Classical equations for quantum squeezing and coherent pumping by the time-dependent quadratic Hamiltonian

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The time evolution, under the general time-dependent quadratic quantum Hamiltonian, of mean values of a class of observables that includes the field quadratures and their dispersions, is obtained exactly in classical Hamiltonian form. Quantum states are described in terms of density operators, so that arbitrary initial states (both pure states and statistical mixtures) are allowed.

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I. INTRODUCTION

The general quadratic Hamiltonian

$$H = f_1 a^\dagger a + f_2 a^\dagger a^\dagger + f_2^* aa + f_3 a^\dagger + f_3^* a \tag{1}$$

with coefficients f_i which may be time dependent and standard bosonic creation and annihilation operators $a^\dagger, a$ is an object of notorious utility [1] particularly in connection with the quantum modeling of the pushing and squeezing of the electromagnetic field of a single mode in a high-Q cavity. Several treatments of the corresponding dynamics, developed in order to bring under control the various effects which can be achieved by driving the parameter functions f_i, are now available [1, 2]. The purpose of this paper is to describe yet another way of approaching this matter which has on its account the following distinctive features. (a) It shows explicitly that the time evolution of the mean values of the field quadratures and of the corresponding dispersions is given exactly in terms of the canonical equations of independent c-number Hamiltonians. These Hamiltonians are expressed in terms of the f_i in addition to one extra non-negative parameter ν which is a constant of motion of (1) determined by the initial state of the field. (b) It uses density matrix language to express quantum states, so that both pure states and statistical mixtures can be handled equally easily. This feature, in particular, may prove useful for the inclusion of the effects of dissipation which are not considered here.

The treatment given below is in fact a simple application of techniques developed before for the treatment of the reduced (in general, nonunitary) dynamics of Gaussian observables of interacting many-boson systems [3–5]. An essential simplifying feature in the present context is the nonautonomous but *noninteracting* character of the general quadratic Hamiltonian (1). Although a definite option is made here to assure as much self-containedness as possible, the reader is directed to the earlier work for additional technical details.

II. QUADRATURES AND DISPERSIONS AS GAUSSIAN OBSERVABLES

The state of the field is generally described in terms of a density operator F (with unit trace) in the Schrödinger picture. Units chosen so that $\hbar = 1$ are used throughout. The mean values q, p of the usual field quadratures relative to a scale parameter μ_0 are then introduced as

$$\langle a\rangle = \mathrm{Tr}\, a F = \sqrt{\frac{\mu_0}{2}}\left(q + i\,\frac{p}{\mu_0}\right) . \tag{2}$$

This allows for the definition of displaced boson operators $b, b^\dagger$ as

$$b \equiv a - \langle a\rangle\ , \quad \left[b, b^\dagger\right] = 1\ ,$$

which can be further Bogolyubov transformed to the operators $\eta, \eta^\dagger$:

$$b \equiv x\eta - y^*\eta^\dagger\ , \quad \left[\eta, \eta^\dagger\right] = 1\ . \tag{3}$$

The preservation of the commutation relations requires as usual that the transformation coefficients x and y be chosen so that $|x|^2 - |y|^2 = 1$. This last transformation will be chosen so that one has

$$\mathrm{Tr}\ \eta\eta F = 0\ .$$

The corresponding coefficients x and y can be determined as follows: Consider the extended one-boson plus pairing density matrix [3, 4],

$$\underline{R} = \begin{pmatrix} \langle a^\dagger a\rangle - |\langle a\rangle|^2 & \langle aa\rangle - \langle a\rangle^2 \\ \langle aa\rangle^* - \langle a\rangle^{*2} & 1 + \langle a^\dagger a\rangle - |\langle a\rangle|^2 \end{pmatrix}\ ,$$

where the brackets denote average values taken with F, and solve the secular problem,

$$\underline{G}\,\underline{R}\,\underline{X} = \underline{X}\,\underline{G}\,\underline{N}\ ,$$

where

$$\underline{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \quad \underline{X} = \begin{pmatrix} x^* & y \\ y^* & x \end{pmatrix}\ ,$$

and

$$\underline{N} = \begin{pmatrix} \nu & 0 \\ 0 & 1+\nu \end{pmatrix}\ .$$

It follows from this that

$$\mathrm{Tr}\ \eta^\dagger\eta\, F = \nu\ ,$$

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so that ν gives the average number of η bosons in F and, consequently, $\nu \geq 0$.

At this point it is useful to introduce a truncated density operator F_0 (also of unit trace) which is completely determined once the quantities p, q, x, y, and ν are given [3, 4]. This operator is constructed so that it reproduces the mean values of the quadratures and of the operators $a^\dagger a$ and aa taken with respect to the full density F. Since it is fully determined by these quantities, irreducible parts of three or more boson operators (e.g., $\langle a^\dagger a\, a^\dagger a\rangle$) are truncated away. The density operator F_0 will, therefore, be called the Gaussian projection of the full density F. It is given, in general, as [3]

$$F_0 = \frac{1}{1+\nu}\left(\frac{\nu}{1+\nu}\right)^{\eta^\dagger \eta} . \tag{4}$$

This density can, in fact, be expressed in the form

$$F_0 = P(t)F \ ,$$

where $P(t)$ is a time-dependent projector [3, 4] which can be explicitly written as

$$P(\cdot) = P^2(\cdot) = \left\{ \left[1 - \frac{\eta^\dagger\eta - \nu}{1+\nu}\right] \mathrm{Tr}(\cdot) + \frac{\eta^\dagger\eta - \nu}{\nu(1+\nu)} \mathrm{Tr}\,(\eta^\dagger\eta\cdot) + \frac{\eta}{\nu}\,\mathrm{Tr}\,(\eta^\dagger\cdot)\, \frac{\eta^\dagger}{1+\nu}\mathrm{Tr}\,(\eta\cdot) + \frac{\eta\eta}{2\nu^2}\,\mathrm{Tr}\,(\eta^\dagger\eta^\dagger\cdot) + \frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}\mathrm{Tr}\,(\eta\eta\cdot) \right\} F_0. \tag{5}$$

The time dependence of P results, in general, from the time dependence of p, q, x, y, and ν. An important property of this operator is that

$$\dot{P}(t)F = 0 \tag{6}$$

so that one has $\dot{F}_0 = P(t)\dot{F}$. These time derivatives involve the dynamic evolution generated by H.

As a result of this construction, the mean value of any observables which is at most quadratic in the operators $a, a^\dagger$ can be fully retrieved from just the Gaussian projection F_0 of the full density. These observables will, therefore, be referred to as Gaussian observables. The general quadratic Hamiltonian and the quadratures q and p are clearly Gaussian observables, the same being true also for the dispersions of the quadratures. A straightforward calculation gives, in fact,

$$\langle H\rangle = h_q + h_{\mathrm{sq}} \ , \tag{7}$$

with

$$h_q = f_1|\langle a\rangle|^2 + f_2\langle a\rangle^{*^2} + f_2^*\langle a\rangle^2 + f_3\langle a\rangle^* + f_3^*\langle a\rangle \ , \tag{8}$$

and

$$h_{\mathrm{sq}} = f_1\left[\frac{1}{2} + |y|^2 + \left(|x|^2 + |y|^2\right)\nu\right] - \left[f_2 x^* y + f_2^* x y^*\right](1+2\nu) \ . \tag{9}$$

For the dispersions of the quadratures, one gets

$$\Delta q^2 = \frac{1}{2\mu_0}[\langle (a^\dagger + a)^2\rangle - (\langle a\rangle^* + \langle a\rangle)^2] = \frac{1}{2\mu_0}[1 + 2|x|^2\nu + 2|y|^2(1+\nu) - (1+2\nu)\,(x^*y + xy^*)] \tag{10}$$

and

$$\Delta p^2 = -\frac{\mu_0}{2}[\langle (a^\dagger - a)^2\rangle - (\langle a\rangle^* - \langle a\rangle)^2] = \frac{\mu_0}{2}[1 + 2|x|^2\nu + 2|y|^2(1+\nu) + (1+2\nu)\,(x^*y + xy^*)] \ . \tag{11}$$

III. EQUATIONS OF MOTION FOR GAUSSIAN OBSERVABLES

The next step is to consider the time evolution of the mean values of Gaussian observables under the general quadratic Hamiltonian (1) in the context of the initial-value problem associated with the Liouville–von Neumann equation,

$$i\dot{F} = [H(t), F] \ .$$

If Γ is a time-independent Gaussian observable, one has

$$i\,\frac{d\langle\Gamma\rangle}{dt} = i\mathrm{Tr}\Gamma\dot{F} = \mathrm{Tr}\,[\Gamma, H(t)]\,F = \mathrm{Tr}\,[\Gamma, H(t)]\,(F_0 + F') \ , \tag{12}$$

where use has been made of the cyclic property of traces and F has been written in terms of F_0 and of the remainder traceless part F'. As stated in the Introduction, important simplifications occur in the case of the quadratic Hamiltonian H. In fact, in this case $\dot{\Gamma}(t) = -i\,[\Gamma, H(t)]$ is itself a Gaussian observable, and, consequently, one has

$$\mathrm{Tr}\ \dot{\Gamma}(t)F = \mathrm{Tr}\ \dot{\Gamma}(t)F_0 \ ,$$

so that the term involving $F'(t)$ in Eq. (12) vanishes.

The time evolution of $\langle\Gamma\rangle$ reduces, therefore, just to

$$i\,\frac{d\langle\Gamma\rangle}{dt} = \mathrm{Tr}\,[\Gamma, H(t)]\,F_0 \ . \tag{13}$$

This result can, furthermore, be immediately extended to Gaussian observables which are explicitly time dependent, but have time derivatives which are also Gaussian observables. In this case, Eq. (13) acquires an extra term

$i\,\mathrm{Tr}\,\frac{\partial \Gamma}{\partial t}\,F_0$.

The task of obtaining equations of motion for the quadratures and their dispersions starting from Eq. (13) is now a straightforward algebraic exercise. For the quadratures p and q take $\Gamma = a$. Substituting this in Eq. (13) and separating real and imaginary terms, one gets

$$\dot q = (f_1 - 2\mathrm{Re} f_2)\,\frac{p}{\mu_0} + 2\mathrm{Im} f_2 q + \sqrt{\frac{2}{\mu_0}}\mathrm{Im} f_3 \;,$$
$$\dot p = -\,(f_1 + 2\mathrm{Re} f_2)\,\mu_0 q + 2\mathrm{Im} f_2 p + \sqrt{2\mu_0}\,\mathrm{Re} f_3 \;. \quad (14)$$

It can easily be checked that these equations can be written in terms of h_q, Eq. (8), as

$$\dot q = \frac{\partial h_q}{\partial p} \;,\;\; \dot p = -\frac{\partial h_q}{\partial q} \quad (15)$$

so that h_q, expressed in terms of p and q, plays the role of a c-number Hamiltonian which generates the time evolution of the quadratures. As for the dispersions, since these are given in terms of x, y, and ν, take $\Gamma = \eta^\dagger\eta^\dagger$. In this case one gets [4]

$$i(\dot x y - x \dot y)(1+2\nu) = \mathrm{Tr}\,\left[\eta^\dagger\eta^\dagger, H_0\right] F_0$$
$$= 2 f_1\, xy - 2 f_2 y^2 - 2 f_2^* x^2 \;. \quad (16)$$

This can be cast in neater form by reparametrizing the Bogolyubov transformation as [3]

$$x = \cosh\sigma + \frac{i\tau}{2} \;,\;\; y = \sinh\sigma + \frac{i\tau}{2} \;,$$

which automatically fulfills the condition $|x|^2 - |y|^2 = 1$. Furthermore, introducing new variables P and Q as [5]

$$P = \tau\sqrt{\frac{1+2\nu}{2}\,\mu_0} \;,\;\; Q = e^{-\sigma}\sqrt{\frac{1+2\nu}{2\mu_0}} \;,$$

one obtains from Eq. (16),

$$\dot Q = (f_1 - 2\mathrm{Re} f_2)\,\frac{P}{\mu_0} + 2\mathrm{Im} f_2\, Q \;,$$

$$\dot P = -\,(f_1 + 2\mathrm{Re} f_2)\,\mu_0 Q + \left(f_1 - 2\mathrm{Re} f_2\right)\frac{(1+2\nu)^2}{4\mu_0 Q^3} - 2\mathrm{Im} f_2\, P \;, \quad (17)$$

which can also be written in terms of $h_{\rm sq}$, Eq. (9), as

$$\dot Q = \frac{\partial h_{\rm sq}}{\partial P} \;,\;\; \dot P = -\frac{\partial h_{\rm sq}}{\partial Q} \;. \quad (18)$$

This shows that $h_{\rm sq}$, expressed in terms of P and Q, plays the role of a c-number Hamiltonian which generates the time evolution of the squeezing variables P and Q. The dispersions Δq^2 and Δp^2 written in terms of these variables appear as

$$\Delta q^2 = Q^2 \;, \quad (19)$$

$$\Delta p^2 = P^2 + \frac{(1+2\nu)^2}{4Q^2} \;, \quad (20)$$

so that the uncertainty product $\Delta\, p \Delta q$ is

$$\Delta p\,\Delta q = \sqrt{\frac{(1+2\nu)^2}{4} + P^2\,Q^2} \geq \frac{1+2\nu}{2} \;. \quad (21)$$

Finally, one must consider the possible time dependence of ν. This is obtained from [4]

$$i\dot\nu = \mathrm{Tr}\,\left[\eta^\dagger\eta, H\right] F_0 = \mathrm{Tr}\, H\left[F_0, \eta^\dagger\eta\right] \;.$$

Since, however, F_0 commutes with $\eta^\dagger\eta$, it follows that $\dot\nu = 0$, i.e., ν is a constant of motion whose value is determined by the particular choice of initial conditions for the density F. Inspection of Eqs. (17) shows, moreover, that the effect of changing the value of ν reduces to a simple rescaling of the functions $Q(t), P(t)$. Let, in fact, $Q(\nu;t)$ and $P(\nu;t)$ be the solutions of these equations for a given value of ν and initial conditions $Q(\nu;0), P(\nu;0)$; it can then be checked immediately that the scaled functions,

$$Q(\nu';t) = \sqrt{\frac{1+2\nu'}{1+2\nu}}\,Q(\nu;t) \;,$$

$$P(\nu';t) = \sqrt{\frac{1+2\nu'}{1+2\nu}}\,P(\nu;t) \;,$$

will also be solutions, corresponding to the value ν' and similarly scaled initial conditions.

IV. DISCUSSION AND NUMERICAL EXAMPLES

The foregoing development has shown that equations of motion determining the time development of Gaussian observables, particularly the quadratures and their dispersions, can be obtained in closed c-number Hamiltonian form for the quantum dynamics generated by the general quadratic Hamiltonian, Eq. (1), and an arbitrary initial state described by a density operator $F(0)$. The quadratures and their dispersions are found from independent sets of equations which are related to each other via their common dependence on the c-number parameters of the underlying quantum quadratic Hamiltonian. The equations that describe the squeezing motion involve also the value of the constant of motion ν which is fixed by the initial state under consideration.

Concerning this quantity, it is easily verified that an initial state given by a density of the form F_0, Eq. (4) [i.e., a density $F(0)$ such that $F'(0) = 0$], will be a pure state (in the sense of having an idempotent density matrix) only when $\nu = 0$. These pure Gaussian states are, in general, one- or two-photon (squeezed) coherent states of the field. Situations in which $\nu \neq 0$ corresponds either to Gaussian mixed states [when $F'(0) = 0$] or to pure or mixed states which are correlated in the sense of giving rise to irreducible contributions to mean values of non-Gaussian observables. The particular form of the quadratic Hamiltonian (in fact, its Gaussian character) leads, however, to the decoupling of the dynamics of Gaussian and non-Gaussian observables, so that the former can still be fully retrieved in closed form from F_0 alone.

For the special case of a coherent state $|v\rangle$, constructed as usual as an eigenstate of the annihilation operator a,

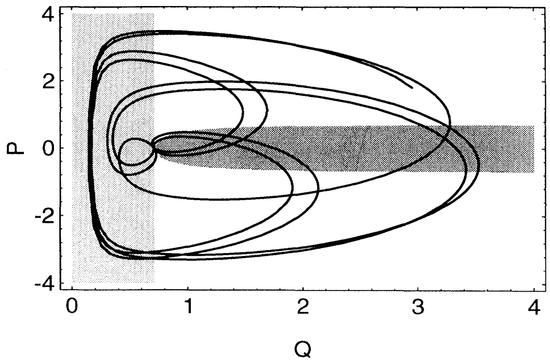

FIG. 1. Phase space (P,Q) trajectory obtained by numerical integration of Eqs. (17) for parameter functions $f_1 = \omega_0$ and $f_2 = 0.2e^{i\omega t}$ with $\omega_0 = \omega\sqrt{2}$ and $\nu = 0$. Initial conditions are $P = 0$, $Q = \sqrt{2}/2$. The scale parameter μ_0 is set equal to one. The highlighted areas correspond to the q- and p-squeezed domains defined in Eq. (22).

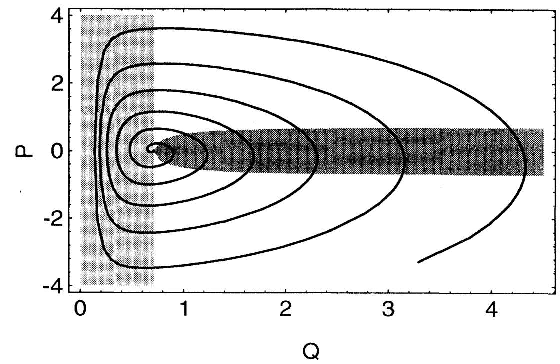

FIG. 2. Same as Fig. 1 but with $f_2 = 0.05e^{i\omega t}$ and $\omega_0 = 0.5\omega$.

one has $\nu = 0, x = 1, y = 0$, which gives $P = 0, Q = 1/\sqrt{2\mu_0}$. The dispersions, given by Eqs. (19) and (20), are $\Delta q^2 = \dfrac{1}{2\mu_0}$ and $\Delta p^2 = \dfrac{\mu_0}{2}$, as appropriate, for a minimum uncertainty packet. If, on the other hand, one considers Fock states $|n\rangle$ (eigenstates of $a^\dagger a$), one finds also that they correspond to $x = 1, y = 0$ but now $\nu = n$. For these states, therefore, $P = 0$ and $Q = \sqrt{\dfrac{1+2\nu}{2\mu_0}}$, so that one recovers from Eqs. (19) and (20) the usual values,

$$\Delta q^2 = \frac{1+2\nu}{2\mu_0} \quad \text{and} \quad \Delta p^2 = \frac{1+2\nu}{2}\mu_0 \ .$$

For a given value of ν one can, therefore, define q-squeezed and p-squeezed (with respect to the usual Fock states) domains in the (P,Q) plane, respectively, as

$$Q^2 < \frac{1+2\nu}{2\mu_0} \quad \text{and} \quad P^2 < \frac{1+2\nu}{4}\,\frac{2\mu_0 Q^2 - (1+2\nu)}{Q^2} \ . \tag{22}$$

These domains meet at the point

$$\left(P = 0,\ Q = \sqrt{\frac{1+2\nu}{2\mu_0}}\right) .$$

They are highlighted in Figs. 1 and 2 (for $\nu = 0$), where phase-space trajectories obtained from numerical integration of Eqs. (17) are also shown. The scale factor μ_0 is set equal to one. The initial conditions used are $P = 0$, $Q = \dfrac{\sqrt{2}}{2}$. In both cases the trajectories correspond to a parametric oscillator with $f_1 = \omega_0$ and $f_2 = \kappa e^{i\omega t}$ with $\kappa = 0.3$, $\omega/\omega_0 = \sqrt{2}/2$ and $\kappa = 0.05$, $\omega/\omega_0 = 2$. Note that the equations do not depend on f_3. The time integrations have been carried through to $t = 15\pi/\omega_0$ and $6\pi/\omega_0$, respectively.

As a final comment, it is worth mentioning that, for the trivial case of the simple harmonic oscillator, $f_1 = \omega_0$, $f_2 = f_3 = 0$, one obtains from Eqs. (14) and (17), respectively,

$$\ddot{q} + \omega_0^2 q = 0$$

and

$$\ddot{Q} + \omega_0^2\left[Q - \frac{(1+2\nu)^2}{4\mu_0 Q^3}\right] = 0 \ . \tag{23}$$

For small oscillations dQ of Q about the equilibrium position $Q_0 + \sqrt{\dfrac{1+2\nu}{2\mu_0}}$, Eq. (23) becomes

$$d\ddot{Q} + 4\omega_0^2 dQ = 0 \ .$$

This doubling of the frequency for the squeezing motion can in fact be related to the treatment of harmonic oscillators in terms of symplectic groups given by Goshen and Lipkin [6]. It can be visualized classically by noting that, since for harmonic oscillators the frequency does not depend on the amplitude of the motion, if a set of independent particles in a harmonic field is symmetrically stretched out for equilibrium, it will subsequently pulsate with frequency $2\omega_0$.

V. CONCLUSIONS

Classical Hamiltonian equations of motion have been obtained that describe completely the time evolution of mean values of Gaussian observables of a quantum system governed by the general time-dependent quadratic Hamiltonian, Eq. (1). These equations, which constitute the main result in this paper, are given in Eqs. (14) and (17). Arbitrary initial quantum states are allowed, both pure states and statistical mixtures. In particular, this reduces the study of squeezing and coherent pumping effects for the field quadratures and their dispersions to the classical dynamics of nonautonomous systems. The treatment given avoids the complete solution of the underlying quantum problem by taking advantage of the decoupling of the dynamics of different classes of observables which result from the noninteracting character of the general quadratic Hamiltonian.

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