bar to distinguish these operators from the ones considered in the text, which are for the infinite-dimensional Hilbert space. Then

$$\begin{aligned}\bar{a}^{\dagger}&=\sqrt{[N]}\exp(-i\hat{\phi}_q)\\&=|1\rangle_{q\,q}\langle 0|+\sqrt{[2]}|2\rangle_{q\,q}\langle 1|+\cdots\\&\quad+\sqrt{[s]}|s\rangle_{q\,q}\langle s-1|\end{aligned} \tag{A5}$$

but $\bar{a}^{\dagger}|s\rangle_q=0$. Notice that the indicial phase factor $e^{i(s+1)\theta_0}$ is absent in Eqs. (A4) and (A5) unlike in Eq. (4.6), because the $\sqrt{[N]}$ operator in Eq. (A4) removes the $|s\rangle_{q\,q}\langle 0|$ projector in Eq. (4.6).

We obtain the q-deformed number operators in $\psi_{\rm PB}$:

$$\bar{a}\,\bar{a}^{\dagger}=[N+1]-[s+1]|s\rangle_{q\,q}\langle s| \ , \tag{A6a}$$

$$\bar{a}^{\dagger}\bar{a}=[N] \ , \tag{A6b}$$

and the q-analog commutation relations

$$\bar{a}\,\bar{a}^{\dagger}-q^{1/2}\bar{a}^{\dagger}\bar{a}=q^{-N/2}-[s+1]|s\rangle_{q\,q}\langle s| \ , \tag{A7a}$$

$$\bar{a}\,\bar{a}^{\dagger}-q^{-1/2}\bar{a}^{\dagger}\bar{a}=q^{N/2}-[s+1]|s\rangle_{q\,q}\langle s| \ . \tag{A7b}$$

Thus, in comparison with the expressions in Sec. II, we find

$$[\bar{a},\bar{a}^{\dagger}]_-=[N+1]-[N]-[s+1]|s\rangle_{q\,q}\langle s| \ , \tag{A8a}$$

$$\{\bar{a},\bar{a}^{\dagger}\}_+=[N+1]+[N]-[s+1]|s\rangle_{q\,q}\langle s| \ . \tag{A8b}$$

Notice that the extra $[s+1]$ term in Eq. (A8a) again serves to render the trace zero for the commutator in $\psi_{\rm PB}^{(q)}$. From Eq. (A6b) the extra term does not change the $[N]$ and N operators. In both the commutator and anticommutator it affects only the $|s\rangle_q$ level. Its effect on the quadratic operator identity Eq. (2.26) is obtained by moving the $[s+1]$ terms to the left-hand side of Eqs. (A8) and substituting for $[N+1]\pm[N]$ in Eq. (2.26).

To study the consequences of the extra terms in Eqs. (A6)–(A8) for a physical state (A1), we consider the operator

$$\hat{O} \equiv g([N]) - [s+1]|s\rangle_{q\,q}\langle s| \ . \tag{A9}$$

Then for a physical state, where I is a non-negative integer,

$$\begin{aligned}\langle p_{\rm PB}|(\hat{O})^I|p_{\rm PB}\rangle &= \langle p_{\rm PB}|\{g([N])\}^I|p_{\rm PB}\rangle \\ &\quad + |c_s|^2 \sum_{r=1}^{I} (-1)^r \begin{bmatrix} I \\ r \end{bmatrix} \{g([s])\}^{I-r}[s+1]^r \ .\end{aligned} \tag{A10}$$

Therefore, for $g([N])$, which has nonsingular expansions in an $[N]$ power series, such as for Eqs. (A6)–(A8), the extra $[s+1]|s\rangle_{q\,q}\langle s|$ term has no effect when finite powers of these operators act on any physical state. So for such operators these extra $[s+1]$ terms have no physically observable consequences; this is the same situation as in the $q=1$ limit [9]. For $|z\rangle_q$ CS's, we find

$${}_q\langle z|\{\hat{O}\}^I|z\rangle_q = {}_q\langle z|\{g([N])\}^I|z\rangle_q + e_q(|z|^2)^{-1}\frac{|z|^{2s}}{[s]!}\left\{\sum_{r=1}^{I}(-1)^r \begin{bmatrix} I \\ r \end{bmatrix}\{g([s])\}^{I-r}[s+1]^r\right\} . \tag{A11}$$

(ii) Similar to in the $q=1$ case [9], the phase-number commutator for the physical state reduces to

$$[\hat{\phi}_q, \hat{N}]_{\rm phy} = -i\{1-(s+1)|\theta_0\rangle_{q\,q}\langle\theta_0|\} \ . \tag{A12}$$

Consequently, for physical states the uncertainty relation is

$$\Delta N \Delta\hat{\phi}_q \geq \tfrac{1}{2}\{1-\bar{P}_q^{(p)}(\theta_0)\} \ , \tag{A13}$$

with

$$\begin{aligned}\bar{P}_q^{(p)}(\theta_0) &\equiv (s+1)\langle p_{\rm PB}|\theta_0\rangle_{q\,q}\langle\theta_0|p_{\rm PB}\rangle \\ &= 2\pi\,\mathrm{Prob}(\theta_0)\end{aligned} \tag{A14}$$

where $\mathrm{Prob}(\phi)d\phi$ is the classical probability to find the phase of a particular oscillator in the range ϕ to $(\phi+d\phi)$. So, for a q-analog CS

$$\Delta N \Delta\hat{\phi}_q = \tfrac{1}{2}\{1-2\pi\delta_q(\theta-\theta_0)\} \ . \tag{A15}$$

(iii) The phase-difference operator is the same as in the $q=1$ case [9,22]:

$$\begin{aligned}(\hat{\phi}_{q1}-\hat{\phi}_{q2})|\theta_{m1}\rangle_q|\theta_{m2}\rangle_q \\ = (\theta_{m1}-\theta_{m2})|\theta_{m1}\rangle_q|\theta_{m2}\rangle_q \ ,\end{aligned} \tag{A16a}$$

where the subscripts 1 and 2 label the modes. The variance of this phase-difference operator $\hat{\phi}_{\rm PD}=\hat{\phi}_{q1}-\hat{\phi}_{q2}$ is

$$(\Delta\hat{\phi}_{\rm PD})^2 = (\Delta\hat{\phi}_{q1})^2 + (\Delta\hat{\phi}_{q2})^2 \tag{A16b}$$

for uncorrelated states, and if both modes are in q-coherent states $|z_{q1}\rangle|z_{q2}\rangle$, where $|z_{q1}|=|z_{q2}|$, then $(\Delta\hat{\phi}_{\rm PD})^2 = 2(\Delta\hat{\phi}_q)^2$, where $(\Delta\hat{\phi}_q)^2$ is given by Eq. (6.8). Similarly, the two-mode vacuum state is unchanged:

$$\begin{aligned}|0,0\rangle_q &= (s_1+1)^{-1/2}(s_2+1)^{-1/2} \\ &\quad \times \sum_{m1=0}^{s_1}\sum_{m2=0}^{s_2}|\theta_{m1}\rangle_q|\theta_{m2}\rangle_q\end{aligned} \tag{A17}$$

so the system is equally likely to be any of the $(s_1+1)(s_2+1)$ phase-difference eigenstates.

(iv) The N and $\hat{\phi}_q$ operators can be used to generate [9] unitary transformations: The ordinary N operator is a generator of increments in the phase

$$\begin{aligned}e^{iN\gamma}|\theta_m\rangle_q &= (s+1)^{-1/2}\sum_{n=0}^{s} e^{in(\theta_m+\gamma)}|n\rangle_q \\ &= |\theta_m+\gamma\rangle_q \ .\end{aligned} \tag{A18}$$

Similarly, $\hat{\phi}_q$ is a generator of increments in the number of q bosons. When λ_I is a positive integer,

$$\begin{aligned}e^{i\lambda_I\hat{\phi}_q}|n\rangle_q &= \{e^{i\hat{\phi}_q}\}^{\lambda_I}|n\rangle_q \\ &= \begin{cases}|(n-\lambda_I)\rangle_q, & n\geq\lambda_I \\ e^{i(s+1)\theta_0}|(s+1-\lambda_I)\rangle_q, & n<\lambda_I \ .\end{cases}\end{aligned} \tag{A19a}$$

More generally,

$$\exp(i\lambda\phi_q) = \exp\left[i\lambda\sum_{n=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|\right] ,$$

so, for λ not an integer,

$$\exp(-i\lambda\hat{\phi}_q)|n\rangle_q = |n+\lambda\rangle_q \ , \tag{A19b}$$

where $|n+\lambda\rangle_q$ is not an eigenstate of N, but it is one of a complete set of $(s+1)$ states that can be used to span $\psi_{\rm PB}^{(q)}$.

With ρ the density matrix for the q-boson quantum field, the phase-moment generating function is

$$X_\theta(\lambda) \equiv \mathrm{Tr}\{\rho e^{i\lambda\hat{\phi}_q}\} \tag{A20}$$

and the ensemble average of the kth power is

$$\begin{aligned}\{(\hat{\phi}_q)^k\} &\equiv \mathrm{Tr}\{\rho(\hat{\phi}_q)^k\} \\ &= \left[-i\frac{\delta}{\delta\lambda}\right]^k X_\theta(\lambda)|_{\lambda=0} \ .\end{aligned} \tag{A21}$$

The Fourier transform is the physical states' phase probability density distribution

$$\bar{P}_q^{(p)}(\theta_m)=\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\theta_m} X_\theta(\lambda) \\ =2\pi \mathrm{Tr}\{\rho\delta(\hat{\phi}_q-\theta_m)\} \ . \tag{A22}$$

Similarly, the number-moment generating function is

$$X_n(\kappa)\equiv \mathrm{Tr}\{\rho \exp(i\kappa\hat{N})\} \ . \tag{A23}$$

For example, for the $|z\rangle_q$ CS's

$$C_q(\kappa,|z|)\equiv \sum_{n=0}^{\infty} e^{i\kappa n}|_q\langle n|z\rangle_q|^2 \\ =e_q(|z|^2)^{-1}e_q(e^{i\kappa}|z|^2) \ , \tag{A24a}$$

with

$$_q\langle z|N^k|z\rangle_q=\left[-i\frac{\delta}{\delta\kappa}\right]^k C_q(\kappa,|z|) \ . \tag{A24b}$$

APPENDIX B: PHASE OPERATORS VIA $|z\rangle_q$ CS EXPANSIONS

Following the approach of Paul [18] in the $q=1$ case, the resolution of unity can be used to construct operators analogous to the classical $e^{\pm ik\phi}$: We define (neglecting the contribution, see Appendix C, from auxiliary q bosons)

$$\mathcal{E}_k\equiv\int \frac{z^k}{|z|^k}|z\rangle_{q\,q}\langle z|d\mu(z) \ , \tag{B1}$$

where the measure

$$d\mu(z)=\frac{1}{2\pi}e_q(|z|^2)e_q(-|z|^2)d_q|z|^2d\theta \ . \tag{B2}$$

Thus, with $\mathcal{E}_{-k}\equiv(\mathcal{E}_k)^\dagger$, the analogs to $\cos k\phi$ and $\sin k\phi$ are $C_k\equiv\frac{1}{2}(\mathcal{E}_k+\mathcal{E}_{-k})$ and $S_k\equiv -i\frac{1}{2}(\mathcal{E}_k-\mathcal{E}_{-k})$. In the $q=1$ case, the recent work of Freyberger and Schleich [18] suggests that the approach of Paul may be able to describe the pioneering Noh-Fougeres-Mandel experiments [10].

APPENDIX C: q-ANALOG P, Q, AND W PHASE-SPACE REPRESENTATIONS [23]

The q-analog $|z\rangle_q$ CS's satisfy a resolution of unity [24,25]

$$\int |z\rangle_{q\,q}\langle z|d\mu(z)+\int|\tilde{z}\rangle_{q\,q}\langle\tilde{z}|d\tilde{\mu}=1 \ , \tag{C1}$$

where $d\mu(z)$ is given by Eq. (B2). The second term consists of a set of q-discrete auxiliary states $|\tilde{z}\rangle_q$ which are eigenstates of an auxiliary q boson $\tilde{a}_k$ annihilation operator (see Refs. [3] and [6])

$$\tilde{a}_k|\tilde{z}_k\rangle=(q^{1/4}\tilde{z}_k)|\tilde{z}_k\rangle \ ,$$

$$|\tilde{z}_k\rangle\equiv M_k\sum_{j=0}^{\infty}\frac{(q^{1/4}\tilde{z}_k)^j}{\sqrt{[j]!}}|j+k\rangle \ , \tag{C2}$$

where

$$|\tilde{z}_k|^2\equiv x_k=q^{k/2}\zeta_i \ ,$$

with $k=0,1,\ldots;$ $M_k=e_q(q^{1/2}x_k)^{-1/2}$. Here $-\zeta_i$ is a zero of $e_q(z)$. The associated discrete measure is

$$d\tilde{\mu}_k=\frac{1}{2\pi M_k^2}e_q(-|\tilde{z}_k|^2)d\theta \ . \tag{C3}$$

For simplicity, in this appendix we omit the explicit contribution from these auxiliary q bosons. Then [3] for the density operator

$$\hat{\rho}=\int d\mu(z)P_N(z,z^*)|z\rangle_{q\,q}\langle z| \tag{C4}$$

there is a P representation for normally ordered operators (subscript-N):

$$\langle (a^\dagger)^r a^s\rangle=\mathrm{Tr}\{\hat{\rho}(a^\dagger)^r a^s\} \\ =\int d\mu(z)(z^*)^r z^s P_N(z,z^*) \ . \tag{C5}$$

There is also a Q representation for antinormally ordered operators (subscript-A):

$$\langle a^r(a^\dagger)^s\rangle=\mathrm{Tr}\{\hat{\rho}a^r(a^\dagger)^s\} \\ =\int d\mu(z)z^r(z^*)^s Q_A(z,z^*) \ , \tag{C6}$$

where the phase-space distributions P_N and Q_A are related by

$$Q_A(z,z^*)\equiv\langle z|\hat{\rho}|z\rangle \\ =\int d\mu(y)P_N(y,y^*)N(y)^2N(z)^2 \\ \times e_q(yz^*)e_q(zy^*) \ . \tag{C7}$$

As opposed to the usual $q=1$ norm, we have absorbed a factor of π into Q_A in the $q\to 1$ limit. Note that

$$0\le|_q\langle y|z\rangle_q|^2=N(y)^2N(z)^2e_q(yz^*)e_q(zy^*)$$

is the q analog of the usual Gaussian $\exp(-|y-z|^2)$ which occurs for $q=1$. That is, by (C7) the expectation value $\langle z|\hat{\rho}|z\rangle$ is the q convolution of $P_N(y,y^*)$ with the q Gaussian $|\langle y|z\rangle|^2$. The norm $\mathrm{Tr}\{\hat{\rho}\}=1$ gives

$$\int d\mu(z)P_N(z,z^*)=1 \ , \tag{C8a}$$

$$\int d\mu(z)Q_A(z,z^*)=1 \ . \tag{C8b}$$

From $\hat{\rho}^\dagger=\hat{\rho}$, P_N and Q_A are real functions of the complex variable z. The Wigner phase distribution can also be defined by

$$\langle\{a^r(a^\dagger)^s\}_{\mathrm{sym}}\rangle\equiv\mathrm{Tr}\{\hat{\rho}\{a^r(a^\dagger)^s\}_{\mathrm{sym}}\} \\ =\int d\mu(z)z^r(z^*)^sW_S(z,z^*) \ . \tag{C9}$$

It is important to note that reordering of a's and $a^\dagger$'s in a general field-theoretic matrix element will introduce q- and q^N-dependent factors as a consequence of Eq. (1.1). This is an additional source of q dependence besides that exhibited by the Q, P, and W phase-space representations in this appendix, and that induced by the mapping $|n\rangle_q\to|z\rangle_q$.

So, for example, for a q-analog CS $|z_0\rangle$,

$$\hat{\rho}|_{\rm CS} \equiv |z_0\rangle\langle z_0| \ ,$$

$$Q_A(z,z^*)|_{\rm CS} = N^2(z)N^2(z_0)e_q(z^*z_0)e_q(zz_0^*) \ , \tag{C10}$$

$$P_N(z,z^*)_{\rm CS} = \frac{1}{|z|}\delta(\theta-\theta_0)\delta(|z|-|z_0|) \ ,$$

where $\delta(|z|-|z_0|)$ is a q-integration delta functional.

For a number state $|n_0\rangle$,

$$\hat{\rho}|_{\rm no.} \equiv |n_0\rangle\langle n_0| \ ,$$
$$Q_A(z,z^*)|_{\rm no.} = \frac{|z|^{2n_0}}{[n_0]!e_q(|z|^2)} \ . \tag{C11}$$

For a thermal state,

$$\hat{\rho}_T \equiv (1/Z)\sum_n |n\rangle\langle n|e^{-n(\hbar\omega/kT)} \ ,$$
$$Q_A(z,z^*)|_T = (1/Z)(|z|^2)e_q(|z|^2)e^{-n(\hbar\omega/kT)} \ . \tag{C12}$$

Since for an arbitrary operator $\hat{O}$,

$$\hat{O} = \int\int |z\rangle_q\, {}_q\langle z|\hat{O}|y\rangle_q\, {}_q\langle y|d\mu(z)d\mu(y) \ , \tag{C13}$$

the density-matrix operator can be written

$$\hat{\rho} = \int\int |z\rangle_q\, {}_q\langle z|\hat{\rho}|y\rangle_q\, {}_q\langle y|d\mu(z)d\mu(y) \ . \tag{C14}$$

The weight function ${}_q\langle z|\hat{\rho}|y\rangle_q$ is simply related to its Fock analog $\rho(n,m)$ for the $|n\rangle_q$ basis by

$${}_q\langle z|\hat{\rho}|y\rangle_q = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\frac{z^{*n}w^m}{\sqrt{[n]![m]!}}N(z)N(y) \ , \tag{C15}$$

where, as usual, in the $|n\rangle_q$ basis

$$\hat{\rho} = \sum_{n,m}\rho(n,m)|n\rangle_q\, {}_q\langle m| \ ,$$

$$\rho(n,m) = {}_q\langle n|\hat{\rho}|m\rangle_q \ ,$$
$$\sum_n \rho(n,n) = 1 \ .$$

Note that for $\hat{\rho}\to\hat{1}$, there is the useful identity

$$\int\int N(z)N(y)e_q(z^*y)|z\rangle_q\, {}_q\langle y|d\mu(z)d\mu(y) = \hat{1} \ . \tag{C16}$$

Alternatively, these two weight functions can be obtained from $P_N(z,z^*)$:

$$\langle z|\hat{\rho}|y\rangle = N(z)N(y) \times \int d\mu(w)P_N(w,w^*)N(w)^2 e_q(z^*w)e_q(yw^*) \ , \tag{C17a}$$

$$\rho(n,m) = \int d\mu(z)P_N(z,z^*)\frac{z^n z^{*m}}{\sqrt{[n]![m]!}}N(z)^2 \ . \tag{C17b}$$

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certainty relations for optical fields agree with theoretical predictions based on the PB operator; see D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, Phys. Rev. A **48**, 3159 (1993). This experiment uses optical homodyne tomography to measure the density matrix of a state of the field, so the results cannot be used to distinguish between different (mathematically consistent) phase operators. Experimental phase measurements involving a low ($\lesssim 10$) average number of detected photons, in some detectors, show that an operational approach to phase operators based on classical optics gives good agreement; see J. W. Noh, A. Fougeres, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); **46**, 2840 (1992). See also M. Freyberger, K. Vogel, and W. Schleich, Quantum Opt. **5**, 2 (1993); Phys. Lett. **176A**, 41 (1993); R. Lynch, Phys. Rev. A **47**, 1576 (1993).

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$$[n]!=\exp\{-\tfrac{1}{4}n(n-1)\ln|q|-n\ln|1-q|\}$$
$$\times\{1-q-q^2+O(q^4)\}$$

for $O<q<1$, $n\geq 3$.

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$$\frac{\omega}{\hbar}(\Delta\hat{Q})^2=\frac{1}{\hbar\omega}(\Delta\hat{P})^2=\lambda(z)\ ,$$

though z dependent and more uncertain than when $q=1$. Thus

$$(\Delta\hat{P})/\langle\hat{P}\rangle=\lambda^{1/2}/(2|z|\sin\theta)$$

and

$$(\Delta\hat{Q})/\langle\hat{Q}\rangle=\lambda^{1/2}/(2|z|\cos\theta)\ .$$

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[16] In contrast to the case of thermal averages for the q-boson free gas, the ratio of CS expectation values

$$\frac{\langle[N+1]\rangle}{\langle[N]\rangle}=q^{1/2}+\frac{\epsilon_1(z)}{|z|^2}$$

is q dependent.

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