Centrifugal force: A gedanken experiment

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A simple gedanken experiment, the motion of a bead inside a rotating linear pipe, is considered. It is shown that at large enough velocities a centrifugal force acting on the bead changes its usual sign and *attracts* towards the rotation axis. The motion of the bead is oscillatory, resembling a well-known case of the mathematical pendulum. A possible relevance of this highly idealized problem to the real motion of charged particles in a pulsar magnetosphere is pointed out.

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I. INTRODUCTION

At present an interest in the study of various physical processes related to and provoked by rotational motion in a number of astrophysical problems has an apparent revival. The increase of interest to the phenomenon of rotation is mostly due to series of highly unusual results of Abramowicz with different collaborators [1-4]. They have discovered some, very strange, dynamical effects of rotation in strong gravitational fields of black holes. In particular, they have found that under certain circumstances, the centrifugal force will *attract* towards the rotation axis even in the case of a nonrotating (Schwarzschild) black hole. This outstanding fact leads to the number of bizarre rotational effects considered and discussed in Refs. [1-4].

The main purpose of this paper is to demonstrate that the centrifugal force reversal effect is not characteristic only for black holes but actually can appear also in the framework of *special relativity.* Such a reversal occurs in a relatively plain mechanical case: for a motion of a bead inside a rotating pipe. This particular kind of motion is considered in detail in the next section of the paper. We derive an equation for the motion of the bead and find solutions of this equation. Actually, we find that if initially (at the moment $t = 0$) the bead is situated on the rotation axis ($r_0 = 0$), and has an initial velocity $v_0 \neq 0$, then the motion of the bead is oscillatory: it moves outwards, reaches the point, where its azimuthal velocity in the laboratory frame becomes equal to the speed of light (a "light cylinder" radius $r \equiv r^* = c/\omega$), turns and moves now towards the rotation axis, passes above the rotation axis and moves in the opposite direction just in the same fashion. Such a motion in the case of zero friction is permanent with a constant period and a constant amplitude (maximum separation from the axis). The most important peculiarity of this idealized gedanken experiment is that according to the equation of motion we can easily prove that under certain conditions the centrifugal force acting on the bead changes its sign and becomes "centripetal," i.e., it is attracted towards the rotation axis. In particular, we find that if the initial velocity of the bead satisfies the following condition:

$$\frac{v_0}{c} > \frac{\sqrt{2}}{2}, \tag{1}$$

then the centrifugal force is always negative (i.e., it is negative for all values of the time coordinate t).

Certainly, this model is more mathematical than physical and is highly idealized since no pipe is rigid enough to remain straight when the bead velocity becomes relativistic. Besides, to keep the "pipe+bead" system in the condition of rotation with a constant angular velocity one certainly needs an infinite amount of energy. At the same time, the centrifugal force reversal effect is physically meaningful since for large enough initial velocities v_0 it occurs at $r \ll r^*$. There are several astrophysical cases where this example may be relevant. In pulsar magnetosphere, for instance, magnetic field lines near the pulsar surface act like "pipes" directing the motion of charged particles, which emerge out of the pulsar surface with relativistic initial velocities. These particles move along the field lines corotating, at the same time, with the dipolar magnetic field of the pulsar. Evidently, when the plasma of the pulsar wind approaches the "light cylinder," the field lines do not remain "rigid," but are carried away by the plasma, giving a way to the matter motion and the analogy ceases to work any longer.

In the concluding section of this paper, we have discussed the results obtained and mention various astrophysical situations and objects where these results may be important.

II. MAIN CONSIDERATION

Let us consider a straight, long, and narrow pipe rotating around an axis normal to the pipe, and a small bead which can move inside the pipe without friction. Let the radii of the bead and the pipe be equal to each other. According to classic mechanics the bead will move with acceleration, and if at the moment $t = 0$ the bead is located at the rotation axis ($r_0 = 0$ at $t = 0$) and has an initial velocity v_0, then the radial distance from the axis will vary with time by the simple mathematical law: $r = (v_0/\omega)\sinh(\omega t)$, while the velocity of the bead will, certainly, vary as $v_0 \cosh(\omega t)$. However, if the pipe is long enough and its walls are absolutely rigid, then the increasing velocity sooner or later will become relativistic. It seems evident that further increase of the velocity will somehow be limited since the total velocity of the bead cannot exceed the speed of light. If the bead reaches the light cylinder (at the radial distance r^*) then the radial velocity shall become zero, so that the velocity, which increases at the initial stage of the motion, must decrease near the light cylinder.

Let us consider this motion properly, in a reference frame of the rotating pipe, where geometry and dynamics of the bead motion are the simplest. Hereafter, we shall use geometrical units, in which $G = c = 1$. If the space-time in the laboratory frame is just Minkowskian with the usual metric,

$$ds^2 \equiv -d\tau^2 = -dT^2 + dX^2 + dY^2, \tag{2}$$

then making the following transformation of variables,

$$T = t, \tag{3a}$$

$$X = r\cos(\omega t), \tag{3b}$$

$$Y = r\sin(\omega t), \tag{3c}$$

we can write the metric of the two-dimensional (in the limit of infinitely narrow pipe) space-time inside the pipe, rotating with constant angular velocity ω, which will be of the following simple form:

$$ds^2 \equiv -d\tau^2 = -(1-\omega^2 r^2)dt^2 + dr^2, \tag{4}$$

where τ is a proper time of the moving bead, defined in the usual way [5].

The Lagrangian for the metric (4) may be defined as [5]

$$L \equiv \frac{1}{2} g_{\alpha\beta}\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right) = \frac{1}{2}\left[-(1-\omega^2 r^2)\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2\right]. \tag{5}$$

Components of the corresponding Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha} \tag{6}$$

may be written in the following way:

$$-(1-\omega^2 r^2)\left(\frac{dt}{d\tau}\right) = \text{const} \equiv -E, \tag{7a}$$

$$\frac{d^2 r}{d\tau^2} = \omega^2 r\left(\frac{dt}{d\tau}\right)^2. \tag{7b}$$

A more simple equation, equivalent to (7b), may be derived by combining (7a) with the usual algebraic relation between four-velocities:

$$-(1-\omega^2 r^2)\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 = -1. \tag{8}$$

The result is

$$\left(\frac{dr}{d\tau}\right)^2 = -1 + \frac{E^2}{1-\omega^2 r^2}. \tag{7c}$$

At the other hand, rewriting (8) as

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1-\omega^2 r^2 - (dr/dt)^2}, \tag{9}$$

and taking into account (7a) we can write a first order differential equation for the function $r(t)$ in the following way:

$$\frac{dr}{dt} = \sqrt{(1-\omega^2 r^2)\left[1 - \frac{(1-\omega^2 r^2)}{E^2}\right]}. \tag{10}$$

A constant parameter $E = -U_t$, coming in view in these equations, may be treated as an energy of the moving bead in the reference frame of the rotating pipe, and may be determined through initial conditions. In the most general case, when at the moment $t = 0$ the bead was at the position $r = r_0$ and had a velocity $v = v_0$, from (10) we can easily find out that

$$E = \frac{(1-\omega^2 {r_0}^2)}{\sqrt{1-\omega^2 {r_0}^2 - {v_0}^2}}. \tag{11}$$

Evidently, $0 < E < \infty$. If, for example, $r_0 = 0$ and $v_0 \neq 0$, then $E = (1-v_0^2)^{-1/2}$ and is more than unity ($E \gg 1$, if $v_0 \approx 1$). While, if $v_0 = 0$ and $r_0 \neq 0$, then we see that $E = (1-\omega^2 r_0^2)^{1/2}$ and, certainly, is less than unity ($E \approx 0$ if $\omega r_0 \approx 1$).

It must be noted that in the nonrelativistic limit the energy of the moving bead, specified by (7) and (10), reduces to the following expression:

$$E_{\text{nr}} = 1 + v^2/2 - \omega^2 r^2/2.$$

Deleting a unity which evidently describes the rest mass energy (per unit mass) of the bead we can see that the remaining terms have a clear "nonrelativistic" physical meaning. In particular, $v^2/2$ is a kinetic energy corresponding to radial motion in the rotating reference frame, while $-\omega^2 r^2/2$ is a so-called "centrifugal energy [6]," known in classical mechanics.

Now, we can rewrite (7) with the intention of getting an equation for a radial acceleration of the bead d^2r/dt^2. Taking into account (7a) and (10) we obtain the following equation:

$$\frac{d^2 r}{dt^2} = \frac{\omega^2 r}{1-\omega^2 r^2}\left[1 - \omega^2 r^2 - 2\left(\frac{dr}{dt}\right)^2\right]. \tag{12}$$

This is an equation of motion for the bead under consideration. On the left hand side appears a radial acceleration of the bead as measured in the reference frame of the rotating pipe. However, note that $T = t$ and $R \equiv \sqrt{X^2+Y^2} = r$, so that this acceleration is equal to the one measured in the nonrotating laboratory frame. The same is true, certainly, for the right hand side of the equation. It represents the force acting on the particle. In the nonrelativistic limit it reduces to the conventional expression for the centrifugal force $f_{cf} = w^2 r$. Therefore, generally, the right hand side of (12) may be treated as a generalized expression for relativistic centrifugal force. We shall return to the analyses of the equation below, after finding solutions for the function $r(t)$. It is easily proved that the solution of (10) may be expressed by means of elliptical functions. For this purpose, originally, let us make in (10) the following, useful substitution of variables:

$$\theta \equiv \arccos(\omega r), \tag{13a}$$

$$\lambda \equiv \omega t, \tag{13b}$$

$$m \equiv 1/E^2, \tag{13c}$$

which reduces (10) to the more simple differential equation:

$$\frac{d\theta}{d\lambda} = -\sqrt{1 - m \sin^2 \theta}. \tag{14}$$

The solution of this equation may be represented as

$$\lambda = \int_0^{\varphi_0} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} - \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \tag{15}$$

where, certainly, $\varphi_0 = \arccos(\omega r_0)$.

If we further introduce one more auxiliary notation

$$\lambda^* \equiv \int_0^{\varphi_0} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \tag{16}$$

we can rewrite (15) as

$$\varphi = \mathrm{am}(\lambda^* - \lambda), \tag{17}$$

where am is an amplitude of Jacobian elliptic functions [7]. Accordingly, for $r(t)$ we should have

$$r(t) = \frac{1}{\omega} cn(\lambda^* - \omega t), \tag{18}$$

where cn is the Jacobian elliptic cosine [7]. Surely, (18) is the general solution of the problem. However, hereafter we shall deal with a restricted symmetrical case specified by the initial condition $r = 0$ and $v_0 \neq 0$ at $t = 0$. For this particular case, it is easy to find that $m = 1 - v_0^2$ and $\lambda^* = K$, where K is a complete elliptical integral of the first kind [7]. These circumstances allow us to rewrite (18) in a somehow different form:

$$r(t) = \frac{1}{\omega} cn(K - \omega t) = \left(\frac{v_0}{\omega}\right) \frac{sn(\omega t)}{dn(\omega t)}, \tag{19}$$

where sn and dn are a Jacobian elliptical sine and a modulus, respectively [7]. The radial velocity of the bead may be calculated directly through (19) and (10) and we get

$$v_r \equiv \frac{dr}{dt} = v_0 \frac{cn(\omega t)}{dn^2(\omega t)}. \tag{20}$$

It is worthwhile considering separately two asymptotic cases of the solutions (19)–(20): $v_0 \ll 1$ and $v_0 \approx 1$. When an initial radial velocity of the bead is sufficiently nonrelativistic, $m \approx 1$, and taking into account an asymptotic behavior of Jacobi's elliptic functions [7] appearing in (19)–(20) we find that $r(t) \approx (v_0/\omega)\sinh(\omega t)$ and $v_r \approx v_0 \cosh(\omega t)$ as it certainly should be (see the Introduction). Another asymptotic case ($v_0 \approx 1$) is less trivial. It corresponds to the $m \approx 0$ case and consequently leads to $r(t) \approx (v_0/\omega)\sin(\omega t)$ and $v_r \approx v_0 \cos(\omega t)$.

In Fig. 1 we have represented the temporal evolution of the radial velocity of the bead v_r and the quantity $v_\varphi \equiv \omega r$, which is equal to the azimuthal velocity of the

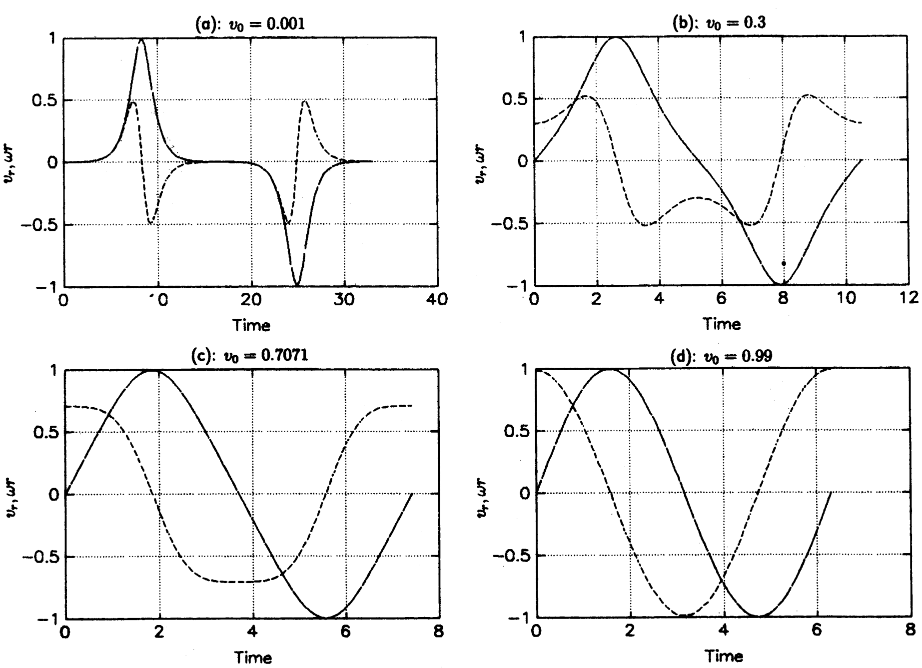

FIG. 1. Temporal evolution of the bead radial velocity v_r (dashed line) and azimuthal velocity $v_\varphi = \omega r$ (solid line) for different values of initial velocity v_0. Namely, (a) $v_0 = 10^{-3}$; (b) $v_0 = 0.3$; (c) $v_0 = \sqrt{2}/2$; (d) $v_0 = 0.99$.

bead in the laboratory frame. The curves correspond to four different values of v_0. Namely, Fig. 1(a) is drawn for the case $v_0 = 0.001$. The motion of the bead may be described as follows: Initially, both velocities (v_r and v_φ) grow almost simultaneously, but later the radial velocity slows its increasing, reaches its maximum value ($v_{max} \approx 0.5$), and begins to decrease, while v_φ continues increasing and reaches the $v_\varphi = 1$ value at the moment $t^* = K/\omega$, when the bead is at a distance $r = r^*$ from the rotation axis and its radial velocity becomes zero. This is a "turning point," since here $v_r(t)$ changes its sign, and the bead begins to move towards the rotation axis with increasing speed. A modulus of radial velocity in the time interval $t^* < t < 2t^*$ varies exactly in the same way as in the previous interval $0 < t < t^*$. At the moment $t = 2t^*$ the bead is in its starting point $r = 0$ just above the rotation axis and it has the same velocity v_0 but directed, this time, in the opposite direction. So, the bead passes over the rotation axis and in the interval $2t^* < t < 4t^*$ repeats in the left half of the pipe the same kind of motion.

Figure 1(b) is drawn for the case $v_0 = 0.3$. Qualitatively, it resembles the previous case. However, a maximum value of v_r is slightly higher and an average radial velocity of the motion is apparently larger.

An interesting "threshold" case $v_0 = \sqrt{2}/2$ is represented in Fig. 1(c). From (10) we can see that for this particular case initial radial acceleration is equal to zero. Thus, originally, the bead moves almost uniformly, but further on "effective centrifugal force" becomes negative and the bead moves with a decreasing radial speed. Other qualities of the motion (i.e., its periodical character) remain the same as in the above mentioned cases.

Finally, Fig. 1(d) represents the case $v_0 = 0.99$. It corresponds to the asymptotic case $m \approx 0$ discussed above and consequently the curves for v_r and v_φ are well approximated by the usual trigonometric cosine and sine, respectively.

Thus, we see that the character of the bead motion is "oscillatory." The period of the "oscillations" $P \equiv 4t^* = 4K/\omega$. In Fig. 2 we have represented the dependence of the function ωP on the value of initial velocity v_0. The period tends to infinity when $v_0 \to 0$ (as it should be), while when $v_0 \to 1$,

$$\lim_{v_0 \to 1} P = \frac{2\pi}{\omega}. \tag{21}$$

It must be noted that from the mathematical point of view the problem we are considering closely resembles the well-known example of a mathematical pendulum motion [6]. In particular, note that period of "oscillations" is proportional to the complete elliptical integral of the first kind in exactly the same way as it is for the pendulum problem.

In order to understand more deeply the qualitative character of these solutions it is useful to introduce the concept of an "effective potential" $U(r)$. Substituting $v = dr/dt$ in (12) from (10), we can rewrite the equation of motion as

$$\frac{d^2r}{dt^2} = \omega^2 r\left[(2m-1) - 2m\omega^2 r^2\right] \equiv -\frac{\partial U(r)}{\partial r}. \tag{22}$$

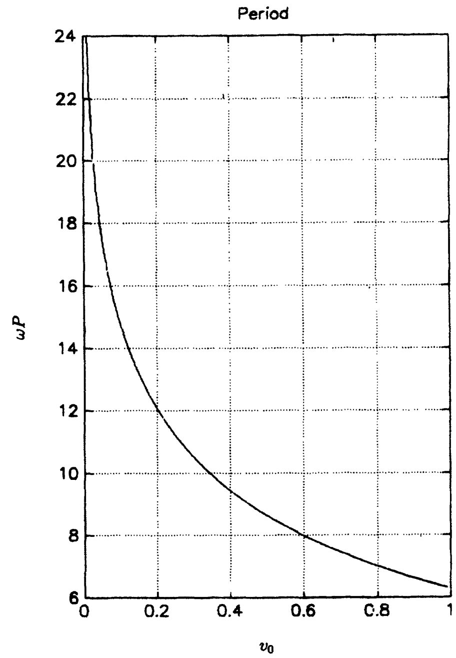

FIG. 2. Dependence of the dimensionless period ωP of the bead "oscillations" on the value of its initial velocity v_0. Note that $\omega P = 4K$, where K is a complete elliptical integral of the first kind.

For our particular case ($x_0 = 0, v_0 \neq 0$), taking into account (13c) and (11) we can rewrite (22) in the following way:

$$\frac{d^2r}{dt^2} = \omega^2 r\left[(1 - 2v_0^2) - 2(1 - v_0^2)\omega^2 r^2\right]. \tag{23}$$

It is evidently seen from (23) that the centrifugal force is really negative for any $r \neq 0$ when $v_0 > \sqrt{2}/2$.

An explicit expression for $U(r)$ is

$$U(r) = \frac{\omega^2 r^2}{2}\left[(1-2m) + m\omega^2 r^2\right]. \tag{24}$$

In Fig. 3 we have represented the function $U(x)$ ($x \equiv \omega r$) for four different values of v_0, corresponding to the cases shown in Figs. 1(a)–1(d). In 1(a) and 1(b) the curves have "secondary" minimums, so that the bead beginning motion from the point $x = 0$ accelerates while "rolling down" to the "secondary" minimum point, and then hampers until it reaches the point $x = 1$, stops and begins to move towards the rotation axis. Figure 1(c) is represented by a "potential well" with an almost plane bottom, where initially, as we have seen earlier, motion should be with almost constant velocity. In 1(d) the form of the potential naturally implies the motion with the centrifugal force always "attracting" towards the rotation axis.

III. CONCLUSION

In this paper, we have investigated the quite simple gedanken experiment: the motion of a bead inside a

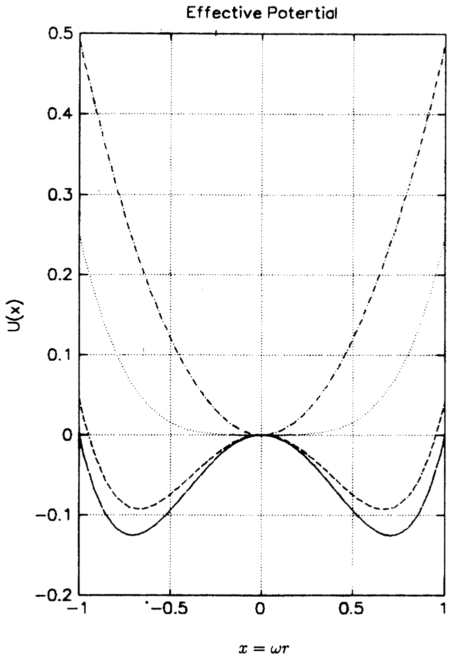

FIG. 3. Effective potential for the bead motion as a function of the quantity $x \equiv \omega \tau = v_\varphi$ for different values of the initial velocity v_0 of the bead. (a) $v_0 = 10^{-3}$; (b) $v_0 = 0.3$; (c) $v_0 = \sqrt{2}/2$; (d) $v_0 = 0.99$.

straight, long enough pipe rotating around an axis normal to the symmetry axis of the pipe. Considering the problem we have shown that under certain circumstances a centrifugal force acting on the bead, and determining the character of its motion, changes its sign and *attracts* towards the rotation axis. Centrifugal force reversal occurs at the distances from the rotation axis close enough to the "light cylinder radius" for arbitrary values of the bead initial position r_0 and velocity v_0. For the particular case, when at the initial moment $t = 0$ the bead was situated on the rotation axis ($r_0 = 0$), we have found that the bead would move between the points $r = \pm r^*$. When $v_0 > \sqrt{2}/2$, the centrifugal force acting on the bead would be negative during the whole course of motion.

Strictly speaking, we have considered a mathematical model of a relativistic body moving under a constraint of a rigidly rotating straight pipe. It must be emphasized that speaking about the "oscillatory" motion of the bead we do not pretend that our model has any physical meaning out to the "velocity of light" circle. Certainly, such a motion in reality is impossible because to ensure the rotation of such a device with a constant angular velocity one surely needs an infinite amount of energy and, besides, no real pipe may be "absolutely rigid" and sooner or later will be broken by the relativistic moving bead. Thus, the described "oscillation" is not actually a physical prediction, but instead must be regarded as a mathematical idealization. That is why we call this model the gedanken experiment and say that it is highly idealized.

However, considering this gedanken experiment we have found a very strange effect — centrifugal force reversal. If the initial velocity of the bead is high enough centrifugal force changes its sign already at $r \ll r^*$ and when $v_0 > \sqrt{2}/2$ the centrifugal force acting on the bead is reversed from the very beginning of the motion—during the whole course of the motion. This result seems to be of principle importance. The effect appears in the more or less usual rotating frame, well described in the framework of special relativity [8]. Thus, we can say that the centrifugal force reversal effect may occur in *special relativity*. Let us remember that up to now it was acknowledged that such an unusual reversal phenomenon was inherent only in such general-relativistic objects as black holes [1–4].

There is one point that must be emphasized in a special way: According to the obtained solutions the moving bead — a body with nonzero mass — reaches at certain moments of its motion the speed of light ($v_\varphi = 1$ when $t = nK/\omega$; $n = 1, 3, 5, ...,$). At a first glance, this fact seriously contradicts the common special relativistic dogma. However, the controversy is, as we think, imaginary. Actually, azimuthal velocity of the bead in the laboratory frame $v_\varphi(t)$ has instantaneous values equal to the speed of light at certain moments of time but for arbitrary *finite* time intervals the average velocity of the motion is always less than 1. Thus, the bead always moves with $v_\varphi < 1$, though the instantaneous value of the velocity may be equal to 1.

We are almost certain that the simple effect found in this paper should have different astrophysical and physical applications. In all those astrophysical objects (such as stellar and accretion disk winds, bipolar outflows in active galactic nuclei, etc.), where a rotation and relativistic velocities are present, and magnetic field lines may act like "pipes" directing the particles motion along themselves, the effect may appear in a natural way. For example, such a situation may exist in a pulsar wind between the pulsar surface and light cylinder of the pulsar. Consideration of concrete applications is beyond the scope of this paper, and is planned to be performed in our forthcoming research.

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