Propagation of mutual coherence in refractive x-ray lasers using a WKB method

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We derive an expression for the mutual coherence function for a lasing medium with spatially varying refraction and gain using a ray-tracing (WKB) formalism. The expression is valid in the limit of large refractive Fresnel number, which is equal to the geometric Fresnel number multiplied by the ratio of laser length to refraction scale length. For quasihomogeneous sources, the coherence function has the form of a Fourier transform of a source function which is modulated by a Gaussian whose width varies inversely with the square root of the gain. This result may be viewed as a generalization of the van Cittert–Zernike theorem to include both refraction and gain. For the case of refractive defocusing, we find the coherence length scales as the square root of the product of gain and the refraction scale length, inversely with the refractive Fresnel number, and exponentially with laser length. For a parabolic profile, this result is exact. We also treat hyperbolic secant refractive and gain profiles. Comparison of our results with numerical computations shows good agreement. We also indicate how to generalize the method to nonideal profiles obtained from measurements or hydrodynamic simulations.

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INTRODUCTION

Coherence is a central issue in the design of current x-ray lasers. It is well known that refractive defocusing by plasma electrons leads to improved coherence compared with that expected from geometrical propagation [1–5]. It is also well established that gain guiding causes mode discrimination which also leads to enhanced coherence over the constant gain case [1–3]. These results were determined by a variety of means, including modal analysis [1–3], numerical solution of the paraxial wave equation [4], ray optics estimates [3,5], and Wentzel-Kramers-Brillouin (WKB) approximations [6,7].

The WKB method is attractive because it allows inclusion of phase information in a ray optics calculation, and ray optics is virtually always an excellent approximation for x-ray lasers. This approach has been taken by Hazak and Bar-Shalom [6], who have used a ray-tracing approach to solve the Maxwell-Bloch equations describing gain buildup and propagation of radiation in x-ray lasers. In a similar spirit, Zahavi, Hazak, and Zinamon [7] recently presented a simple generalization of the van Cittert–Zernike theorem to include refraction (but not gain), in which the exponential scaling of the coherence length with propagation distance is manifested in the exponential divergence of nearby ray orbits.

The thrust of this paper is somewhat different: we develop a WKB methodology using an asymptotic parameter which reflects the highly refractive nature of current laboratory x-ray lasers. The most important rays contributing to the WKB coherence function are those with largest gain length, which allows us to evaluate the coherence function approximately by expanding the ray trajectories about the maximum gain-length ray. In this manner we are able to obtain an analytic expression for the coherence function in the presence of both refraction and spatially varying gain. This expression shows explicitly how a nonconstant gain profile leads to enhanced coherence due to an effective narrowing of the source. The mutual coherence function takes the form of a Fourier transform of a Gaussian modulated source function, a result which may be viewed as a generalization of the van Cittert–Zernike theorem to include both refraction and gain. Going beyond this expansion technique, the WKB methodology may also be applied directly to a system for which the ray trajectories are known either analytically or numerically.

This paper is organized as follows. In Sec. I we derive a formula for the mutual intensity in the WKB approximation, valid for large refractive Fresnel number (to be defined below). While we do not give a detailed derivation of the WKB propagator, which can be found in the literature, we review the basic steps of the derivation and its geometrical interpretation. In Sec. II we consider the special case of gain proportional to electron density. We then show how to evaluate the coherence function approximately for quasihomogeneous profiles by expanding the ray trajectories about the maximum gain-length ray and derive the Fourier transform relation noted above. In Sec. III we apply the method to the case of parabolic transverse refraction and gain and derive a scaling law for the coherence length. We then treat approximately the more realistic $\operatorname{sech}^2(x)$ density and gain profiles and compare the results to the parabolic profile. This is followed by a discussion of the implementation of the methodology to more general density and gain profiles which might be obtained from either experiment or hydrodynamic and atomic physics simulations.

I. BASIC FORMULATION

Our starting point is the mutual intensity $\Gamma(x,x';z)$, which is the two-point spatial correlation function of the electric field:

$$\Gamma(x,x';z)=\langle E^*(x,z)E(x',z)\rangle \ , \tag{1.1}$$

where the angular brackets denote an ensemble average and the scalar quantity E refers to any polarization component of the field. The propagation direction is z and x is a transverse coordinate. If we express the electric field in the form $E(x,z)=\exp(ikz-\omega t)\tilde{E}(x,z)$ and make the paraxial assumption $\partial^2\tilde{E}/\partial z^2 \ll k(\partial\tilde{E}/\partial z)$, then $\tilde{E}(x,z)$ satisfies the steady-state paraxial wave equation [8]

$$\left[2i\frac{\partial}{\partial z}+\frac{1}{k}\frac{\partial^2}{\partial x^2}-h(x)+ig(x)\right]\tilde{E}(x,z)=-4\pi k P_{\rm sp}(x,z) \ , \tag{1.2}$$

where $k=\omega/c$, ω is the angular frequency of the laser light, $h(x)=\omega_p^2(x)/kc^2$, $\omega_p(x)$ is the electron plasma frequency, $g(x)$ is the atomic gain of the plasma, and $P_{\rm sp}(x)$ is the spontaneous atomic polarization, or source. We will also be interested in the complex degree of coherence

$$\gamma(x,x';z)=\frac{\Gamma(x,x';z)}{\sqrt{I(x;z)I(x';z)}} \ , \tag{1.3}$$

where $I(x;z)=\Gamma(x,x;z)$ is the intensity, and its magnitude or degree of coherence $\mu(x,x';z)=|\gamma(x,x';z)|$.

The solution for $\tilde{E}(x,z)$ can be expressed in terms of the Green's function $K(x,z;x_0,z_0)$ for the paraxial wave equation as

$$\tilde{E}(x,z)=\int\int_{\rm source} dx_0 dz_0 K(x,z;x_0,z_0)P(x_0,z_0) \ , \tag{1.4}$$

where $K(x,z;x_0,z_0)$ is a solution of Eq. (1.2) without the source term, with the initial condition

$$K(x,z_0;x_0,z_0)=\delta(x-x_0) \ . \tag{1.5}$$

The spontaneous polarization is assumed to be δ function correlated in space[1] and proportional to the source intensity at each point,

$$\langle P(x_0,z_0)^* P(x_0',z_0')\rangle = \text{const}\times I(x_0,z_0)\delta(x_0-x_0')\delta(z_0-z_0') \ . \tag{1.6}$$

If we substitute Eq. (1.4) into Eq. (1.1) and use (1.5), we obtain for the mutual intensity[2]

$$\Gamma(x,x';z)=\text{const}\times\int\int_{\rm source} dx_0 dz_0 I(x_0,z_0) \times K(x,z;x_0,z_0)^* \times K(x',z;x_0;z_0) \ . \tag{1.7}$$

To motivate the approximation we intend to use, it is convenient to recast Eq. (1.2) in dimensionless form. Suppose the peak gain and maximum density occur at $x=0$. We define a refraction angle $\theta_r=\omega_p(0)/ck$; a refraction length $L_r=a/\theta_r$, where a is the characteristic laser half-width; normalized coordinates $x\to x/a$ and $z\to z/L_r$; normalized refraction strength $\bar{h}(x)=h(x)/k\theta_r^2=\omega_p^2/\omega_p^2(0)$; normalized gain $\bar{g}(x)=g(x)/g(0)$; a *refractive Fresnel number* $F_r=ka^2/L_r$;[3] *a refractive gain length* $G_r=g(0)L_r$; and a refraction parameter $\eta=F_r/G_r$. Equation (1.2) with $P_{\rm sp}=0$ may now be written as

$$\frac{i}{F_r}\frac{\partial E}{\partial z}=\frac{1}{2}\left[\frac{1}{F_r^2}\frac{\partial^2}{\partial x^2}-\bar{h}(x)+i\eta^{-1}\bar{g}(x)\right]E(x,z) \ . \tag{1.8}$$

This form allows us to identify the important parameter F_r for this problem, which may be written numerically as

$$F_r=5.96\times 10^2\left[\frac{n_e}{10^{21}}\right]^{1/2}\left[\frac{a}{100\ \mu\text{m}}\right] \ , \tag{1.9}$$

where n_e is the electron density. This expression shows that F_r is large (≈ 600) for the typical x-ray laser parameters $a=100\ \mu$m and $n_e=10^{21}$ cm^{-3}.

The physical meaning of $F_r \gg 1$ is just that the fractional variation of the transverse wave number k_x is small over a transverse wavelength scale $1/k_x$ or

$$(1/k_x^2)\partial k_x/\partial x \ll 1 \ . \tag{1.10}$$

To see this, we use

$$k_x \sim \theta_r k=\frac{ka}{L_r} \tag{1.11}$$

and

$$\frac{\partial k_x}{\partial x}\sim\frac{k}{L_r} \ , \tag{1.12}$$

so we then have

$$\frac{1}{k_x^2}\frac{\partial k_x}{\partial x}=\frac{L_r}{ka^2}=\frac{1}{F_r} \ , \tag{1.13}$$

which justifies our assertion. For the case of weak refraction $L/L_r<1$, we would replace the refraction angle θ_r by the geometric angle $\theta_0\equiv a/L$ and obtain a condition on the conventional Fresnel number. Naturally this is a heuristic argument which will break down if the field suffers wavelength scale amplitude variations, as would occur at a focal (caustic) surface.

Our method involves a WKB expansion in $1/F_r$ rather than in the conventional Fresnel number ($F=ka^2/L$ $=1.05\times 10^2[(\lambda/200\ \text{Å})(L/3\ \text{cm})]^{-1}[a/100\ \mu\text{m}]^2$, for a laser of length L). This approach is advantageous for several reasons. First of all, the latter is usually smaller, i.e., $L/L_r>1$. Second, it will become apparent that F_r

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[1]The assumption of δ correlation is only an idealization; in fact, the correlation length can be no less than a wavelength and paraxiality requires that it be many wavelengths.

[2]We must exercise caution in evaluating Eq. (1.6) in the limit of small z, due to the singular source correlation (1.5). If the intensity is taken simply as $I(x,z)=\Gamma(x,z;z)$, then the source intensity is singular. It is physically more plausible to define $I(x,z)=\lim_{\varepsilon\to 0}\int_{-\varepsilon/2}^{\varepsilon/2}\Gamma(x,x+\xi;z)d\xi$, which gives the correct source intensity.

[3]We note here that, because $\phi_r=\omega_p(0)/ck$, F_r is actually independent of k: $F_r=\omega_p(0)a/c$.

figures prominently in solutions of (1.8) and therefore is a useful parameter for refractive x-ray lasers. Third, and most fundamentally, because the plasma refractive index depends on the wavelength, an attempt to formally solve Eq. (1.8) based on an expansion in $1/F$ will spoil the usual ordering. For example, if we hold all laser parameters constant except for the wavelength and seek a large F solution, the refraction term will be $O(1/k^2)=O(1/F^2)$, whereas in Eq. (1.8) it is $O(1)$; we would conclude that the WKB Green's function, which is a first-order expansion, is just that for free propagation. [In either case the gain term is $O(1/F)$ or $O(1/F_r)$.] While the coefficient of the term proportional to $1/F^2$ is also typically large (as pointed out in Ref. [6]), it is more natural, in a strictly formal sense, to obtain an asymptotic solution involving the single parameter F_r. This simple picture breaks down, however, if the geometric angle θ_0 exceeds the refraction angle θ_r (i.e., $L/L_r<1$), in which case, as noted above, the appropriate Fresnel number is the conventional one. These considerations are most relevant for field calculations within and on the output face of the laser, for which the geometric propagation outside the plasma is not important. Indeed, for propagation within the laser plasma, the validity of the large F_r approximation is independent of the radiation wavelength, provided paraxiality remains valid. [Note that all the wavelength dependence in Eq. (1.8) is through the propagation coordinate z.]

Having now confirmed the relevance of the refractive Fresnel number, we are motivated to seek a solution of Eq. (1.8) for $F_r \gg 1$; this is the WKB solution. We will now draw on standard results from nonstationary (z-dependent) WKB theory, which we summarize here in the context of x-ray lasers for completeness. The WKB Green's function $K(x_0,x;z-z_0)$ for a z-invariant medium is just the ray optics scalar field at x created by a point source at x_0 in the paraxial approximation and has the form [9–12]

$$K^{\mathrm{WKB}}(x_0,x,z-z_0)=\left[\frac{\partial^2 S(x_0,x;z-z_0)}{\partial x\,\partial x_0}\right]^{1/2} \times\exp\left\{iF_r S(x_0,x;z-z_0)-i\frac{\pi}{2}\mu\right\} \times\exp\{\tfrac{1}{2}G(x_0,x;z-z_0)\} . \quad (1.14)$$

The term $F_r S(x_0,x;z)$ is, to within a constant, the optical path length along a ray connecting the points (x_0,z_0) and (x,z),

$$S=\int_{\mathrm{ray}} n(s)ds =\int_{\mathrm{ray}} L[x(z),\dot{x}(z)]dz , \quad (1.15)$$

where we have identified the optical Lagrangian

$$L(x,\dot{x})=n(x)\sqrt{1+\dot{x}^2} =n(x)(1+\tfrac{1}{2}\dot{x}^2)+O(\dot{x}^4) , \quad (1.16)$$

with $\dot{x}=dx/dz$, and the refractive index n is defined by

$$n^2(x)=1+\omega_p^2(x)/c^2k^2 =1+\theta_r^2\bar{h}(x) . \quad (1.17)$$

In the paraxial approximation we thus have

$$L(x,\dot{x})=1+\tfrac{1}{2}(\dot{x}^2+\theta_r^2\bar{h}(x)] . \quad (1.18)$$

The ray path of integration is understood to satisfy $x(0)=x_0$ and $x(z)=x$. The term $G(x_0,x;z) =\int_{\mathrm{ray}} g[x(z')]dz'$ is the gain length. As we will show shortly, the prefactor $R=(\partial^2 S/\partial x\,\partial x_0)^{1/2}$ in Eq. (1.14) is required to conserve rays. Finally, the *Maslov index* μ in Eq. (1.14) represents the phase shift due to focal points or caustics along a family of the rays, i.e., where the prefactor diverges.

It will be convenient in the following to use the Hamiltonian form of the ray equations. We therefore define, in the usual way, the momentum (equivalent to angle of propagation) $\theta=\partial L/\partial\dot{x}$ and the optical Hamiltonian

$$H(x,\theta)=\theta\dot{x}-L =\tfrac{1}{2}\{\theta^2-\theta_r^2[\bar{h}(x)+i\eta^{-1}\bar{g}(x)]\} , \quad (1.19)$$

where we have dropped the constant L which plays no role in subsequent calculations. The ray trajectories are then generated by Hamilton's equations

$$\frac{dx}{dz}=\frac{\partial H}{\partial\theta} , \quad \frac{d\theta}{dz}=-\frac{\partial H}{\partial x} . \quad (1.20)$$

While we do not prove it here, the function $S(x_0,x;z)$ in Eq. (1.14) is then equal to Hamilton's principal function [13], which has the important property of being the generating function for the dynamical canonical transformation taking x_0 into x in the distance z, and therefore satisfies the relations [13]

$$\frac{\partial S(x_0,x;z)}{\partial x}=\theta(x_0,x;z) , \quad (1.21\mathrm{a})$$

$$\frac{\partial S(x_0,x;z)}{\partial x_0}=-\theta_0(x_0,x;z) . \quad (1.21\mathrm{b})$$

In light of Eqs. (1.21), we now see that the prefactor $R=(\partial^2 S/\partial x\,\partial x_0)^{1/2}=i(\partial\theta_0/\partial x)^{1/2}$ in Eq. (1.14) manifests ray conservation: the rays penetrating a small segment dx at x originated from an angular range $d\theta_0$ at the source.

The angle function $\theta=\theta(x_0,x;z)$ on the right-hand side of Eq. (1.21a) may be viewed as the surface in (x,z) space specifying the angle for the family of rays with fixed initial coordinate x_0 but unrestricted initial angle θ_0 (point-source initial condition). On this surface ("Lagrangian manifold") the function S is in fact independent of path; one need not integrate along a ray at all [12]. Hamilton's principal function S can now be written in terms of the angle function and the Hamiltonian using a path of integration particularly useful for later work:

$$S(x_0,x;z)=S_0(x_0)+\int_{x_0}^{x}\theta(x_0,q;z)dq -\int_0^{z}H[x_0,\theta(x_0,x_0;z')]dz' , \quad (1.22)$$

where we have integrated first from $z'=0$ to holding x fixed at $x=x_0$ and then from $x'=x_0$ to x keeping z constant. Note that for this path, the term involving H is independent of x.

The WKB coherence function now follows from Eqs. (1.7) and (1.14):

$$\Gamma(x,x';z)=\int\int I(x_0)\left[\frac{\partial\theta_0}{\partial x}(x_0,x;z')\frac{\partial\theta_0}{\partial x}(x_0,x';z)\right]^{1/2} \times\exp\{iF_r\delta S(x_0,x,x';z') +\tfrac{1}{2}[G(x_0,x;z')+G(x_0,x';z')]\} \times dx_0dz' , \quad (1.23)$$

where we have used (1.21), set $z'=z-z_0$, and have defined the phase difference δS using Eq. (1.22),

$$\delta S(x_0,x,x';z)=S(x_0,x;z)-S(x_0,x';z) =\int_x^{x'}\theta(x_0,q;z)dq , \quad (1.24)$$

and $G(x_0,x;z)=\int_{\rm ray} g[x(z')]dz'$ is the gain length.[4]

While we have noted that the optical path length S is a path-independent quantity on the surface $[x,\theta(x_0,x;z),z]$, the gain length G is not. However, for the important special case where the gain is proportional to electron density, there is a direct relation between the gain along a ray and the optical path length S. Take $\bar{g}=\bar{h}$ in Eq. (1.8); we can then show that the gain along a ray is given by the formula

$$G(x_0,x;r)=G_r\left[z+\frac{\partial}{\partial z}[zS(x_0,x;z)]\right] =G_r[z+\hat{T}S(x_0,x;z)] , \quad (1.25)$$

where $\hat{T}\equiv 1+z\partial/\partial z$ (see the Appendix for derivation). This formula is useful because it gives a simple differential relation between the functional quantities S and G; it shows that S serves as a "potential" function for the gain. In practice, however, it may be no easier to evaluate the gain length from Eq. (1.25) than by computing it directly along a ray, but this formula will be useful for the analytical development of the following sections.

II. EXPANSION ABOUT THE MAXIMUM GAIN-LENGTH RAY

For this section we will assume gain is proportional to refractive index, so that $\bar{g}=\bar{h}$. Equation (1.24) can be evaluated approximately if we assume the rays giving a significant contribution arise from a small range of initial conditions (x_0,θ_0); small in the sense that the ray equations can be linearized about some reference ray. This in turn will be the case if the coherence scale length x_c is much smaller than the transverse intensity scale length. This condition is a good approximation for nearly all cases considered in this paper. Stated mathematically, we assume that the mutual intensity has the quasihomogeneous form [14]

$$\Gamma(x,x')=I(\bar{x})f(\Delta x) , \quad (2.1)$$

where $\bar{x}=(x+x')/2$ and $\Delta x=(x-x')/2$.[5] Here the function $I(\bar{x})$ is slowly varying with respect to the coherence length and $f(\Delta x)$ is rapidly varying with respect to the scale length for intensity variation. For a given source point x_0 we are interested in the rays in the neighborhood of the maximum gain-length ray ending at $\bar{x}$. This ray satisfies

$$x_{\rm max}(\zeta_0;z)=\bar{x} , \quad \theta_{\rm max}(\zeta_0;z)=\bar{\theta} , \quad (2.2)$$

where $\zeta_0=(x_0,\theta_0)$ gives the initial position and angle of the maximum gain-length ray and $\bar{\theta}$ is the exit angle of the maximum gain-length ray. We now linearize about this ray

$$\begin{bmatrix}\Delta x\\ \Delta\theta\end{bmatrix}=\begin{bmatrix}x-\bar{x}\\ \theta-\bar{\theta}\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\Delta x_0\\ \Delta\theta_0\end{bmatrix} , \quad (2.3)$$

where we have introduced the $ABCD$ matrix of Gaussian optics [8]

$$A(x_0,\bar{x};z)=\frac{\partial x_g}{\partial x_0} , \quad B(x_0,\bar{x};z)=\frac{\partial x_g}{\partial\theta_0} , \quad C(x_0,\bar{x};z)=\frac{\partial\theta_g}{\partial x_0} , \quad D(x_0,\bar{x};z)=\frac{\partial\theta_g}{\partial\theta_0} \quad (2.4)$$

and $(\Delta x_0,\Delta\theta_0)$ signifies departure from the maximum gain-length initial conditions. All partial derivatives in Eq. (2.5) are evaluated at the fixed initial point of the reference ray

$$\zeta_0(x_0,\bar{x};z)=(x_0,\theta_0(x_0,\bar{x};z)) , \quad (2.5)$$

where the initial angle θ_0 is considered to be a function of the initial and exit positions and distance z. The ray orbits (2.4) may be viewed as arising from the quadratic z-dependent Hamiltonian [15]

[4]Equation (1.23) is equivalent to a result obtained in Ref. [6] using a conventional WKB expansion (in $1/F$), coupled with estimates of the sizes of various terms in the wave equation. It is worthwhile pointing out that if the angle function is single valued or finitely multivalued, Eq. (1.23) is formally identical to an integral over *coherent* wavelets from point sources of intensity x_0. For the case of no refraction or gain, this is the essence of the van Cittert–Zernike theorem.

[5]Note that, for reasons which will become apparent, our definition of Δx is one-half the value used in Ref. [14].

$$H(\Delta x,\Delta\theta;z)=\frac{1}{2}\left[(\Delta\theta)^2+\frac{d^2}{dx^2}n^2[x(z)](\Delta x)^2\right] \tag{2.6}$$

evaluated along the reference ray $x_g(z)$. The optical path length in terms of the *ABCD* matrix elements is then [8,11]

$$S(\Delta x_0\Delta x;x_0,\bar{x},z)=S(x_0,\bar{x};z)+\frac{1}{2B}(A\Delta x_0^2-2\Delta x\Delta x_0+D\Delta x^2) \tag{2.7}$$

and the gain length, from (1.25), is

$$G(\Delta x_0,\Delta x;x_0,z)=G(x_0,\bar{x};z)+\frac{G_r}{2B^2}\{[AB+z(A_zB-AB_z)]\Delta x_0^2-2[B-zB_z]\Delta x\Delta x_0+[BD+z(BD_z-B_zD)]\Delta x^2\}\ , \tag{2.8}$$

where the z subscripts denote partial derivatives and

$$G(x_0,\bar{x};z)=G_r\left[z+\frac{\partial}{\partial z}(zS_0)\right]\ . \tag{2.9}$$

The source point for the maximum gain-length ray may be found by differentiating $G(x_0,\bar{x};z)$ with respect to x_0. For density profiles symmetric about and peaking at $x_0=0$, the maximum gain-length ray will be that with $\bar{x}_0=0$, where the density and gain are both flat. For this important special case we have for the coherence function

$$\Gamma(x,x';z)=\text{const}\times\frac{\exp[G(\bar{x}_0=0,\bar{x};z)]}{B}\exp\{-\tfrac{1}{8}G_rc(z)\Delta x^2+iF_r(D/2B)\bar{x}\Delta x\}$$
$$\times\int_{-\infty}^{\infty}d\Delta x_0 I_0(\Delta x_0)\exp\{-G_r[\tfrac{1}{4}a(z)\Delta x_0^2-\tfrac{1}{2}b(z)(x+x')\Delta x_0]+iF_r(x-x')\Delta x_0\}\ , \tag{2.10}$$

where

$$a(z)=\frac{2}{B^2}\{[AB+z(A_zB-AB_z)]\}\ , \tag{2.11a}$$

$$b(z)=4[B-zB_z]\ , \tag{2.11b}$$

$$c(z)=\frac{2}{B^2}\{[BD+z(BD_z-B_zD)]\}\ , \tag{2.11c}$$

and we have made the large gain-length approximation $\exp[G(x_0=0,\bar{x};z)]/B\gg1$, which allows us to replace a source distributed in z with a line source situated at $z=0$, viz.,

$$I(x,z)=I(x,0)\delta(z)$$
$$=I_0(x)\delta(z)\ . \tag{2.12}$$

(This approximation will be discussed further in Sec. III.) For parabolic profiles, which will be treated in Sec. III, Eq. (2.10) is an exact result. If we now use $\eta=F_r/G_r\gg1$, we obtain[6]

$$\mu(x,x';z)=\left|\frac{\int_{-\infty}^{\infty}d\Delta x_0I_0(\Delta x_0)\exp[-\frac{1}{4}G_r\bar{a}(z)\Delta x_0^2]\exp[iF_r(x-x')\Delta x_0/B]}{\int_{-\infty}^{\infty}d\Delta x_0I_0(\Delta x_0)\exp[-\frac{1}{4}G_r\bar{a}(z)\Delta x_0^2]}\right|\ . \tag{2.13}$$

The significance of this equation is that it has the form of a normalized Fourier transform of a "gain narrowed" source function $I(\Delta x_0)\exp[-\frac{1}{4}G_r\bar{a}(z)\Delta x_0^2]$, with transform variable $\kappa(z)=F_r(x-x')/B$. Note that (2.13) depends only on the A and B matrix elements, that is, on the first of Eq. (2.2). These derivatives may be easily evaluated numerically for a general profile or analytically if the ray trajectory is also known analytically.

III. PARABOLIC REFRACTION AND GAIN

As the simplest application of the results of the previous sections, we now consider the elementary but important example of parabolic density and gain. We take

$$n^2(x)=\begin{cases}1-\theta_r^2(1-i\eta^{-1})(1-x^2)\ , & |x|<1\\ 1\ , & |x|\geq1\ ,\end{cases} \tag{3.1}$$

which could represent a quadratic expansion of the re-

[6]In fact we assume here that $\eta(x-x')/(x+x')\gg1$, which is violated if (2.10) is used to evaluate the intensity $I(x,z)=\Gamma(x,x;z)$, $x\neq0$. In this case the term proportional to $b(z)$ must be retained.

fractive index about its minimum. Such a profile is appropriate for, e.g., an exploding-foil x-ray laser after burnthrough of the optical pump laser and has been adopted for this purpose in the literature [1]. As remarked above, it is well known that for an untruncated parabolic profile, the WKB Green's function is an exact solution to the wave equation (1.8), so the results of this section are expected to be very accurate for $x \ll 1$. The ray trajectories for $|x|,|x_0|<1$ are then just

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \end{bmatrix}, \tag{3.2}$$

where θ is in units of refraction angle θ_r; this well-known result may be derived from Eqs. (1.17), (1.19), (1.20), and (3.1). Thus in Eq. (2.4) we have $A=D=\cosh z$ and $B=C=\sinh z$. We ignore side rays (those which exit the plasma, $|x|$ or $|x_0|>1$) in this section, so our treatment is valid only for calculation of coherence lengths $l_c<1$ for which side rays make no contribution to the integral in (1.24). We can evaluate the degree of coherence from Eq. (2.13) directly, but it is instructive to obtain the same results directly from (1.23). We first obtain the function $S(x_0,x;z)$ from Eq. (1.15) or (2.7):

$$S(x_0,x;z)=\frac{1}{2}\frac{(x_0^2+x^2)\cosh z-2xx_0}{\sinh z} . \tag{3.3}$$

Using Eq. (1.25) for the gain length, we find

$$G(x_0,x;z)=G_r\{\tfrac{1}{4}(x^2+x_0^2)a(z)+\tfrac{1}{2}xx_0b(z)\} , \tag{3.4}$$

where

$$a(z)=\frac{\sinh(2z)-2z}{\sinh^2 z} \tag{3.5a}$$

and

$$b(z)=\frac{z\coth z-1}{\sinh z} \tag{3.5b}$$

are identical to the functions defined in Eqs. (2.11). We can then get the phase difference δS in Eq. (1.23) using $\delta S=S(x_0,x;z)-S(x_0,x';z)$:[7]

$$\delta S(x_0,x,x';z)=\frac{1}{2}\frac{(x^2-x'^2)\cosh z-2(x-x')x_0}{\sinh z} . \tag{3.6}$$

The prefactor R is easily obtained from Eq. (3.3): we have

$$\frac{\partial\theta_0}{\partial x}=\frac{\partial^2 S}{\partial x\partial x_0}=\frac{1}{\sinh z} , \tag{3.7}$$

which is equal to $1/B$, consistent with Eq. (2.10).

Substituting Eqs. (3.4), (3.6), and (3.7) in Eq. (1.23), the integral expression for the mutual intensity is found to be

$$\Gamma(x,x';z)=\text{const}\times\int_0^z dz'\frac{\exp(G_r z')}{i\sinh z'}\exp\left\{iF_r\frac{x^2-(x')^2}{2\tanh z'}-\tfrac{1}{8}G_r a(z')[x^2+(x')^2]\right\}$$
$$\times\int_{-\infty}^{\infty}dx_0 I(x_0,z')\exp\left\{-G_r[\tfrac{1}{4}a(z')x_0^2-\tfrac{1}{2}b(z')(x+x')x_0]+iF_r\frac{(x-x')x_0}{\sinh z'}\right\} . \tag{3.8}$$

The Maslov index does not enter this expression because the amplitude factor is monotonically decreasing with z: families of rays do not focus. Equation (3.8) is equivalent to Eq. (2.10) if we note, from (3.4), that $G(0,\bar{x};z)=G_r a(z)\bar{x}^2/4$ and $c(z)=a(z)$ for the parabolic profile.

Let us consider the degree of coherence with respect to distance from the optical axis, i.e., for $x'=0$. For now, we also approximate the distributed source by a line source is situated at $z=0$. This is justified if $\exp(G_r z)\gg 1$ since, from Eq. (3.8), the integral will then be dominated by values of z' near z. If we also have $\sinh z\gg 1$, this condition should be modified to $\exp[(G_r-1)z]=\exp[L(g_0-1/L_r)]\gg 1$ for a laser of length L. Thus we use (2.12), and from (3.8) we get

$$\Gamma(x,0;z)=\text{const}\times\frac{\exp(G_r z)}{i\sinh z}\exp\{[iF_r\coth z-\tfrac{1}{8}G_r a(z)]x^2\}\int_{-\infty}^{\infty}dx_0 I_0(x_0)\exp[-\tfrac{1}{4}G_r a(z)x_0^2-\kappa(z)xx_0] , \tag{3.9}$$

where $\kappa(z)=-[F_r\operatorname{csch} z+iG_r b(z)]x\approx -F_r x\operatorname{csch} z$, and the approximate equality occurs because $F_r/G_r=\eta\gg 1$. The degree of mutual coherence now takes the simple form

$$\mu(x,0;z)\approx\left|\frac{\int_{-\infty}^{\infty}dx_0 I_0(x_0)\exp[-\tfrac{1}{4}G_r a(z)x_0^2]\exp[i\kappa(z)x_0]}{\int_{-\infty}^{\infty}dx_0 I(x_0)\exp[-\tfrac{1}{4}G_r a(z)x_0^2]}\right| , \tag{3.10}$$

which is the normalized Fourier transform of a gain narrowed source of Eq. (2.13).

Let us now consider the evaluation of the coherence function for constant source intensity

[7] If we did not need S to calculate the gain length (as in the case of constant gain), we could evaluate the phase using Eq. (1.25) without computing the optical path length at all. We can solve Eq. (1.28) for $\theta(x_0,x;z)$, giving $\theta(x_0,x;z)=(x\cosh z-x_0)/\sinh z$, and do the integral in Eq. (1.24): $\delta S(x_0,x,x';z)=\int_x^{x'}\theta(x_0,q;z)dq=[(x^2-x'^2)\cosh z-2(x-x')x_0]/2\sinh z$, which is consistent with Eq. (3.3).

$$I_0(x)=\begin{cases}\bar{I}\ , & |x|\le 1\\ 0\ , & |x|>1\ .\end{cases} \quad (3.11)$$

The integral over x_0 in Eq. (3.9) can then be evaluated exactly in terms of error functions, but first we examine certain limiting cases. If $G_r a(z) \ll 1$, $\exp[-\frac{1}{4}G_r a(z)x_0^2]\approx 1$ and we find

$$\mu(x,0;z)=\left[\frac{2G_r\xi}{\sinh(2G_r\xi)}\frac{\sin^2(F_r\xi)+\sinh^2[G_r(z\coth z-1)\xi^2]}{\xi^2[F_r^2+G_r^2 z\coth z-1)^2]}\right]^{1/2} . \quad (3.12)$$

where $\xi=x/\sinh z$ is a scaled coordinate.

Note from Eq. (3.13) that for $G_r\to 0$, we obtain the coherence function for parabolic reaction and no gain [16], $\mu(x,0;z)=\mathrm{sinc}(F_r x\,\mathrm{csch} z)$, which gives a coherence length $x_c=\sinh z/F_r$. In the opposite limit $G_r a(z)\gg 1$, the integrand is effectively zero at $|x_0|>1$, so the source and hence limits of integration may be extended to $\pm\infty$ We then find

$$\mu(x,0;z)=\exp\left[-\frac{F_r^2+G_r^2(z\coth z-1)^2}{G_r[\sinh(2z)-2z]}x^2\right] . \quad (3.13)$$

This last form of the coherence function allows us to solve explicitly for the coherence length x_c as a function of the three parameters F_r, G_r, and z. We define the coherence length by $\mu(x_c,0;z)=\sin(1)\approx 0.84$ as in Ref. [2] and obtain, for $\eta\gg 1$,

$$x_c\approx 0.42\{G_r[\sinh(2z)-2z)\}^{1/2}/F_r\ . \quad (3.14)$$

This result is in agreement with the ray optics model of Refs. [3] and [5]. Note that for $\exp(z)\gg 1$, we have the scaling

$$\begin{aligned}x_c&\approx 0.30F_r^{-1}G_r^{1/2}e^z\\ &\approx 0.30F^{-1}G^{1/2}e^z z^{-3/2}\\ &\approx 0.30(\eta^3 FG)^{-1/4}\exp(\eta G/F)^{1/2}\ ,\end{aligned} \quad (3.15)$$

where the last two lines use the conventional gain length and Fresnel number. Conversely, for the weakly refractive limit $z\ll 1$ we obtain

$$\begin{aligned}x_c&\approx 0.48F_r^{-1}G_r^{1/2}z^{3/2}\\ &\approx 0.48F^{-1}G^{1/2}\ ,\end{aligned} \quad (3.16)$$

which is valid for $\exp(-G/3)\ll 1$, since $G_r a(z)\approx G/3$ for $z\ll 1$. As we might expect, refractive variables provide a simpler description for the strongly refracting case, while conventional variables produce a simpler formula for the weakly refracting case.

Now we turn to the exact coherence function for this model, which we have said can be expressed in terms of error functions. Performing the integral in Eq. (3.9) we find

$$\begin{aligned}\Gamma(x,0;z)&=\mathrm{const}\times\frac{\exp(G_r z)}{\sqrt{G_r[\sinh(2z)-2z]}}\\ &\times\exp\left\{\frac{[G_r(z\coth z-1)+iF_r]^2x^2}{G_r[\sinh(2z)-2z]}+i\frac{F_r}{2}\coth z-\tfrac{1}{4}G_r a(z)\right\}\\ &\times\sum_{l=0}^{1}(-1)^l\mathrm{erf}\left\{\frac{G_r(z\coth z-1)+iF_r}{\sqrt{G_r[\sinh(2z)-2z]}}x+(-1)^l\frac{\sqrt{G_r[\sinh(2z)-2z]}}{2\sinh z}\right\} .\end{aligned} \quad (3.17)$$

Using this result, we can witness the transition from weakly to strongly gain guided behavior. In Fig. 1 we show the coherence function for $F_r=500$ and $z=5$, as G_r varies between 0.001 and 5.0. For G_r less than about 2.0, we see the characteristic sinc oscillations due to the hard edges of the laser [Eq. (3.12)]. As G_r is increased, these edges are effectively softened by the falloff of gain off axis ($x=0$), leading to weaker oscillations as the degree of coherence approaches the Gaussian profile described by Eq. (3.13).

In Fig. 2 we plot the coherence length as a function of Fresnel number for two gain lengths (a) $G=10$ and (b) $G=15$. We have shown the result for the line source approximation $I(x,z)=I_0(x)\delta(z)$, as well as for a distributed source. The latter is obtained by numerically calculating

$$\bar{\Gamma}(x,0;z)=-\int_0^z\Gamma(x,0;z-z')dz'\ , \quad (3.18)$$

where $\Gamma(x,0;z)$ is given by Eq. (3.17). The scaling law result [Eq. (3.14)] is also shown on these plots [also see the line of Eq. (3.15)]. The scaling law underestimates the line source coherence length because it overestimates the source size when $\exp[-G_r a(z)/4]$ is not sufficiently small [see Eq. (3.10)]; this quantity decreases with increasing Fresnel number so the scaling law becomes more accurate. For example, at $F=100$ and $G=10$, we have $\exp[-G_r a(z)/4]\approx\exp(-1.1)\approx 0.32$, but for $F=1000$,

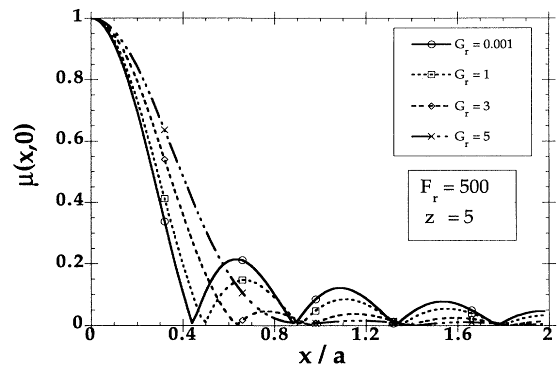

FIG. 1. WKB coherence function for $F_r = 500$, $z = 5$, and four values of G_r.

$\exp[-G_r a(z)/4] \approx \exp(-3.5) \approx 0.029$. We also see the importance of including the distribution of the source for lower gains and smaller Fresnel number, the latter because refraction leads to smaller effective gain as F becomes smaller: the prefactor in Eq. (3.17) becomes, for $\exp(z) \gg 1$,

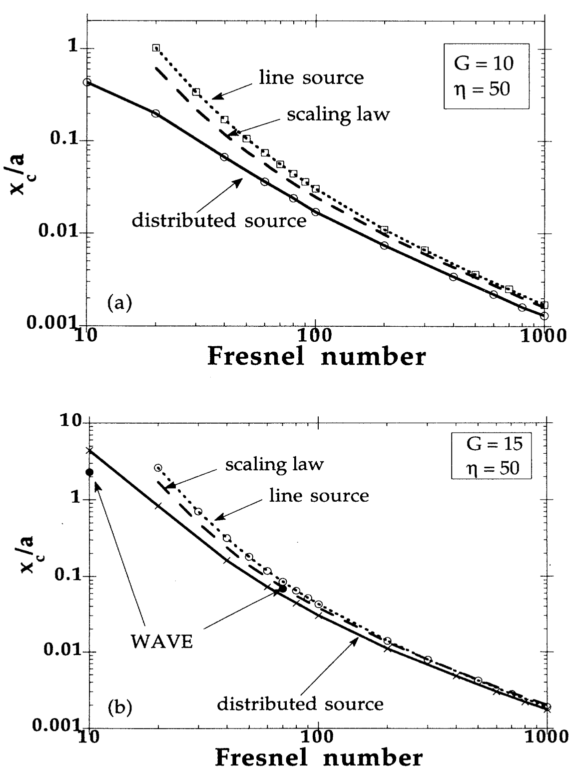

FIG. 2. WKB coherence length against Fresnel number, comparing a distributed source, line source, and scaling law, for $\eta = 50$ and (a) $G = 10$ and (b) $G = 15$. The solid circles in (b) are the results from the WAVE code.

$$\exp(G_r z)[\sinh(2z) - 2z]^{-1/2} \xrightarrow{z \gg 1} 2^{-1/2}\exp\{(G_r - 1)z\}$$
$$= 2^{-1/2}\exp\{G - \sqrt{\eta G/F}\}\ , \tag{3.19}$$

which shows explicitly that the gain length is effectively smaller for smaller F. Figure 2(b) also shows calculations using the wave optics code WAVE [4], at two Fresnel numbers $F = 10$ and 50. The WAVE calculations were performed using a parabolic distribution of both density and gain which were cut off at $x = \pm 2$. Also, the source was taken proportional to the gain. At $F = 50$ the WAVE result is $\sim 15\%$ larger than our result, probably because the parabolic source function is effectively smaller than the constant source employed in our calculation. At $F = 10$, the WAVE result is a factor of 2 smaller than our calculation; at this coherence length, the differences in cutoff of density profiles is likely an important factor. In Fig. 3 we show the evolution of the coherence function itself with Fresnel number for the case $G = 15$.

IV. HYPERBOLIC SECANT PROFILE

We now apply the method to a somewhat more realistic model for an x-ray laser, the refractive profile

$$n^2(x) = 1 - \phi_r^2(1 - i\eta^{-1})\mathrm{sech}^2(x)\ , \tag{4.1}$$

which goes smoothly to zero for $x \gg 1$ and reduces to the parabolic case above for $x \ll 1$. Our approach will be to evaluate the quantities in Eq. (1.23) directly, since the gain and optical path length can be obtained analytically. The ray trajectories in this case are

$$\begin{aligned} \sinh x &= \sinh x_0 \cosh(\beta_0 z) + (\theta_0/\beta_0)\cosh x_0 \sinh(\beta_0 z)\ , \\ \theta &= [q_0 \sinh x_0 \sinh(\beta_0 z) + \theta_0 \cosh x_0 \cosh(\beta_0 z)]\mathrm{sech} x\ , \end{aligned} \tag{4.2}$$

where $\beta_0 = \sqrt{\theta_0^2 + \mathrm{sech}^2 x_0}$ is the ray invariant.

The gain along a ray in this case can be evaluated to yield

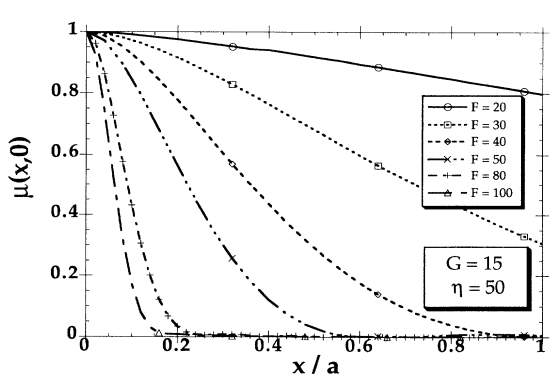

FIG. 3. WKB coherence functions for parabolic profiles, for $G = 15$, $\eta = 50$, and Fresnel numbers of 20, 30, 40, 50, 80, and 100.

$$G(x_0,\theta_0;z)=G_r\text{arctanh}\{[\beta_0-(\theta_0^2/\beta_0)\cosh^2 x_0]\times\tanh(\beta_0 z)-\tfrac{1}{2}\theta_0\sinh(2x_0)\}+G_r\text{arctanh}[\tfrac{1}{2}\theta_0\sinh(2x_0)]\ . \tag{4.3}$$

The optical path length can be expressed in terms of the gain and a function $R=R(x_0,\theta_0)$, which we define as follows: if we set $\sigma\equiv q_0\tanh x_0/\theta_0$ and

$$\alpha\equiv\begin{cases}\text{arctanh}(\sigma)\ , & \sigma<1\\ \text{arctanh}(1/\sigma)\ , & \sigma\geq 1\ ,\end{cases} \tag{4.4}$$

then we have

$$R=\begin{cases}\left[\left|\left[\dfrac{\theta_0\cosh x_0}{\beta_0\cosh\alpha}\right]^2-1\right|\right]^{1/2}, & \sigma<1\\ \left[\left[\dfrac{\theta_0\cosh x_0}{\beta_0\sinh\alpha}\right]^2+1\right]^{1/2}, & \sigma\geq 1\ .\end{cases} \tag{4.5}$$

The optical path length is then

$$S_r(x_0,\theta_0;z)=\tfrac{1}{2}[(1+\beta_0^2)z-(1+\beta_0^2R^2)G(x_0,\theta_0;z)/G_r]\ . \tag{4.6}$$

In each of the expressions (4.3)–(4.6), we are to interpret the initial angle θ_0 as a function of x, x_0, and z, arising from the inversion of the first of Eq. (4.2). Unlike the parabolic case, this inversion is not tractable analytically and must be carried out numerically. Explicitly, we need to evaluate

$$\Gamma(x,x';z)=\int dx_0 g(x_0)\left[\frac{\partial\theta_0}{\partial x}(x_0,\theta_0(x_0,x,z),z)\frac{\partial\theta_0}{\partial x}(x_0,\theta_0(x_0,x',z),z)\right]^{1/2}\times\exp\{iF_r[S(x_0,\theta_0(x_0,x,z),z)-S(x_0,\theta_0(x_0,x',z),z)]\}\times\exp\{\tfrac{1}{2}[G(x_0,\theta_0(x_0,x,z),z)+G(x_0,\theta_0(x_0,x',z),z)]\}\ , \tag{4.7}$$

where we have taken the source proportional to the gain and assumed a large gain-length approximation $I(x_0,z_0)\propto g(x_0)\delta(z_0)$. The integral can then be easily computed numerically, using a Newton-Raphson root-finding method to do the inversion.

In Fig. 4 we depict the WKB coherence function for the $\text{sech}^2 x$ profile for a gain length $G=10$ and Fresnel numbers $F=50$ and 100. Also shown on this plot are the coherence functions obtained in the parabolic approximation to the $\text{sech}^2 x$ profiles, as well as the scaling law of Eq. (3.15). We note that the $\text{sech}^2 x$ profile gives a smaller coherence length, as expected from the wider gain distribution (and effective source size) and weaker refractive spatial filtering away from the x-ray laser axis ($x=0$). In addition, the $\text{sech}^2 x$ coherence function approaches zero monotonically with x, while the parabolic coherence

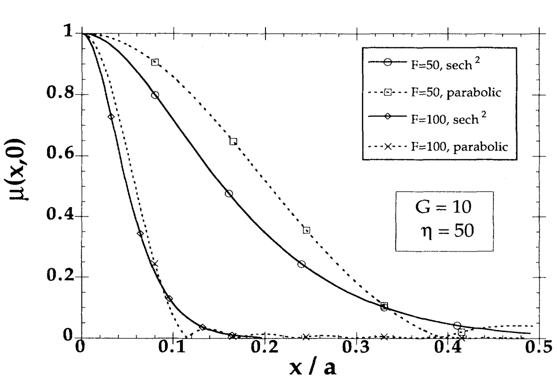

FIG. 4. Comparison of parabolic and $\text{sech}^2 x$ profiles for $G=10$, $\eta=50$, and Fresnel numbers $F=50$ and 100.

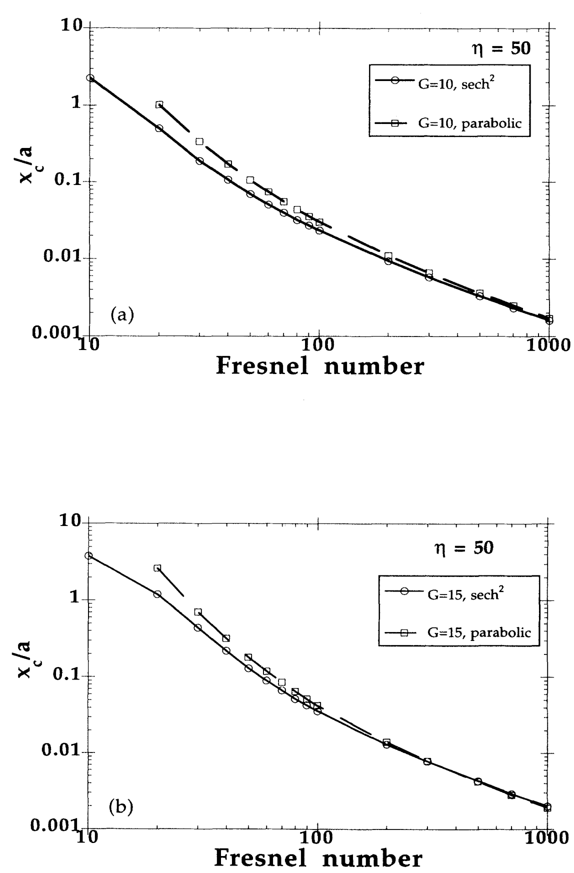

FIG. 5. WKB coherence length against Fresnel number, comparing parabolic and $\text{sech}^2 x$ profiles for $\eta=50$ and (a) $G=10$ and (b) $G=15$. A line source was used in all cases.

function has weak oscillations. These oscillations are due to the sharply defined spatial localization of the parabolic profile in contrast to the broader $\mathrm{sech}^2 x$ distribution.

Figure 5 shows the variation of coherence length for the $\mathrm{sech}^2 x$ profile as the Fresnel number varies between 10 and 1000. Two gain lengths are depicted: (a) $G=10$ and (b) $G=15$. The coherence length for the $\mathrm{sech}^2 x$ profile approaches that of the parabolic profile in the limit of large Fresnel number (small coherence length) for both cases, because in the regime only rays with $x < x_c \ll 1$ are important.

CONCLUSION

We have shown, using WKB methods, that the coherence function in the presence of spatially varying refraction and gain may be expressed in the form of a Fourier transform of a Gaussian modulated source. The derived formula is valid for large values of refractive Fresnel number $F_r = ka^2/L_r$ and for spatial points restricted transversely to the laser cross section. This formula allows for the simple calculation of the coherence length for approximately parabolic refractive and gain profiles and confirms previous experience that refractive defocusing leads to exponential scaling of the coherence length with length and that gain guiding leads to effective narrowing of the source. The more general formula can be applied to ray trajectories obtained either analytically or from numerical integration of the ray equations through density profiles obtained from hydrodynamic simulations or experiment. We also applied the method to a $\mathrm{sech}^2 x$ density and gain profile, which gave a coherence length which approached that of the parabolic profile for large Fresnel number.

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APPENDIX: DERIVATION OF EQ. (1.25)

In this appendix we derive Eq. (1.25), which relates the gain length along a ray to the optical path length, for the case where gain is proportional to electron density. First we note that the optical Lagrangian can be written in terms of $\theta = dx/dz$ and the normalized gain length $\bar{g}$ as

$$L = \tfrac{1}{2}\theta^2 - \theta_r^2 \bar{g}(x) , \tag{A1}$$

where we have used $\bar{g} = \bar{h}$ in Eq. (1.8). Similarly, the optical Hamiltonian can be expressed as

$$H = \tfrac{1}{2}[\theta^2 + \theta_r^2 \bar{g}(x)] . \tag{A2}$$

Since $G = g_0 \int_{\mathrm{ray}} \{1 + \bar{g}[x(z)]\} dz$, we have

$$\begin{aligned} G/G_r &= z + \int_{\mathrm{ray}} [L(x(z),\theta(z)) - H(x(z),\theta(z))] dz \\ &= z + S(x_0,x;z) - H(x_0,x;z)z , \end{aligned} \tag{A3}$$

where we have used the fact that H is constant along a ray. We now note that Hamilton's principal function S satisfies

$$\frac{\partial S(x_0,x;z)}{\partial z} = -H(x_0,x;z) , \tag{A4}$$

which gives

$$\begin{aligned} G(x_0,x;z) &= G_r \left[z + S(x_0,x;z) + z\frac{\partial S(x_0,x;z)}{\partial z} \right] \\ &= G_r \left[z + \frac{\partial}{\partial z}[zS(x_0,x;z)] \right] , \end{aligned} \tag{A5}$$

which is Eq. (1.25).

[1] R. A. London, M. D. Rosen, and M. Strauss, Phys. Rev. Lett. **65**, 563 (1990).
[2] P. Amendt, R. A. London, and M. Strauss, Phys. Rev. A **44**, 7478 (1991); **47**, 4348 (1993).
[3] G. Hazak and A. Bar-Shalom, Phys. Rev. A **40**, 7055 (1989).
[4] M. D. Feit and J. A. Fleck, Jr., J. Opt. Soc. Am. B **7**, 2048 (1990).
[5] R. A. London, Phys. Fluids B **5**, 2707 (1993).
[6] G. Hazak and A. Bar-Shalom, Phys. Rev. A **38**, 1300 (1988).
[7] O. Zahavi, G. Hazak, and Z. Zinamon, J. Opt. Soc. Am. A **9**, 1807 (1992); J. Opt. Soc. Am. B **10**, 271 (1993).
[8] E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986).
[9] J. H. Van Vleck, Proc. Natl. Acad. Sci. U. S. A. **14**, 178 (1928).
[10] A. Schulman, *Techniques and Applications of Path Integration* (Wiley-Interscience, New York, 1981).
[11] M. Nazarathy and J. Shamir, J. Opt. Soc. Am. **72**, 1398 (1982).
[12] R. G. Littlejohn, J. Stat. Phys. **68**, 7 (1992).
[13] H. Goldstein, *Classical Mechanics,* 2nd ed. (Addison-Wesley, Reading, MA, 1980).
[14] W. H. Carter and E. Wolf, J. Opt. Soc. Am. **65**, 1067 (1975).
[15] R. G. Littlejohn, Phys. Rep. **138**, 193 (1986).
[16] M. D. Feit, Lawrence Livermore National Laboratory Report No. UCRL-JC-109183, 1991 (unpublished).