Normal ordering of the SU(1,1) and SU(2) squeeze operators

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We have derived the normal-ordered form of the SU(1,1) and SU(2) squeeze operators in the multimode bosonic representation and evaluated the coherent-state matrix elements of these operators. The analysis can be applied to any exponential functions of the generators of the Lie algebras su(1,1) and su(2) without difficulty.

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In the past decade the single- and two-mode bosonic realizations of the su(1,1) Lie algebra as well as the (two-mode) Schwinger bosonic representation of the su(2) Lie algebra have been receiving a lot of attention in the study of the nonclassical properties of light in quantum optics. For instance, the linear dissipative processes in quantum optical systems can be studied with the su(1,1) Lie algebra in the framework of the Liouville space formulation [1], while beam splitters [2–4], interferometers [5], and linear directional couplers [6] are successfully described by the su(2) Lie algebra. Recently Lo and coworkers constructed the multimode bosonic realization of the Lie algebras su(1,1) and su(2) as well as the associated generalized coherent states in the multimode Fock space [7,8]. The su(1,1) generalized coherent state is, in fact, the generalized multimode squeezed vacuum state discussed by Lo and Sollie [9] and the su(1,1) unitary displacement operator can be identified as the generalized multimode squeeze operator. Similar to the single- and two-mode cases, the multimode bosonic realization of the su(1,1) Lie algebra has immediate relevance to the squeezing properties of boson fields. On the other hand, in the su(2) generalized coherent state, which exhibits the SU(2) squeezing for the su(2) generators, each bosonic mode has the sub-Poissonian statistics and the su(2) unitary displacement operator behaves like a generalized multimode rotation operator. Accordingly, these two displacement operators can be regarded as the SU(1,1) and SU(2) squeeze operators. In the present work we are interested in deriving the normal-ordered form of the SU(1,1) and SU(2) squeeze operators because normal ordering of operators is very useful in calculating the coherent-state matrix elements of the operators.

The su(2) Lie algebra consists of three generators J_0, J_+, and J_- satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = 2J_0 \quad . \tag{1}$$

In the multimode bosonic realization the three generators are defined as [7]

$$J_0 = \sum_{i,j=1}^{N} \Lambda_{ij} a_i^\dagger a_j \quad , \quad J_+ = \sum_{i,j=1}^{N} \beta_{ij} a_i^\dagger a_j \quad ,$$
$$J_- = J_+^\dagger = \sum_{i,j=1}^{N} \beta_{ji}^* a_i^\dagger a_j = \sum_{i,j=1}^{N} \beta_{ij}^\dagger a_i^\dagger a_j \quad , \tag{2}$$

where $\Lambda_{ij} = \Lambda_{ij}^\dagger = \sum_{k=1}^{N}(\beta_{ik}\beta_{kj}^\dagger - \beta_{ik}^\dagger\beta_{kj})/2$ with the constraint that $\beta_{ij} = \sum_{k=1}^{N} \Lambda_{ik}\beta_{kj} - \beta_{ik}\Lambda_{kj}$. The corresponding SU(2) unitary displacement operator is given by $D_1(\alpha) = \exp(\alpha J_+ - \alpha^* J_-) = \exp\{i\sum_{j,k=1}^{N} \Phi_{jk} a_j^\dagger a_k\}$, where $i\Phi_{jk} = \alpha\beta_{jk} - \alpha^*\beta_{jk}^\dagger$ and $\boldsymbol{\Phi} = -i(\alpha\boldsymbol{\beta} - \alpha^*\boldsymbol{\beta}^\dagger)$ is Hermitian. In order to cast the operator $D_1(\alpha)$ into the normal-ordered form, we shall proceed in the following way. First of all we realize that the operator $D_1(\alpha)$ transforms the annihilation operator of each bosonic mode as follows:

$$D_1(\alpha)^\dagger \, a_j \, D_1(\alpha) = \sum_{k=1}^{N} [\exp(i\boldsymbol{\Phi})]_{jk} a_k \quad , \tag{3}$$

which implies that

$$\begin{aligned}[a_j, D_1(\alpha)] &= \frac{\partial}{\partial a_j^\dagger} D_1(\alpha) \\ &= \sum_{k=1}^{N} \{[\exp(i\boldsymbol{\Phi})]_{jk} - \delta_{jk}\} D_1(\alpha) a_k \\ &= \sum_{k=1}^{N} [\exp(i\boldsymbol{\Phi}) - \boldsymbol{I}]_{jk} D_1(\alpha) a_k \quad . \end{aligned} \tag{4}$$

Then let us define the operator $\mathcal{N}(\{a_j^\dagger\}, \{a_j\}; \alpha)$ to be the normal-ordered form of the unitary displacement operator $D_1(\alpha)$, whose expectation value in the ordinary multimode Glauber coherent state $|\{z_j\}\rangle$ is simply given by $\langle\{z_j\}|D_1(\alpha)|\{z_j\}\rangle = \mathcal{N}(\{z_j^*\}, \{z_j\}; \alpha)$. It is straightforward to show that

$$\begin{aligned}\langle\{z_j\}|[a_k, D_1(\alpha)]|\{z_j\}\rangle &= \sum_{l=1}^{N} [\exp(i\boldsymbol{\Phi}) - \boldsymbol{I}]_{kl} z_l \\ &\quad \times \mathcal{N}(\{z_j^*\}, \{z_j\}; \alpha) \\ &= \frac{\partial}{\partial z_k^*} \mathcal{N}(\{z_j^*\}, \{z_j\}; \alpha) \quad , \end{aligned} \tag{5}$$

with the condition that $\mathcal{N}(\{z_j^* = 0\}, \{z_j = 0\}; \alpha) = 1$. By direct integration one can easily obtain the desired solution of the above differential equation:

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$$\begin{aligned}\mathcal{N}(\{z_j^*\},\{z_j\};\alpha) &= \exp\left(\sum_{k,l=1}^{N} z_k^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{kl}z_l\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k,l=1}^{N} z_k^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{kl}z_l\right)^n\\ &= \sum_{n=0}^{\infty}\sum_{k_{11}=0}^{n}\sum_{k_{12}=0}^{n-k_{11}}\sum_{k_{13}=0}^{n-k_{11}-k_{12}}\cdots\sum_{k_{N,N-1}=0}^{n-k_{11}-k_{12}-\cdots-k_{N,N-2}}\frac{1}{k_{11}!k_{12}!k_{13}!\cdots k_{N,N-1}!k_{NN}!}\\ &\quad\times\{z_1^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{11}z_1\}^{k_{11}}\\ &\quad\times\{z_1^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{12}z_2\}^{k_{12}}\\ &\quad\times\{z_1^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{13}z_3\}^{k_{13}}\\ &\quad\times\cdots\{z_N^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{N,N-1}z_{N-1}\}^{k_{N,N-1}}\\ &\quad\times\{z_N^*[\exp(i\mathbf{\Phi})-\mathbf{I}]_{NN}z_N\}^{k_{NN}}\\ &= \sum_{n=0}^{\infty}\sum_{\{k_{lm}\}}\left(\prod_{r,s=1}^{N}\frac{\{[\exp(i\mathbf{\Phi})-\mathbf{I}]_{rs}\}^{k_{rs}}}{k_{rs}!}\right)\left(\prod_{p=1}^{N}(z_p^*)^{L_p}\right)\left(\prod_{q=1}^{N}(z_q)^{M_q}\right)\ , \end{aligned} \tag{6}$$

where $\sum_{l,m=1}^{N} k_{lm} = n$, $L_p = \sum_{j=1}^{N} k_{pj}$, $M_q = \sum_{j=1}^{N} k_{jq}$, and $\sum_{\{k_{lm}\}}$ denotes the summation over all possible partitions of $n = \sum_{l,m=1}^{N} k_{lm}$. As a result, the normal-ordered form of the displacement operator $D_1(\alpha)$ is simply given by

$$\begin{aligned}D_1(\alpha) &= \mathcal{N}(\{a_j^\dagger\},\{a_j\};\alpha)\\ &= \sum_{n=0}^{\infty}\sum_{\{k_{lm}\}}\left(\prod_{r,s=1}^{N}\frac{\{[\exp(i\mathbf{\Phi})-\mathbf{I}]_{rs}\}^{k_{rs}}}{k_{rs}!}\right)\\ &\quad\times\left(\prod_{p=1}^{N}(a_p^\dagger)^{L_p}\right)\left(\prod_{q=1}^{N}(a_q)^{M_q}\right)\ .\end{aligned} \tag{7}$$

Next we consider the case of su(1,1) Lie algebra whose three generators obey the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \quad , \quad [K_+, K_-] = -2K_0 \quad . \tag{8}$$

In the multimode bosonic representation the three generators are given by [8]

$$\begin{aligned}K_+ &= \frac{1}{2}\sum_{i,j=1}^{N}\mu_{ij}a_i^\dagger a_j^\dagger \ , \quad K_- = \frac{1}{2}\sum_{i,j=1}^{N}\mu_{ij}^* a_i a_j \ ,\\ K_0 &= \sum_{i,j=1}^{N}\gamma_{ij}(2a_i^\dagger a_j + \delta_{ij}) \ ,\end{aligned} \tag{9}$$

where $\mu_{ij} = \mu_{ji}$ and $\gamma_{ij} = \gamma_{ij}^\dagger = 4^{-1}\sum_{k=1}^{N}\mu_{ik}\mu_{kj}^*$ with the requirement that $\mu_{ij} = \sum_{k,l=1}^{N}\mu_{ik}\mu_{kl}^\dagger\mu_{lj}$. The SU(1,1) unitary displacement operator is given by $D_2(\alpha) = \exp(\alpha K_+ - \alpha^* K_-)$, which can be cast into the following disentangled form [10]:

$$\begin{aligned}D_2(\alpha) &= \exp\{\Gamma_+K_+\}\exp\{\ln(\Gamma_0)K_0\}\exp\{\Gamma_-K_-\}\\ &= \Gamma_0^{\mathrm{Tr}\{\gamma\}}\exp\left(\frac{1}{2}\Gamma_+\sum_{i,j=1}^{N}\mu_{ij}a_i^\dagger a_j^\dagger\right)\\ &\quad\times\exp\left(2\ln(\Gamma_0)\sum_{i,j=1}^{N}\gamma_{ij}a_i^\dagger a_j\right)\\ &\quad\times\exp\left(\frac{1}{2}\Gamma_-\sum_{i,j=1}^{N}\mu_{ij}^* a_i a_j\right)\ ,\end{aligned} \tag{10}$$

where $\alpha = re^{i\theta}$, $\Gamma_0 = [\cosh(r)]^{-2}$, $\Gamma_\pm = \pm e^{\pm i\theta}\tanh(r)$, and Tr $\{\gamma\} = \sum_{i=1}^{N}\gamma_{ii}$. It is clear that by applying the analysis of the SU(2) case, this disentangled form will then yield the normal-ordered form readily:

$$\begin{aligned}D_2(\alpha) &= \Gamma_0^{\mathrm{Tr}\{\gamma\}}\exp\left(\frac{1}{2}\Gamma_+\sum_{i,j=1}^{N}\mu_{ij}a_i^\dagger a_j^\dagger\right)\\ &\quad\times\sum_{n=0}^{\infty}\sum_{\{k_{lm}\}}\left(\prod_{r,s=1}^{N}\frac{(\{\exp[2\ln(\Gamma_0)\gamma]-\mathbf{I}\}_{rs})^{k_{rs}}}{k_{rs}!}\right)\\ &\quad\times\left(\prod_{p=1}^{N}(a_p^\dagger)^{L_p}\right)\left(\prod_{q=1}^{N}(a_q)^{M_q}\right)\exp\left(\frac{1}{2}\Gamma_-\sum_{i,j=1}^{N}\mu_{ij}^* a_i a_j\right)\ .\end{aligned} \tag{11}$$

Furthermore, with the SU(1,1) and SU(2) displacement operators in the normal-ordered form, it is straightforward to evaluate the coherent-state matrix elements of these operators as follows:

$$\begin{aligned}\langle\{\xi_j\}|D_1(\alpha)|\{z_j\}\rangle &= \exp\left(\sum_{k,l=1}^{N}\xi_k^*[\exp(i\mathbf{\Phi})-\boldsymbol{I}]_{kl}z_l\right)\langle\{\xi_j\}|\{z_j\}\rangle\ ,\\ \langle\{\xi_j\}|D_2(\alpha)|\{z_j\}\rangle &= \Gamma_0^{\mathrm{Tr}\{\boldsymbol{\gamma}\}}\exp\left(\frac{1}{2}\Gamma_+\sum_{i,j=1}^{N}\mu_{ij}\xi_i^*\xi_j^*\right)\exp\left(\sum_{k,l=1}^{N}\xi_k^*\{\exp[2\ln(\Gamma_0)\boldsymbol{\gamma}]-\boldsymbol{I}\}_{kl}z_l\right)\\ &\quad\times\exp\left(\frac{1}{2}\Gamma_-\sum_{i,j=1}^{N}\mu_{ij}^*z_iz_j\right)\langle\{\xi_j\}|\{z_j\}\rangle\ ,\end{aligned} \tag{12}$$

where

$$\begin{aligned}\langle\{\xi_j\}|\{z_j\}\rangle &= \prod_{j=1}^{N}\langle\xi_j|z_j\rangle\\ &= \prod_{j=1}^{N}\exp\{\xi_j^*z_j-(|\xi_j|^2+|z_j|^2)/2\}\ .\end{aligned}$$

In summary, we have derived the normal-ordered form of the SU(1,1) and SU(2) squeeze operators in the multimode bosonic representation and evaluated the coherent-state matrix elements of these operators. The analysis can be applied to any exponential functions of the generators of the Lie algebras su(1,1) and su(2) without difficulty.

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