

Distance between density operators: Applications to the Jaynes-Cummings model

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A measure for the distance between two arbitrary density operators, based on the Hilbert-Schmidt norm, is investigated. It is applied to the Jaynes-Cummings model where the question of the extent to which a given state is close to the initial state or to another state of interest has commanded a lot of attention. This problem is studied in detail for the whole system as well as for the field and the atom subsystems. The behavior of the distance for some important particular cases is numerically evaluated. Some interesting aspects of the field and the atom dynamics are discovered.

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I. INTRODUCTION

The Jaynes-Cummings model (JCM) describing a single mode of the quantized electromagnetic field coupled to a two-level atom in a lossless cavity [1–6] is an important fundamental theoretical model of the interaction between two quantum systems. In spite of its simplicity, the model predicts many nonclassical effects and exhibits reasonably complicated behavior. Some of these effects have recently been observed experimentally using highly excited Rydberg atoms in high- Q microwave cavities [7]. It should be noted that details of the interaction strongly depend on the initial conditions [8–18]. Surprisingly enough, the most transparent quantum features such as, e.g., collapses and revivals of the Rabi oscillations of the atomic inversion [8,9], are especially striking if the field is at the beginning in the coherent state. Just in this case we also have another interesting effect: the state preparation by quantum apparatus. It has been shown that the atom and field most closely return to pure states during the collapse region but not at the peak of revivals as may be expected [10–13]. However, this does not mean that the atom or the field is as close as possible to the *initial* state.

The state preparation effect was investigated using both the purity parameter and the von Neumann entropy. Another aspect of the evolution of quantum systems can be highlighted using the concept of Wehrl's entropy [19,20]. However, to investigate the question of how close a given state is to the initial state we need a proper measure of the distance between quantum states. In the case of a compound system such a distance should be calculated *separately* for the whole system as well as for subsystems. We cannot *a priori* assume that we will get

similar results for subsystems even if we start evolution in a pure state. Such a distance between density operators is another way to study the time development of the JCM. If the distance between two density operators is small, these two density operators can be considered as very similar to one another, and on the other hand a large distance means very different density operators. It enables us to compare the state of the system at any time with the initial state or another reference state of interest. It is possible to define the distance between two quantum states described by density operators in many ways. Although different in details, the most natural definitions of distance are based on the norms defined on the space of density operators, such as e.g., the Hilbert-Schmidt norm or the trace norm [21–24]. Also more sophisticated distances often use the standard, norm-based definitions as building blocks [25,26]. Quite recently, another distance between density operators has been considered in the context of measurements which optimally resolve neighboring quantum states [27] (see also [28–32]). Applications of the concept of distance to the JCM provide a deeper insight into many aspects of the time dynamics of the field, the atomic system as well as the whole system. We can precisely see, e.g., how the field evolves into a special state or how closely it will reach such a special state.

This paper is organized as follows. In Sec. II, a definition of the distance between density operators, based on the Hilbert-Schmidt norm, is presented and some basic facts and consequences are discussed. A short discussion of the distance based on the trace norm is also given. In Sec. III, the JCM is considered in the form suitable for our objectives. General properties of the distance in the JCM are presented. More detailed case studies are performed in Sec. IV, where time dynamics of various distances is numerically evaluated. Results concerning distances are compared with other quantities such as inversion and the von Neumann entropy. In Sec. V we summarize our main results and mention possible gener-

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alizations. To make the paper self-contained we collect crucial formulas and facts about the JCM in the Appendix.

II. DISTANCE BETWEEN DENSITY OPERATORS

The standard distance between two real numbers (or two points on the real line) x, y is something very simple: $|x - y|$. Also the distance between two points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in Euclidean space R^n ,

$$D(x, y) = \left(\sum_{i=0}^n (x_i - y_i)^2 \right)^{1/2}, \quad (2.1)$$

is a familiar concept. It is very useful, however, to introduce an abstract definition of distance which could be applied also for more complex sets of objects such as, e.g., vectors, functions, matrices, and operators. Such an abstract definition is a basis for the general theory of metric spaces [22]. An arbitrary set of objects of any nature is called a metric space if there is a real, non-negative function $D(x, y)$, called the metric function or the distance between objects x and y of M , which satisfies the following conditions: (i) $D(x, y) \geq 0$, (ii) $D(x, y) = 0$ if and only if $x = y$, (iii) symmetry: $D(x, y) = D(y, x)$, and (iv) triangle inequality: $D(x, y) \leq D(x, z) + D(z, y)$. The above four requirements seem obvious if we think in terms of the archetypical examples given by real numbers or Euclidean distance (2.1) but they actually provide a very general and powerful theoretical framework, far beyond the capabilities of these illustrative examples. For a given set of objects, the distance satisfying (i)-(iv) can be defined in a number of different ways. For example, any set can be made a metric space if we use the discrete metric $D_0(x, y)$ equal to zero for $x = y$ and equal to one otherwise. Of course it is not very useful. Fortunately, in many cases we are able to develop a more natural concept of distance making use of the detailed structure of the set of interest. It is indeed the case for the density operators. By definition, they are linear, bounded, positive-definite, and self-adjoint operators. Moreover they are the Hilbert-Schmidt operators [22–24]. Therefore we can build from them a *normed space* with the Hilbert-Schmidt norm

$$\|\hat{\rho}\|_2 = \sqrt{\text{Tr} \hat{\rho}^\dagger \hat{\rho}} = \sqrt{\text{Tr} \hat{\rho}^2}. \quad (2.2)$$

The corresponding distance can now be easily introduced as

$$D_2(\hat{\rho}_1, \hat{\rho}_2) = \|\hat{\rho}_1 - \hat{\rho}_2\|_2 = \sqrt{\text{Tr}(\hat{\rho}_1 - \hat{\rho}_2)^2}. \quad (2.3)$$

As density operators are also *trace class* operators [21–24] we can define another norm, called the trace norm,

$$\|\hat{\rho}\|_1 = \text{Tr}|\hat{\rho}| = \text{Tr} \sqrt{\hat{\rho}^\dagger \hat{\rho}}, \quad (2.4)$$

and corresponding distance

$$D_1(\hat{\rho}_1, \hat{\rho}_2) = \|\hat{\rho}_1 - \hat{\rho}_2\|_1. \quad (2.5)$$

Let us note that just the trace norm was used by Hillery [25,26] to introduce the concept of nonclassical distance. In this paper we will use the distance based on the Hilbert-Schmidt norm. This norm is a generalization of the Frobenius norm known from the theory of finite matrices [33]. The corresponding distance strongly resembles the Euclidean distance (2.1) and is more suitable for detailed computations of concrete examples than distance based on the trace norm. A comparison of these two distances will be presented elsewhere.

It will be convenient to use rescaled distance that differs from the standard one by a factor $1/\sqrt{2}$. This choice is equivalent to the normalization of the distance: it assures that the distance is a number between 0 and 1. Introducing this scaling factor and carrying out the square in Eq. (2.3) we have

$$\begin{aligned} D(\hat{\rho}_1, \hat{\rho}_2) &= \frac{1}{\sqrt{2}} D_2(\hat{\rho}_1, \hat{\rho}_2) \\ &= \frac{1}{\sqrt{2}} \left[\text{Tr} \hat{\rho}_1^2 + \text{Tr} \hat{\rho}_2^2 - 2 \text{Tr}(\hat{\rho}_1 \hat{\rho}_2) \right]^{1/2}. \end{aligned} \quad (2.6)$$

In the above definition, quantities like $\text{Tr}(\hat{\rho}_i^2)$, related to purities, occur explicitly. Thus the distance is determined by purities of both states and by the trace of the product of the two density operators. The latter is a measure of orthogonality of states. Any density operator $\hat{\rho}_i$ can be diagonalized, i.e., we can find its eigenvalues π_α^i and eigenstates $|\psi_\alpha^i\rangle$, providing the following decomposition:

$$\hat{\rho}_i = \sum_\alpha |\psi_\alpha^i\rangle \pi_\alpha^i \langle \psi_\alpha^i|, \quad i = 1, 2. \quad (2.7)$$

Eigenstates $|\psi_\alpha^i\rangle$ form an orthonormal basis

$$\langle \psi_\alpha^i | \psi_\beta^i \rangle = \delta_{\alpha, \beta}, \quad (2.8)$$

and the sum of the corresponding eigenvalues π_α^i is normalized to unity:

$$\sum_\alpha \pi_\alpha^i = 1, \quad i = 1, 2. \quad (2.9)$$

Making use of diagonal representations (2.7) we find for the distance between two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ the expression

$$\begin{aligned} D(\hat{\rho}_1, \hat{\rho}_2) &= \frac{1}{\sqrt{2}} \left(\sum_\alpha (\pi_\alpha^1)^2 + \sum_\beta (\pi_\beta^2)^2 \right. \\ &\quad \left. - 2 \sum_{\alpha, \beta} |\langle \psi_\alpha^1 | \psi_\beta^2 \rangle|^2 \pi_\alpha^1 \pi_\beta^2 \right)^{1/2}. \end{aligned} \quad (2.10)$$

We see that the distance can be easily calculated if eigenvalues and eigenfunctions of corresponding density operators are known. Let us consider two special but important cases. When both systems are in pure states, i.e.,

$$\hat{\rho}_i = |\psi^i\rangle \langle \psi^i| \quad \text{for } i = 1, 2, \quad (2.11)$$

the distance should be a function of the scalar product of two state vectors. Indeed, we find

$$D(\hat{\rho}_1, \hat{\rho}_2) = \left(1 - |\langle \psi^1 | \psi^2 \rangle|^2\right)^{1/2}. \quad (2.12)$$

The distance depends on the absolute value of the scalar product between kets $|\psi^1\rangle$ and $|\psi^2\rangle$. The overlap of two wave functions, which is a measure of a similarity of two states, is thus also an indicator of the distance between them. If only one density operator represents a pure state (let us take $\hat{\rho}_1 = |\psi^1\rangle\langle\psi^1|$ for definiteness) we have

$$D(\hat{\rho}_1, \hat{\rho}_2) = \left(\frac{1}{2}(1 + \text{Tr}\hat{\rho}_2^2) - \langle \psi^1 | \hat{\rho}_2 | \psi^1 \rangle\right)^{1/2}. \quad (2.13)$$

In applications to the JCM we will represent the density operators in diagonal forms corresponding to Eq. (2.7), which allows us to calculate numerically all distances of interest in a relatively simple way.

It is seen that our distance has a maximum value (equal to 1) for any two orthogonal normalized states. Therefore two orthogonal states are as far apart as possible. We find this property an advantage because it is very natural from the geometrical point of view: two orthogonal states have really very little in common. Let us stress that our distance does not take into account any ordering of the orthogonal states with respect to the eigenvalues of any operator. Therefore, e.g., all different Fock states are equally distant from each other.

III. DISTANCE IN THE JAYNES-CUMMINGS MODEL

The JCM describes the interaction between a two-level atom and one mode of the quantized radiation field (see, e.g., [6] for a recent review). Our aim is to study the distance between states of the JCM and properly chosen reference states as a function of time. This will be done for the whole system as well as for the atomic and the field subsystems. Assuming the exact resonance we have the following well-known Hamiltonian of the JCM in the dipole and rotating wave approximations:

$$\hat{H} = \hat{H}_0 + \hat{H}_{Int} = \hat{H}_0 + \lambda(\hat{a}^\dagger|g\rangle\langle e| + \hat{a}|e\rangle\langle g|), \quad (3.1)$$

$$\hat{H}_0 = \hat{H}_{0F} + \hat{H}_{0A} = \omega\hat{a}^\dagger\hat{a} - \omega|g\rangle\langle g|, \quad (3.2)$$

where \hat{a}^\dagger and \hat{a} are the standard creation and annihilation operators of monochromatic photons with frequency ω , the excited and the ground state of the atom are denoted by $|e\rangle$ and $|g\rangle$, respectively, and λ is the atom-field coupling constant which may be treated as real and positive. We set $\hbar = 1$ and energy of the excited atomic state to be equal to zero. We assume that the system is prepared in an uncorrelated product state of the atom and the field at the initial time $t = 0$,

$$|\Psi(0)\rangle = |A\rangle \otimes |F\rangle, \quad (3.3)$$

where the atomic state $|A\rangle$ is an arbitrary coherent superposition of the excited and the ground state,

$$|A\rangle = \cos(\sigma/2)|e\rangle + \sin(\sigma/2)e^{i\tau}|g\rangle. \quad (3.4)$$

We assume also that the field is at the beginning in a coherent state

$$|F\rangle = |\alpha\rangle := \exp(-\bar{n}/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha = \sqrt{\bar{n}}. \quad (3.5)$$

Here $|n\rangle$ denotes a Fock state with n photons present in the considered quantized field mode and \bar{n} is the mean photon number in the coherent state $|\alpha\rangle$. According to [13], the state of the system at time t can be represented as a superposition of time-dependent orthonormal atomic and field states (see Appendix for more details):

$$|\Psi(t)\rangle = |A^+\rangle \otimes \sqrt{\pi^+}|F^+\rangle + |A^-\rangle \otimes \sqrt{\pi^-}|F^-\rangle, \quad (3.6)$$

where

$$\langle A^+ | A^+ \rangle = 1 = \langle A^- | A^- \rangle, \quad \langle A^+ | A^- \rangle = 0, \quad (3.7)$$

$$\langle F^+ | F^+ \rangle = 1 = \langle F^- | F^- \rangle, \quad \langle F^+ | F^- \rangle = 0. \quad (3.8)$$

The atomic and field states $|A^\pm\rangle$ and $|F^\pm\rangle$ as well as the quantities π^\pm (with $0 \leq \pi^\pm \leq 1$) are time dependent. The resulting density operators of the whole system $\hat{\rho}(t)$, the atom $\hat{\rho}_A(t)$, and the field $\hat{\rho}_F(t)$ are given by

$$\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|, \quad (3.9)$$

$$\hat{\rho}_A(t) := \text{Tr}_F \hat{\rho}(t) = |A^+\rangle\pi^+\langle A^+| + |A^-\rangle\pi^-\langle A^-|, \quad (3.10)$$

$$\hat{\rho}_F(t) := \text{Tr}_A \hat{\rho}(t) = |F^+\rangle\pi^+\langle F^+| + |F^-\rangle\pi^-\langle F^-|, \quad (3.11)$$

where Tr_A (Tr_F) means the trace over atomic (field) states. It is seen from the above equations that eigenvalues of the atom and the field density operators are exactly the same: π^\pm . It is a general property of two component systems that both subsystems have the same eigenvalues [34]. It can be traced back to the Araki-Lieb inequalities for the von Neumann entropies [35].

The reference state of the whole system is assumed to be initially an uncorrelated product state of atom and field states:

$$|\tilde{\Psi}\rangle = |\tilde{A}\rangle \otimes |\tilde{F}\rangle, \quad (3.12)$$

where again $|\tilde{A}\rangle$ and $|\tilde{F}\rangle$ denote (reference) states of the atom and the field, respectively. As we are interested in dynamics of distances caused by interaction, we choose the time-dependent reference state describing *free* evolution of the initial state (3.12),

$$|\tilde{\Psi}(t)\rangle = \exp(-i\hat{H}_0 t) |\tilde{\Psi}\rangle = |\tilde{A}(t)\rangle \otimes |\tilde{F}(t)\rangle, \quad (3.13)$$

where

$$|\tilde{A}(t)\rangle = \exp(-i\hat{H}_{0A} t) |\tilde{A}\rangle, \quad (3.14)$$

$$|\tilde{F}(t)\rangle = \exp(-i\hat{H}_{0F} t) |\tilde{F}\rangle.$$

The corresponding density operators for the whole system $\hat{\sigma}$, the atom $\hat{\sigma}_A$, and the field $\hat{\sigma}_F$ are given by pure states

$$\hat{\sigma}(t) = |\tilde{\Psi}(t)\rangle\langle\tilde{\Psi}(t)|, \quad (3.15)$$

$$\hat{\sigma}_A(t) = \text{Tr}_F \hat{\sigma} = |\tilde{A}(t)\rangle\langle\tilde{A}(t)|, \quad (3.16)$$

$$\hat{\sigma}_F(t) = \text{Tr}_A \hat{\sigma} = |\tilde{F}(t)\rangle\langle\tilde{F}(t)|. \quad (3.17)$$

We will investigate the following metric functions:

$$D(t) = D(\hat{\sigma}(t), \hat{\rho}(t)), \quad (3.18)$$

$$D_A(t) = D(\hat{\sigma}_A(t), \hat{\rho}_A(t)), \quad (3.19)$$

$$D_F(t) = D(\hat{\sigma}_F(t), \hat{\rho}_F(t)), \quad (3.20)$$

representing distances between density operators of the whole system, the atom, and the field, respectively. As a direct consequence of Eqs. (3.9)–(3.11) and (3.15)–(3.17) we obtain

$$D(t) = (1 - |\gamma_A^+ \gamma_F^+ + \gamma_A^- \gamma_F^-|^2)^{1/2}, \quad (3.21)$$

$$D_A(t) = (g - \sqrt{\pi^+} |\gamma_A^+|^2 - \sqrt{\pi^-} |\gamma_A^-|^2)^{1/2}, \quad (3.22)$$

$$D_F(t) = (g - \sqrt{\pi^+} |\gamma_F^+|^2 - \sqrt{\pi^-} |\gamma_F^-|^2)^{1/2}, \quad (3.23)$$

where

$$\gamma_A^\pm = \langle \tilde{A}(t) | A^\pm \rangle (\pi^\pm)^{1/4}, \quad (3.24)$$

$$\gamma_F^\pm = \langle \tilde{F}(t) | F^\pm \rangle (\pi^\pm)^{1/4}, \quad (3.25)$$

and

$$g = \frac{1}{2} [1 + (\pi^+)^2 + (\pi^-)^2] = 1 - \pi^+ \pi^-. \quad (3.26)$$

The parameter g denotes half of the sum of purities of density matrices for which the distance is calculated. In our case the purity of the reference states and of the state of the whole system is always equal to 1. For the atomic and the field state purity is the same: $(\pi^+)^2 + (\pi^-)^2$. Thus g can vary only between $\frac{3}{4}$ and 1, where the minimum is reached for $\pi^+ = \frac{1}{2} = \pi^-$.

In the investigated cases of the Jaynes-Cummings model, the distance between density operators of the whole system is always greater than or equal to the corresponding distance for the subsystem density operators

$$D(t) \geq D_A(t) \quad \text{and} \quad D(t) \geq D_F(t). \quad (3.27)$$

However, the more general problem, as to whether the distance between two density operators of the compound system is always greater than or equal to the distance of the corresponding density operators of any of the subsystems, is an open question.

IV. RESULTS OF CASE STUDIES

This section contains many illustrative examples, presenting in detail the time dynamics of distances in the JCM. For completeness of our discussion and to place results in a broader context we include also results related to other parameters useful in investigations of some aspects of the JCM, such as atomic inversion and the von Neumann entropy. To provide better understanding of our numerical results we start with some estimations concerning revival times and approximate solutions.

A. Preliminaries

Apart from the distance, we shall consider also two additional quantities: the atomic inversion and the von Neumann entropy. The atomic inversion I of the system is defined as

$$I = \langle \Psi(t) | (|e\rangle\langle e| - |g\rangle\langle g|) | \Psi(t) \rangle. \quad (4.1)$$

The von Neumann entropy S of a system with a density operator $\hat{\rho}$ is given by

$$S := -\text{Tr}(\hat{\rho} \ln \hat{\rho}). \quad (4.2)$$

The entropy of a system in a pure state is always zero [34]. But for a system composed of two interacting subsystems it is usually greater than zero for each subsystem, even if the whole system remains in the pure state. In this case, it follows from the Araki-Lieb inequalities that entropies of both subsystems are equal one to another. As a result we have

$$S(t) = 0, \quad S_A(t) = -\pi^+ \ln \pi^+ - \pi^- \ln \pi^- = S_F(t). \quad (4.3)$$

To understand qualitatively results for distances it is instructive to look at approximate solutions of the JCM with special initial conditions, which are valid for large intensities of the radiation field [11]:

$$|\Psi^\pm(t)\rangle \approx \exp(-i\hat{H}_0 t) \frac{1}{\sqrt{2}} \left[\pm \exp\left(\mp i \frac{\lambda t}{2\sqrt{\bar{n}}}\right) |e\rangle + |g\rangle \right] \otimes \sum_{n=0}^{\infty} \sqrt{p_n} \exp(\mp i \lambda t \sqrt{n}) |n\rangle, \quad (4.4)$$

where $p_n = \exp(-\bar{n}) \bar{n}^n / n!$. For $t = 0$, Eq. (4.4) reduces to

$$|\Psi^\pm(0)\rangle = \frac{1}{\sqrt{2}} (\pm |e\rangle + |g\rangle) \otimes |\alpha\rangle. \quad (4.5)$$

The atomic wave functions of these initial states represent the eigenstates of the semiclassical Hamiltonian, which describes the interaction of a two-level atom with a classical monochromatic radiation field. As it is shown in Eq. (4.4), the corresponding states of the whole system will be disentangled approximately at all times. Initial states of the system can be written as a linear combina-

tion of the states $|\Psi^\pm(0)\rangle$. Therefore Eq. (3.6) can be approximated by

$$|\Psi(t)\rangle \approx \frac{1}{\sqrt{2}} \left[\cos(\sigma/2) + \sin(\sigma/2)e^{i\tau} \right] |\Psi^+(t)\rangle + \frac{1}{\sqrt{2}} \left[-\cos(\sigma/2) + \sin(\sigma/2)e^{i\tau} \right] |\Psi^-(t)\rangle. \quad (4.6)$$

In this paper we consider the following atomic reference states:

$$|\tilde{A}\rangle = \cos(\tilde{\sigma}/2)e^{i\tilde{\nu}}|e\rangle + \sin(\tilde{\sigma}/2)e^{i\tilde{\tau}}|g\rangle. \quad (4.7)$$

For the field we take coherent reference states:

$$|\tilde{F}\rangle = |\tilde{\alpha}\rangle, \quad \tilde{\alpha} = \sqrt{\tilde{n}}e^{i\tilde{\phi}}. \quad (4.8)$$

The field parts of the scalar product of $|\Psi^\pm(t)\rangle$ with the reference state $|\tilde{\Psi}(t)\rangle$ have revival times t_μ^\pm given by

$$\lambda t_\mu^\pm = 2\sqrt{\tilde{n}}(2\pi\mu \mp \tilde{\phi}), \quad \mu = 0, 1, \dots \quad (4.9)$$

In contrast, the revival times t_ν of the atomic inversion I are given by

$$\lambda t_\nu = 2\pi\nu\sqrt{\tilde{n}}, \quad \nu = 1, 2, \dots \quad (4.10)$$

At half of the first revival time of the atomic inversion I we have

$$|\Psi^\pm(t_1/2)\rangle \approx \exp(-i\hat{H}_0 t_1/2) \frac{1}{\sqrt{2}} (-i|e\rangle + |g\rangle) \otimes \sum_{n=0}^{\infty} \sqrt{P_n} \exp(\mp i\pi\sqrt{n\tilde{n}})|n\rangle. \quad (4.11)$$

It shows that independently of the initial atomic state, the state of the atomic system at half of the revival time is approximately given by $-i|e\rangle + |g\rangle$ (see, e.g., [11]). Following [13], we call this state the atomic attractor state. For a method to calculate revival times see, e.g., [9].

B. Numerical results

Using different sets of parameters for the initial state of the system and for the reference state we calculate numerically distances $D(t)$, $D_A(t)$, and $D_F(t)$ as well as the atomic inversion $I(t)$ and the von Neumann entropy $S_A(t)$ of the atom. The latter is equal to the entropy of the field $S_F(t)$. All results are plotted as functions of scaled time λt . For each set of parameters three pictures are displayed. In the first picture we present the atomic inversion $I(t)$. The second one shows the von Neumann entropy $S_A(t)$. The third picture contains distances for the atom $D_A(t)$, the field $D_F(t)$, and the whole system $D(t)$. In all cases, curves denoted by *a*, *b*, and *c* correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively. As we mentioned, the distance $D(t)$ is always greater than or equal to $D_A(t)$ and $D_F(t)$. We will see that in most cases also $D_F(t)$ is greater than $D_A(t)$. It is not, however, a general rule. All figures are plotted for a mean photon number $\tilde{n} = 25$.

In Fig. 1, the reference state is equal to the initial

state of the system, and the atom is initially fully inverted. At half of the first revival time of the atomic inversion, $\lambda t_1/2 = \pi\sqrt{\tilde{n}}$, the von Neumann entropy has a local minimum: states of both subsystems are as close as possible to some pure states. However, this does not mean that these states are as close as possible to *initial* states. In fact, both the atomic and the field distances have at this time well-developed maxima. The field distance and, similarly, the distance of the whole system exhibit pronounced revivals at two times the revival time of the atomic inversion, in agreement with Eq. (4.9). The atomic distance “oscillates” with a double frequency compared to the whole system. The field distance follows more closely than the atomic distance, the behavior of the whole system.

In Figs. 2–4, reference states are also equal to initial states, but now the atom is prepared in some coherent superpositions of ground and excited states. These states enter with equal weights, but phases are different for different figures. The initial state in Fig. 2 corresponds to the state $|\Psi^-(0)\rangle$ in Eq. (4.5). This state is one of the eigenstates of the semiclassical Jaynes-Cummings Hamiltonian and, as a result, the atomic inversion remains close to zero and shows small intrinsic oscillations only. Also because the state of the whole system is approximately disentangled some time after $t = 0$, the von Neumann

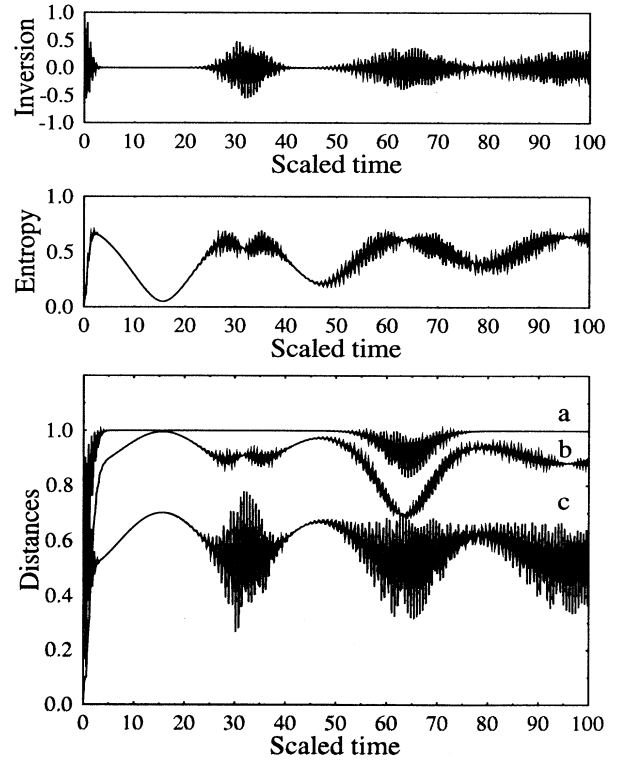


FIG. 1. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\tilde{n} = 25$, $\sigma = 0$, and $\tau = 0$, i.e., the atom is initially fully inverted. The reference state is equal to the initial state. Curves *a*, *b*, and *c* correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

entropy rises rather slowly. Taking $|\Psi^+(0)\rangle$ as the initial state we obtain a very similar picture. Quite different behavior is seen in Fig. 3. Here the atomic initial state corresponds to the atomic attractor state [11–14]. This state is (approximately) reached in the collapse region, after half of the first revival time of the atomic inversion, practically independently from the initial atomic state. Thus in this case we know *a priori* that at the time $\lambda t_1/2 = \pi\sqrt{\bar{n}}$ the atom is very close to its *initial* state. Indeed, the atomic distance $D_A(t)$ shows a pronounced minimum at this time. It clearly confirms that we use a reasonable measure for the distance. In Fig. 4, the initial atomic state is orthogonal to the attractor state and the atomic distance shows a pronounced maximum, as expected. Let us note that, in contrast, inversion and entropy are not able to distinguish between these two cases.

In Figs. 5–7, atomic and field components of reference states are different from the initial state. In all cases, the latter is determined by $\sigma = \pi/2$ and $\tau = 0$, i.e., we use $|\Psi^+(0)\rangle$. In Fig. 5, parameters defining the reference state are $\tilde{\sigma} = \sigma$, $\tilde{\nu} = -1$, and $\tilde{\phi} = 2$. The first minimum of the atomic distance $D_A(t)$ occurs at

a time smaller than one-half of the first revival time of the atomic inversion. The field distance has its first minimum at $\lambda t_1^+ = 2\sqrt{\bar{n}}(2\pi - 2)$, according to Eq. (4.9). The minus sign in that equation must be used, because the initial state of the system is $|\Psi^+(0)\rangle$. In Fig. 6, we change one of the parameters defining the reference state a little, taking $\tilde{\phi} = -2$. The first minimum of $D_F(t)$ occurs at $\lambda t_0^+ = 2\sqrt{\bar{n}}(0 + 2) = 20$. It is seen from Fig. 5 and Fig. 6 that the minimum of the field distance is more pronounced at smaller times. This fact can be explained as follows. The field states in Eq. (4.4) can be further approximated for small times compared with $\lambda t_1/2 = \pi\sqrt{\bar{n}}$ to be coherent states with amplitudes $\sqrt{\bar{n}} \exp[\mp i\lambda t/(2\sqrt{\bar{n}})]$, and this approximation is the better the smaller the scaled time λt is. Now this coherent state with the minus sign is equal to the field reference state at the first revival time $\lambda t_0^+ = 2\sqrt{\bar{n}}(0 + 2) = 20$ of the field distance $D_F(t)$, and the minus sign must be used, because the initial state of the system is again $|\Psi^+(0)\rangle$. To plot Fig. 7, we use another set of parameters: $\tilde{\nu} = -1/4$ and $\tilde{\phi} = -1/2$. A comparison of Fig. 7 with Fig. 5 and Fig. 6 shows a small shift of the extrema of $D_A(t)$. The first minimum of D_F , which is very pronounced, is now situated at $\lambda t_0^+ = 2\sqrt{\bar{n}}(0 + 1/2) = 5$.

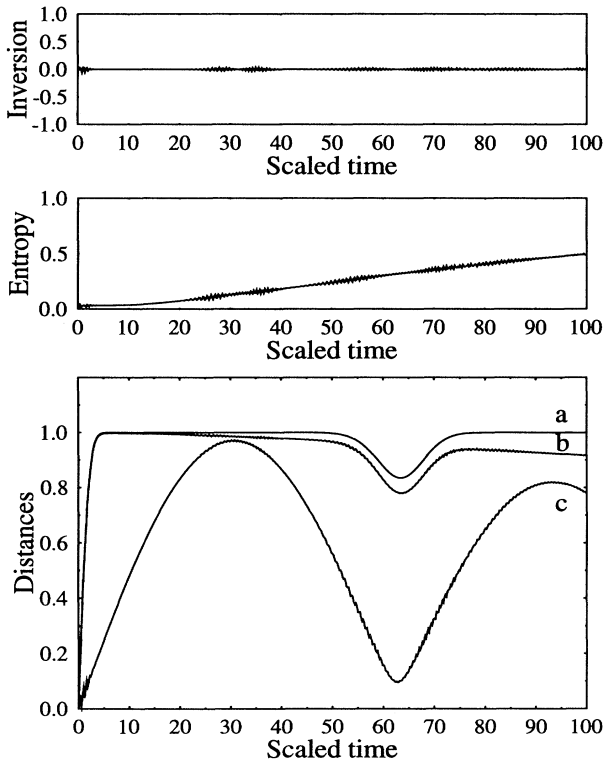


FIG. 2. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The initial state of the system is defined by $\sigma = \pi/2$ and $\tau = \pi$, i.e., the atom is initially in the eigenstate $|\Psi^-(0)\rangle$ of the semiclassical Hamiltonian. The reference state is equal to the initial state. Curves a, b, and c correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

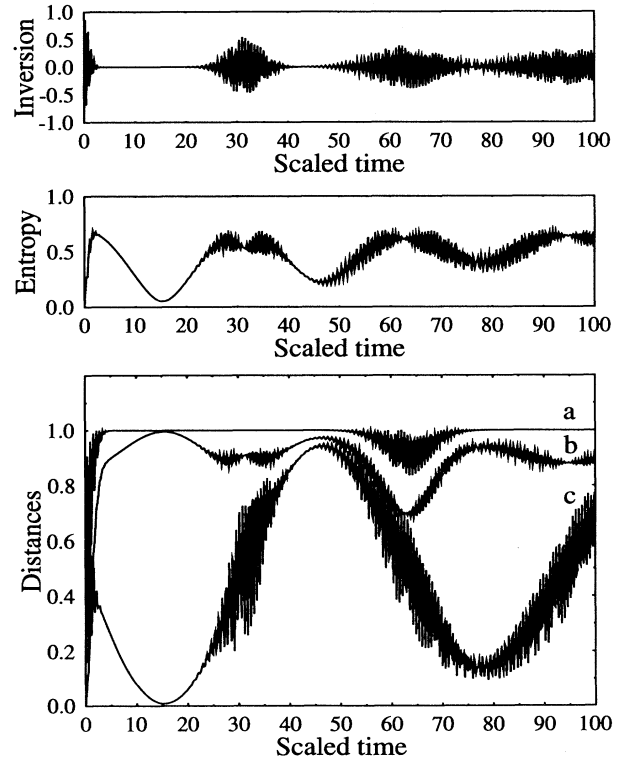


FIG. 3. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\sigma = \pi/2$ and $\tau = \pi/2$, i.e., the atom is initially in the attractor state. The reference state is equal to the initial state. Curves a, b, and c correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

V. CONCLUDING REMARKS

We have considered a reasonable measure for the distance between two arbitrary density operators, based on the Hilbert-Schmidt norm. Some general properties of this definition as well as its advantage compared to the distance based on the trace norm have been discussed. The introduced distance has been applied to the Jaynes-Cummings model. We have compared density operators resulting in this model with special reference states for the whole system, the atomic subsystem, and the field subsystem. Reference states have been chosen as the initial states of the system or other states of interest. We have shown that the problem of determining when and to what extent a given state is close to the initial state is not necessarily equivalent to the problem of when and to what extent it is close to any pure state, even if we start evolution from pure states from the beginning. We have proven that the distance can provide more detailed information about the system behavior than some global quantities such as, e.g., the von Neumann entropy. It could be of interest to perform similar calculations for other systems, such as, for example, the two-photon Jaynes-Cummings model. Properties of other definitions

of the distance and their applications to the JCM are also worth exploration. These and related problems are currently under consideration and results will be presented elsewhere.

APPENDIX

The only purpose of this appendix is to collect definitions and important results about the JCM used in our study. It makes the paper self-contained. Up to some details it is based on [10,13,14]. The time-dependent solution of the Schrödinger equation with initial conditions (3.3)–(3.5) is given by

$$|\Psi(t)\rangle = |g\rangle \otimes |F_g(t)\rangle + |e\rangle \otimes |F_e(t)\rangle, \quad (\text{A1})$$

where the field states $|F_g(t)\rangle$ and $|F_e(t)\rangle$ are defined as

$$|F_g(t)\rangle = (q(t)\hat{C}'(t) + p\hat{S}(t))|F(t)\rangle, \quad (\text{A2})$$

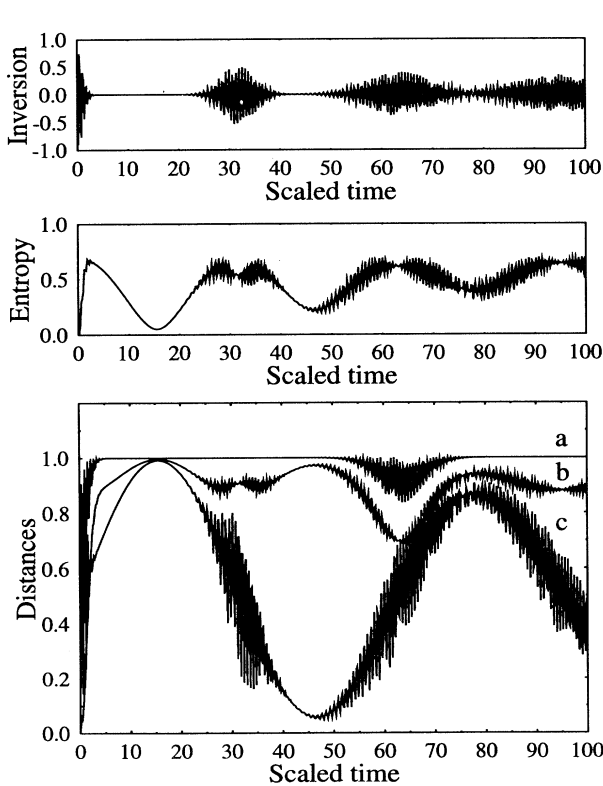


FIG. 4. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\sigma = \pi/2$ and $\tau = -\pi/2$, i.e., the initial atomic state is orthogonal to the attractor state. The reference state is equal to the initial state. Curves *a*, *b*, and *c* correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

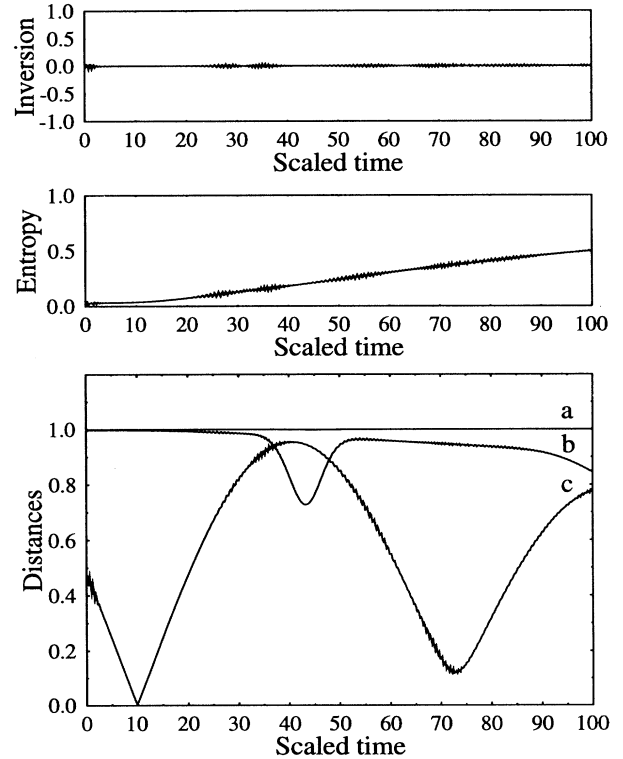


FIG. 5. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\sigma = \pi/2$ and $\tau = 0$, i.e., the atom is initially in the eigenstate $|\Psi^+(0)\rangle$ of the semiclassical Hamiltonian. The reference state is now different from the initial state: $\tilde{\sigma} = \pi/2$, $\tilde{\nu} = -1$, and $\tilde{\phi} = 2$. Curves *a*, *b*, and *c* correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

$$|F_e(t)\rangle = [p\hat{C}(t) + q(t)\hat{S}'(t)]|F(t)\rangle, \quad (\text{A3})$$

with

$$\hat{C} = \cos(\lambda t \sqrt{\hat{a}\hat{a}^\dagger}), \quad \hat{C}' = \cos(\lambda t \sqrt{\hat{a}^\dagger\hat{a}}), \quad (\text{A4})$$

$$\hat{S} = -i\hat{a}^\dagger \frac{\sin(\lambda t \sqrt{\hat{a}\hat{a}^\dagger})}{\sqrt{\hat{a}\hat{a}^\dagger}}, \quad \hat{S}' = -i\hat{a} \frac{\sin(\lambda t \sqrt{\hat{a}^\dagger\hat{a}})}{\sqrt{\hat{a}^\dagger\hat{a}}}, \quad (\text{A5})$$

and

$$|F(t)\rangle = \exp(-\bar{n}/2) \sum_{n=0}^{\infty} \frac{[\alpha \exp(-i\omega t)]^n}{\sqrt{n!}} |n\rangle, \quad (\text{A6})$$

$$p = \cos(\sigma/2), \quad q(t) = \sin(\sigma/2) \exp[i(\tau + \omega t)]. \quad (\text{A7})$$

The solution $|\Psi(t)\rangle$ can be written with the help of orthonormal atomic and field states as follows:

$$|\Psi(t)\rangle = |A^+\rangle \otimes \sqrt{\pi^+} |F^+\rangle + |A^-\rangle \otimes \sqrt{\pi^-} |F^-\rangle, \quad (\text{A8})$$

where the following definitions are used:

$$|A^\pm\rangle = \mathcal{N}_A^\pm \exp\left(-\frac{1}{2}(i\Phi \mp \Theta)\right) \{|e\rangle \pm \exp[(i\Phi \mp \Theta)]|g\rangle\}, \quad (\text{A9})$$

$$|F^\pm\rangle = \mathcal{N}_F^\pm \exp\left(\frac{1}{2}(i\Phi \pm \Theta)\right) \times \{|F_e(t)\rangle \pm \exp[-(i\Phi \pm \Theta)]|F_g(t)\rangle\}. \quad (\text{A10})$$

The quantities not yet defined have the following meaning:

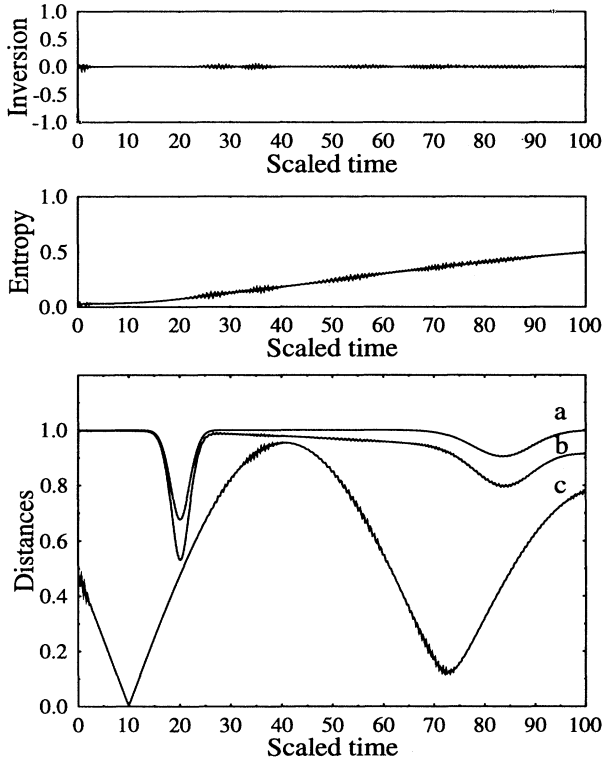


FIG. 6. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\sigma = \pi/2$ and $\tau = 0$, i.e., the atom is initially in the eigenstate $|\Psi^+(0)\rangle$ of the semiclassical Hamiltonian. The reference state is now different from the initial state: $\bar{\sigma} = \pi/2$, $\bar{\nu} = -1$, and $\bar{\phi} = -2$. Curves a , b , and c correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

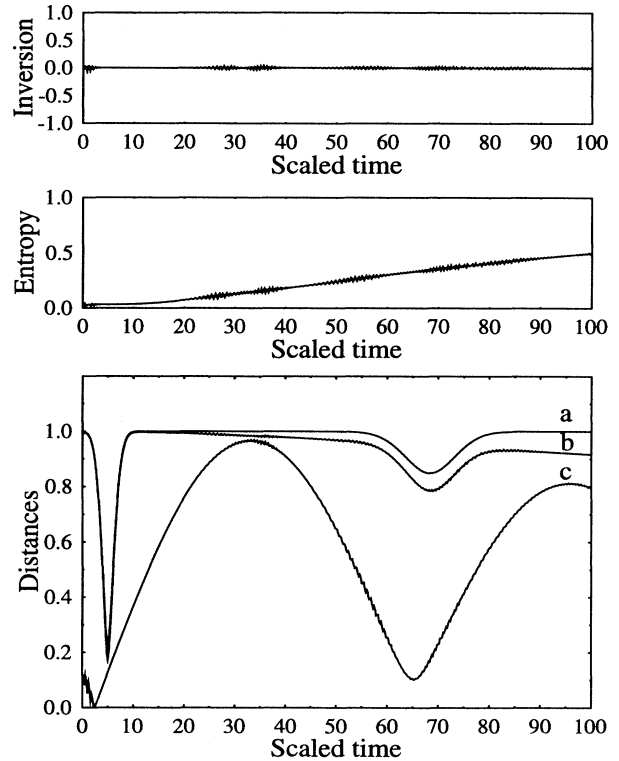


FIG. 7. The atomic inversion $I(t)$, the von Neumann atomic entropy $S_A(t)$, and distances as functions of scaled time λt . The parameters of the initial state are $\sigma = \pi/2$ and $\tau = 0$, i.e., the atom is initially in the eigenstate $|\Psi^+(0)\rangle$ of the semiclassical Hamiltonian. The reference state is now different from the initial state: $\bar{\sigma} = \pi/2$, $\bar{\nu} = -1/4$, and $\bar{\phi} = -1/2$. Curves a , b , and c correspond to $D(t)$, $D_F(t)$, and $D_A(t)$, respectively.

$$\exp(i\Phi) := \frac{\langle F_e(t)|F_g(t)\rangle}{|\langle F_e(t)|F_g(t)\rangle|}, \quad (\text{A11})$$

$$\Omega := \frac{\langle F_e(t)|F_e(t)\rangle - \langle F_g(t)|F_g(t)\rangle}{|\langle F_e(t)|F_g(t)\rangle|},$$

$$\Theta := \sinh^{-1}(\Omega/2), \quad (\text{A12})$$

$$\begin{aligned} \pi^\pm &:= \langle F_e(t)|F_e(t)\rangle \pm \exp(\mp\Theta)|\langle F_e(t)|F_g(t)\rangle| \\ &= \langle F_g(t)|F_g(t)\rangle \pm \exp(\pm\Theta)|\langle F_e(t)|F_g(t)\rangle|, \end{aligned} \quad (\text{A13})$$

$$\mathcal{N}_A^\pm := [2 \cosh(\Theta)]^{-1/2}, \quad \mathcal{N}_F^\pm := [2\pi^\pm \cosh(\Theta)]^{-1/2}. \quad (\text{A14})$$

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