Models of particle detection in regions of space-time

Donald Marolf*
Physics Department, The Pennsylvania State University, University Park, Pennsylvania 16803
(Received 8 July 1993)

We investigate two models of measuring devices designed to detect a nonrelativistic free particle in a given region of space-time. These models predict different probabilities for a free quantum particle to enter a space-time region R; therefore, this notion is device dependent. The first model is of a von Neumann coupling, which we present as a contrast to the second model. The second model is shown to be related to probabilities defined through partitions of configuration-space paths in a path integral. This study thus provides insight into the physical situations to which such definitions of probabilities are appropriate.

PACS number(s): 03.65.Bz

I. INTRODUCTION

The theory of measurement in quantum mechanics has been a source of discussion and controversy for as long as the subject has existed. Indeed, the Syracuse University Library Catalogue system returns a list of 13 books published since 1968 in response to the keyword inquiry "quantum measurement." Such a list may at best be considered abbreviated as it neglects works predating 1968, such as von Neumann's *Mathematical Foundations of Quantum Mechanics* [2] and numerous research papers not found in books. Within this immense volume of material, a number of important contributions have been made by investigating specific examples (or at least models) of experimental situations, in other words, through a study of measuring devices. These contributions include the classic works of Mott [1], von Neumann [2], Bohr and Rosenfeld [3], and DeWitt [4] as well as the more recent studies of detectors in quantum field theories which are summarized by Birrell and Davies in [5].

The basic procedures followed in these studies is to first describe an appropriate classical measuring device and to then quantize the coupled system consisting of both the apparatus and the system to be measured. In the above examples, the couplings considered single out operators associated with the measured system that can be said to be "measured" so that the "experiment" may then be given an interpretation by the textbook procedure in which the total state of the system is decomposed as a sum of eigenvectors of this operator and the "probabilities" to measure various eigenvalues are given by the coefficients in this sum.

At this point a comment is in order concerning the use of quotation marks in this paper. It is the author's intention to use such punctuation to avoid a discussion of the interpretation of quantum mechanics as the precise meaning of the words "probabilities," "measurement," and even "outcome" for each reader will be influenced by the interpretation to which he or she subscribes. Rather than single out one set of such definitions for use here or use complicated notation to distinguish between the various meanings present in the literature at large, we choose to use these words for any of the possible meanings but to recognize that an ambiguity exists by enclosing the words in quotation marks for *every* such use. When such a term appears in the text, the intended meaning should be clear from the context (as in the above paragraph) and is ultimately determined by the mathematical steps used to calculate the corresponding "probabilities."

Suppose then that we wish to predict whether a quantum free particle will be found in some region R of spacetime—a case that has received some attention [6–8]. An analysis in the spirit of [1–5] would begin with the description of a classical detector and then proceed with a quantization of the coupled detector-particle system. A study of the quantum detector after the experiment is completed determines the "probabilities" of the outcomes through the norms of the relative states [9][1] associated with possible responses of the device and it may be that the results are highly dependent on the device.

We therefore study models of two such detectors. The first such model (A) is presented in Sec. II and leads to a definition of probability of the familiar type given by projections of the free particle state onto parts of the spectrum of some Hermitian operator. On the other hand, the model (B) presented in Sec. III is related to a definition of such probabilities which has been previously proposed [6–8] in terms of partitions of paths in a path integral. We note that these two definitions are not equivalent, so that this study may be used to develop an understanding of the physical situations to which such definitions of probabilities are and are not appropriate.

This discussion will constitute the bulk of the text,

*Electronic address: marolf@phys.psu.edu

[1]We note that, while the terminology of Everett [9] is convenient here, the use of such terminology does not imply that the following analysis assumes the Everett interpretation of quantum mechanics.

after which we will close with a discussion of the main points and two appendixes. Appendix A presents a brief discussion of bubble chambers in order to contrast this device with models A and B. Appendix B reviews the "ideal measurement limit," which we use in our models.

II. SINGLE MEASUREMENT MODEL

For definiteness, we would like to analyze the detection of a free nonrelativistic particle in 1+1 dimensions in the space-time region R to the right of $x = 0$ between the times T_1 and T_2. Our detector will be defined by a pointer which moves to the right when the particle is present in region R. We will then say that the particle has or has not entered R based on whether or not the pointer has moved between times T_1 and T_2.

Specifically, we will assume that the coupling is of a von Neumann type associated with measurement of $\chi = \int_{T_1}^{T_2} dt\ \theta(x)$ during the time interval (T_1, T_2) so that if $A_g(t)$ is the pointer value at time t when coupled with strength g to the particle and under the retarded boundary conditions $A_g(T_1) = A_0(T_1)$ we have $A_g(T_2) = A_0(T_2) + g\chi_0$. Similarly, χ_0 is the undisturbed (g=0) value of χ.[2] Note that χ is just the total time spent by the particle in region R so that this is consistent with our decision to say that the particle entered region R if and only if the pointer moves during the experiment.

We will also assume that the free (g=0) pointer evolution is trivial: $A_0(t) = A_0(t')$ and $\pi_0(t) = \pi_0(t')$, where π is the quantity canonically conjugate to A. This may be regarded as either an unphysical but useful mathematical model or as the limit in which the kinetic energy term in the Hamiltonian would cause the pointer to move a distance that is much less than the accuracy Δ to which the pointer can be read, i.e., such that $P(T_2 - T_1)/M \ll \Delta$. Alternatively, we might take the free pointer Hamiltonian to be A^2 (A might be the momentum of some other free particle) in which case A would still be conserved when the device and free particle do not interact. We require conservation of A for the uncoupled device since a pointer set initially to zero should remain at zero if no particle is present.

The quantization of this model is then straightforward. As we will want to compare the disturbed (finite g) and undisturbed ($g = 0$) cases, we will in fact construct a quantum theory for each g. We will use the Heisenberg picture and choose representations carried by Hilbert spaces isomorphic to each other and to $\mathcal{L}^2(\mathcal{R}^2) = \mathcal{L}^2(\mathcal{R}; x) \otimes \mathcal{L}^2(\mathcal{R}; A)$. Of particular use will be the isomorphism I_g^- from the Hilbert space $\mathcal{H}_0$ of the uncoupled system to the Hilbert space $\mathcal{H}_g$ of the system with coupling strength g that satisfies

$$(I_g^-)^{-1}\hat{\mathcal{O}}_g(t)I_g^- = \hat{\mathcal{O}}_0(t) \tag{2.1}$$

for any operator $\hat{\mathcal{O}}_0(t)$ built from the basic operators $\hat{X}_0(t)$, $\hat{P}_0(t)$, $\hat{A}_0(t)$, and $\hat{\pi}_0(t)$ (the particle and pointer positions and momenta at the time t in the uncoupled system) and the operator $\hat{\mathcal{O}}_g(t)$ built in exactly the same way but from the basic operators $\hat{X}_g(t)$, $\hat{P}_g(t)$, $\hat{A}_g(t)$, and $\hat{\pi}_g(t)$ at some time $t \leq T_1$. Because the coupling begins only at time T_1, the same isomorphism satisfies (2.1) for all $t \leq T_1$ and this isomorphism corresponds to our classical use of retarded boundary conditions (a fixed g-independent initial condition) in comparing the coupled and uncoupled systems.

Because our coupling is to be of the von Neumann type associated with measurements of χ, we take the quantum dynamics to be defined so that

$$\hat{\mathcal{O}}_g(T_2) = I_g^- \exp(-ig\hat{\pi}_0\hat{\chi}_0)\ \hat{\mathcal{O}}_0(T_2)\ \exp(ig\hat{\pi}_0\hat{\chi}_0)(I_g^-)^{-1}. \tag{2.2}$$

Here $\hat{\chi}_0$ is the operator $\int_{T_1}^{T_2} dt\ \hat{\theta}_0(x(t))$ in $\mathcal{H}_0$ and $\hat{\theta}_0(x(t))$ is the projection onto the positive spectrum of $\hat{X}_0(t)$. Note that $I_g^-\hat{\chi}_0(I_g^-)^{-1} \neq \hat{\chi}_g$ since χ is built from the basic operators between times T_1 and T_2.

As is consistent with our use of retarded boundary conditions, we will assume that the state of our system contains no correlations between the particle and detector at time T_1. That is, we take $|\psi\rangle$ to be of the form

$$|\psi\rangle = |\phi_p\rangle \otimes |\phi_D\rangle \tag{2.3}$$

in terms of the factorization $\mathcal{H}_g = \mathcal{H}_{g,p:T_1} \otimes \mathcal{H}_{g,D:T_1}$ of the total Hilbert space $\mathcal{H}_g$ into a Hilbert space $\mathcal{H}_{g,p:T_1}$ associated with the particle at time T_1 and a Hilbert space $\mathcal{H}_{g,D:T_1}$ associated with the detector at time T_1. Here $|\phi_p\rangle$ and $|\phi_D\rangle$ are normalized states in $\mathcal{H}_{g,p:T_1}$ and $\mathcal{H}_{g,D:T_1}$.

We now note that since $\hat{\chi}_0$ is defined for the uncoupled system (in which the factorization $\mathcal{H}_0 = \mathcal{H}_{0,p:t} \otimes \mathcal{H}_{0,D:t}$ is t independent), it can be written as the direct product

$$\hat{\chi}_0 = \hat{\chi}_f \otimes \mathbb{1}_{0,D} \tag{2.4}$$

for some $\hat{\chi}_f : \mathcal{H}_{0,p:T_1} \to \mathcal{H}_{0,p:T_1}$ where $\mathbb{1}_{0,D}$ is the identity operator in $\mathcal{H}_{0,D:T_1}$. Similarly, for any operator $\mathcal{O}_0$ of the form (2.4), it will be convenient to define a corresponding operator $\mathcal{O}_f$ by $\mathcal{O}_0 = \mathcal{O}_f \otimes \mathbb{1}_{0,D}$.

The evolution (2.2) then leads in the usual way (see [2,10]) to the statement that in the ideal measurement limit[3] we find

$$\langle\psi|\hat{\Pi}_{A(T_2)=A'}dA'|\psi\rangle \to |\langle\chi_f = A'|I_{g,f}^-|\phi_p\rangle|^2 d\mu_{\hat{\chi}_f}(A') \tag{2.5}$$

[2] It is interesting to note that it is difficult to construct such couplings through an action or Hamiltonian principle without first expressing χ_0 in terms of the operators $P(T_1)$ and $X(T_1)$ through the uncoupled equations of motion, that is, without essentially reducing it to an operator defined at the single time T_1.

[3] A brief review of this limit is presented in Appendix B for comparison with our discussion of model B in Sec. III B.

of "probability densities," where $|\chi_f = A'\rangle\langle\chi_f = A'|d\mu_{\hat{\chi}_f}(A')$ and $\hat{\Pi}_{A(T_2)=A'}dA'$ are the spectral measures evaluated at A' of the operators $\hat{\chi}_f$ and $\hat{A}(T_2)$, respectively, and $I^-_{g,f}$ is the isomorphism from $\mathcal{H}_{g,p:T_1}$ to $\mathcal{H}_{0,p:T_1}$ induced by the isomorphism $I^-_g:\mathcal{H}_0 \to \mathcal{H}_g$. Equivalently, the decoherence functional $\langle\psi|\hat{\Pi}_{A=A'}dA'\ \hat{\Pi}_{A=A''}dA''|\psi\rangle$ converges in this limit to

$$\mathcal{D}(A', A'') = \langle\phi_p|\hat{\Pi}_{\chi_f=A'}d\mu_{\hat{\chi}_f}(A')\ \hat{\Pi}_{\chi_f=A''}d\mu_{\hat{\chi}_f}(A'')|\phi_p\rangle. \tag{2.6}$$

A general review of decoherence functionals is given in [6,11,12] and in other works. The results are equivalent and decoherence is trivial because each decoherence functional involves only commuting projection operators.

As defined by the distribution (2.5), the "probability" of finding $A = 0$ with arbitrarily high accuracy vanishes unless the spectral distribution of $\hat{\chi}_f$ is singular. That is, it vanishes unless $\hat{\chi}_f$ has a normalizable eigenstate in $\mathcal{H}_{0,p:T_1}$ with eigenvalue zero. That this is *not* the case can be seen by taking the expectation value of $\hat{\chi}_f$ in any state $|\phi\rangle \in \mathcal{H}_{0,p:T_1}$:

$$\begin{aligned}\langle\phi|\hat{\chi}_f|\phi\rangle &= \langle\phi|\int_{T_1}^{T_2} dt\ \hat{\theta}_f(x(t))|\phi\rangle \\ &= \int_{T_1}^{T_2} dt\ \langle\phi|\hat{\theta}_f(x(t))|\phi\rangle \\ &= \int_{T_1}^{T_2} dt \int_0^\infty dx\phi^*(x;t)\phi(x;t),\end{aligned} \tag{2.7}$$

where $\phi(x;t) = \langle\phi|x;t\rangle$, where $|x;t\rangle$ is the eigenvector of $\hat{X}_f(t)$ in $\mathcal{H}_{0,p:T_1}$ with eigenvalue x. Since an arbitrary state $|\phi\rangle$ can have $\phi(x;t)$ vanish for all $x \geq 0$ only for isolated times t, this integral is always greater than zero so that $\hat{\chi}_f$ does not annihilate any state in $\mathcal{H}_{0,p:T_1}$ and therefore does not have any normalizable eigenvectors with eigenvalue zero. It follows that in the limit of ideal measurement, the particle is detected by device A with "probability" one independent of the initial state $|\phi_p\rangle$.

III. A MULTIPLE MEASUREMENT MODEL

While we based device A on the von Neumann measurement of a single classical quantity χ_0, we will follow a different strategy for device B and build it instead from *many* von Neumann measurements. While this may seem like an unnatural procedure from the viewpoint of measurement theory in quantum mechanics (where we usually follow a "one question, one measurement" principle), this approach will lead to a definition of the probability for a free particle to enter the region R given by partitions of paths in a path integral [6–8]. We therefore pursue it primarily as a way of understanding the type of physical situations to which such a definition is appropriate.

Subsection A gives an overview of the classical model which is quantized in subsection B. Subsection B then shows that the corresponding "probabilities" can be related to a path integral that sums only over paths which avoid the region R. Subsection C relates model B to the discussion given in [6–8] and subsection D analyzes the difference between our approach and that of [6–8] in light of the fact that device B disturbs the system it measures.

A. Model

We define model B by coupling $\mathcal{P}_{\lambda,\tau}(t) = \delta(x(t) - \lambda)\delta(t-\tau)$ to a pointer for each (λ, τ) in R. Note that these are "explicitly time dependent operators" in the sense that $\frac{d\mathcal{P}}{dt} \neq \{\mathcal{P}, H\}$ and that they are related to χ of Sec. II through

$$\chi = \int_{-\infty}^{\infty} dt \int_R d\lambda d\tau\ \mathcal{P}_{\lambda,\tau}. \tag{3.1}$$

It follows that χ is nonzero exactly when

$$\int_R d\lambda d\tau\ \phi(\lambda,\tau)\mathcal{P}_{\lambda,\tau} \tag{3.2}$$

is nonzero for some smooth function ϕ and that classically devices A and B detect particles under identical conditions.

Thus, for each $(\lambda, \tau) \in R$, we introduce a pointer coordinate $A_{\lambda,\tau}$ and a conjugate momentum $\pi_{\lambda,\tau}$ and define model B through the action

$$\begin{aligned}S_B = S_{\text{free}} &- g\int_{-\infty}^{\infty} dt \int_{(\lambda,\tau)\in R} d\lambda d\tau\ \pi_{\lambda,\tau}\mathcal{P}_{\lambda,\tau} \\ &+ \int_{-\infty}^{\infty} dt \int_{(\lambda,\tau)\in R} d\lambda d\tau\ \pi_{\lambda,\tau}\dot{A}_{\lambda,\tau},\end{aligned} \tag{3.3}$$

where again we have assumed that the pointers have trivial free evolution. We then interpret the result of this "experiment" by saying that the particle entered region R if and only if some pointer is disturbed during the time interval. Note that each coupling takes place instantaneously and, as a result, each pointer $A_{\lambda,\tau}$ coupled to $\mathcal{P}_{\lambda,\tau}$ responds to the same value of $\mathcal{P}_{\lambda,\tau}$ as it would have if the (λ, τ) coupling were not present. However, the (λ, τ) coupling will effect the value of any $\mathcal{P}_{\lambda',\tau'}$ measured afterwards since from [13] we see that the change induced in $I(\lambda_2, \tau_2) = \int_\infty^\infty dt\ \mathcal{P}_{\lambda_2,\tau_2}$ by the (λ_1, τ_2) coupling is given by

$$D^-_{I(\lambda_1,\tau_1)}I(\lambda_1,\tau_1) = (I(\lambda_1,\tau_1), I(\lambda_2,\tau_2))\ \theta(\tau_1,\tau_2) \neq 0, \tag{3.4}$$

where the bracket in (3.4) is the Peierls bracket[4] and we have used the notation of [14]. In language that is more familiar from quantum mechanics, this experiment attempts to measure a set of operators that do not com-

[4]The Peierls bracket (A,B) may be thought of as the Poisson bracket extended to quantities evaluated at different times by using the equations of motion.

mute with each other and the result is that the measurement of one necessarily effects the outcome of the others. In this sense then, model B describes a device which always measures a disturbed system.

B. Quantization

We again define the quantum system for each value of the coupling constant g in terms of a Hilbert space $\mathcal{H}_g$ isomorphic to $\mathcal{L}^2(X)\otimes\mathcal{L}^2(A)$, although now $\mathcal{L}^2(A)$ is the Hilbert space $\mathcal{L}^2(\{A{:}R\to\mathcal{R}\})$ defined by some measure on the space of real functions on R. Since the coupling exists only for $T_1\leq t\leq T_2$, we have another "retarded" isomorphism $I_g^-:\mathcal{H}_0\to\mathcal{H}_g$ defined by

$$(I_g^-)^{-1}\mathcal{O}_g(t)I_g^-=\mathcal{O}_0(t) \tag{3.5}$$

for all operators $\mathcal{O}_0(t)$ and $\mathcal{O}_g(t)$ such that they are built in the same way from $\{X_0(t),P_0(t),A_{\lambda,\tau,0}(t),\pi_{\lambda,\tau,0}(t)\}$ and $\{X_g(t),P_g(t),A_{\lambda,\tau,g}(t),\pi_{\lambda,\tau,g}(t)\}$, respectively. The state of the system will again be assumed to be uncorrelated at time T_1: $|\psi\rangle=|\phi_p\rangle\otimes|\phi_D\rangle\in\mathcal{H}_{g,p:T_1}\otimes\mathcal{H}_{g,D:T_1}$ for normalized states $|\phi_p\rangle$ and $|\phi_D\rangle$.

However, as we are not considering the reaction of a pointer to a single undisturbed quantity, we cannot simply use the standard results to draw conclusions about "probabilities" (or "decoherence") in the ideal measurement limit but must derive the appropriate results ourselves. We will model this derivation after the discussion in Appendix B and begin by assuming that the initial state $|\phi_D\rangle$ of our device is characterized by a width σ and is given explicitly by

$$|\phi_D\rangle=\frac{1}{N}\int\prod_{\lambda,\tau}dA_{\lambda,\tau}\exp\left(-\int_R d\lambda d\tau\ A^2/2\sigma^2\right)|A;T_1\rangle \tag{3.6}$$

in terms of the eigenstates $|A;T_1\rangle$ of $A_{\lambda,\tau}(T_1)$ and where N is chosen so that $\langle\phi_D|\phi_D\rangle=1$.

We would like to say (in some sense) that detection of a particle in the region R at a confidence level n occurs when the response of the device is greater than $n\sigma$. This means that we will need some measure of the response, i.e., some choice of a function ϕ in (3.2). In order to weight all pointers equally, we choose $\phi(\lambda,\tau)=1$.

We therefore associate the "detection" of a particle in R with the projection $\Pi_>$ onto the part of the spectrum of $\int_R d\lambda d\tau\ A_{\lambda,\tau}$ greater than or equal to $n\sigma$ and we associate the projection $\Pi_<=\mathbb{1}-\Pi_>$ with the lack of detection of a particle in region R. We then define the "probabilities" for the particle to be (or not to be) detected in R by an ideal measurement to be the limit of the norms of these projections of $|\psi\rangle$ in which $n\to\infty$ but $n\sigma\to0$. Equivalently, we could define the "probabilities" for the alternatives through the corresponding limit of the decoherence functional

$$\mathcal{D}^B_{\alpha,\alpha'}=\langle\psi|\Pi_\alpha\Pi_{\alpha'}|\psi\rangle \tag{3.7}$$

for $\alpha\in\{>,<\}$, although decoherence is trivial even before the limits are taken since $\Pi_>$ and $\Pi_<$ are commuting projection operators.

In order to compute these "probabilities," we note that the expansion of $\Pi_>|\psi\rangle$ (or $\Pi_<|\psi\rangle$) in terms of the basis $|x,A;T_2\rangle$ of simultaneous eigenvectors of $\hat{X}_g(T_2)$ and $\hat{A}_{\lambda,\tau,g}(T_2)$ is given by the coefficients

$$\begin{aligned}\langle x_2,A_2;T_2|\Pi_\alpha|\psi\rangle=\theta_\alpha&\left(\int_R d\lambda d\tau\ A_{\lambda,\tau,2}-n\sigma\right)\\&\times\int\prod_t dx\int\prod_{\lambda,\tau,t}(dA_{\lambda,\tau}d\pi_{\lambda,\tau})e^{iS_B}\\&\times\frac{1}{N}\exp\left(-\int_R d\lambda d\tau A_1^2/2\sigma^2\right)\\&\times\phi_p(x_1),\end{aligned} \tag{3.8}$$

where $\phi_p(x)=\langle x;T_1|\phi_D\rangle$, $\theta_>$ and $\theta_<$ are step functions with support on the positive and negative axes, respectively, and the sum is over all paths $x(t)$ from (x_1,T_1) to (x_2,T_2) and all pointer configurations and momenta subject to the boundary conditions $A_{\lambda,\tau}(T_1)=A_{\lambda,\tau,1}$ and $A_{\lambda,\tau}(T_2)=A_{\lambda,\tau,2}$. Because of the trivial evolution that we have assumed for the pointer degrees of freedom, the integrations over the fields A and π at intermediate times are particularly simple and their only effect is to require that $A_{\lambda,\tau}(T_2)-A_{\lambda,\tau}(T_1)=g\int_{T_1}^{T_2}dt\ \mathcal{P}_{\lambda,\tau}=gI_{\lambda,\tau}[x(t)]$. It follows that we can calculate the "probabilities" $\mathcal{D}_\gg$ and $\mathcal{D}_\ll$ from a path integral expression which, after a shift of the integration variables $A_{\lambda,\tau}$ by the amount $I_{\lambda,\tau}[x(t)]$, takes the form

$$\begin{aligned}\mathcal{D}_\ll=&\int\prod_t(dx_1dx_2)\ e^{i(S_{\rm free}[x_1]-S_{\rm free}[x_2])}|\phi_p(x_1)|^2\\&\times\frac{1}{N^2}\int\prod_{\lambda,\tau}dA_{\lambda,\tau}\ \exp\left(\int_R d\lambda d\tau\ A^2_{\lambda,\tau}/\sigma^2\right)\\&\times\theta_<\left(\int_R d\lambda d\tau\ (A_{\lambda,\tau}+gI_{\lambda,\tau})-n\sigma\right).\end{aligned} \tag{3.9}$$

Note that for a given path $x(t)$ that spends a total time T in the region R, the integral over the pointer fields (specifically, the expression on the second line) gives just the Gaussian measure (with width σ) of the set $S_{gT-n\sigma}$ of all configurations such that $\int_R d\lambda d\tau\ A_{\lambda,\tau}+gT-n\sigma<0$, which we will denote by $\mu_\sigma(S_{gT-n\sigma})$.

Now consider a path such that $T>0$. Note that for small $n\sigma$ we have $\mu_\sigma(S_{gT-n\sigma})<\mu_\sigma(S_{gT/2})$ and that as $\sigma\to0$, the measure μ_σ is concentrated on configurations near $A_{\lambda,\tau}=0$. It follows that $\mu_\sigma(S_{gT-n\sigma})\to0$ in the ideal measurement limit where $n\to\infty$ but $n\sigma\to0$ and that paths which enter R do not contribute to (3.9) in this limit. Note that we must, however, keep the coupling constant g fixed or send it to zero more slowly than σ in order for this conclusion to be reached. In this way, the limit of small coupling does not commute with the limit of ideal measurement.

However, for a path with $T=0$ the integral over pointer configurations gives the value $\mu_\sigma(S_{-n\sigma})$, which approaches 1 in the limit $n \to \infty$, even when $n\sigma \to 0$. Thus we find that in the "ideal measurement limit" the "probability for the particle to avoid the region R" is given by the expression

$$\mathcal{D}_{\ll} = \int_{\text{paths} \,\cap\, R=\emptyset} \prod_t (dx_1 dx_2)\ e^{i(S_{\text{free}}[x_1]-S_{\text{free}}[x_2])} \times |\phi_p(x_1)|^2, \tag{3.10}$$

which can be related to the analysis of [6–8].

C. Probabilities by partitions of paths

The above expression was presented in Refs. [6–8] as part of a proposal for the definition of probabilities for the particle to enter or avoid the region R through partitions of paths in a path integral. Specifically [6–8], associate the operator $\mathcal{C}_R$ ($\mathcal{C}_{\overline{R}}$) in the Hilbert space $\mathcal{H}_{F:I}$ of an isolated free particle with the alternative "particle does (not) enter R," where

$$\langle x_2; t_2 | \mathcal{C}_{\overline{R}} | x_1; t_1 \rangle = \int_{\text{paths} \,\cap R=\emptyset} e^{iS_{\text{free}}} \tag{3.11}$$

and $\mathcal{C}_R \equiv \mathbb{1} - \mathcal{C}_{\overline{R}}$. Here the sum is over all paths that begin at (x_1, t_1) and end at (x_2, t_2) without passing through the region R, S_{free} is the action for the nonrelativistic free particle, $|x;t\rangle$ is an eigenstate of the time-dependent position operator $\widehat{X(t)}$ with eigenvalue x, and t_1 and t_2 are any times respectively to the past and future of R.

Probabilities are then defined through the decoherence functional

$$\mathcal{D}^{\text{HYT}}_{\alpha,\alpha'} = \langle \phi | \mathcal{C}^\dagger_\alpha \mathcal{C}_\alpha | \phi \rangle \tag{3.12}$$

for $\alpha \in \{R, \overline{R}\}$. Although $\mathcal{C}_R \equiv \mathbb{1} - \mathcal{C}_{\overline{R}}$, decoherence is not immediate since neither $\mathcal{C}_R$ nor $\mathcal{C}_{\overline{R}}$ is a projection operator. Since we will mention this decoherence functional (3.12) several times, it will be convenient to refer to it as the "Hartle-Yamada-Takagi (HYT) decoherence functional."

We note that $\mathcal{D}^B_{\ll} = \mathcal{D}^{\text{HYT}}_{\overline{R},\overline{R}}$, so that $\mathcal{D}_{\gg}$ must also agree with $\mathcal{D}^{\text{HYT}}_{RR}$ whenever the alternatives decohere for both decoherence functionals. Since the integral (3.11) has been calculated before, we quote the result of [6,7] that any "probability" between zero and one may be found, depending on the state $|\psi\rangle$. We then note that this result is quite different from what we found in Sec. II, but do not concern ourselves further with the calculation of expression 3.10.

As mentioned above, $\Pi_>$ and $\Pi_<$ always decohere since they are commuting projection operators, although the same is not true of $\mathcal{C}_R$ and $\mathcal{C}_{\overline{R}}$. As a result, $\mathcal{D}^B$ and $\mathcal{D}^{\text{HYT}}$ are not identical. This difference is investigated in the following subsection and may be summarized by saying that the decohering effect of device B is not included in the HYT decoherence functional.

D. The disturbing nature of device *B*

At first glance it may be tempting to say that, since (3.7) is a decoherence functional for the device while the HYT result concerns the free particle, we should not be surprised that the two decoherence functionals $\mathcal{D}^B$ and $\mathcal{D}^{\text{HYT}}$ do not agree. However, this statement is not entirely satisfactory since, in the ideal measurement limit, we *do* find exact agreement for the corresponding decoherence functionals in a von Neumann measurement (see Appendix B for an illustration involving device A). The explanation lies in the fact that, as pointed out in [7], we in general expect a measurement that takes place over an extended time to disturb the system being measured. We now show in detail how this occurs in model B and how it accounts for the discrepancy between $\mathcal{D}^B$ and $\mathcal{D}^{\text{HYT}}$.

To do so, we first note that model A associates with $\Pi_>$ the projection operator $\Pi_{I_g^- \hat{\chi}_0 (I_g^-)^{-1} \leq 0}$ and with $\Pi_<$ the projection operator $\Pi_{I_g^- \hat{\chi}_0 (I_g^-)^{-1} > 0}$ in the sense that

$$\mathcal{D}^A_{\alpha,\alpha'} \equiv \lim_{m\to i} \langle \psi | \Pi_\alpha \Pi_{\alpha'} | \psi \rangle = \lim_{\substack{(n\sigma)\to 0 \\ n\to\infty}} \langle \psi | \int \theta_\alpha(\chi' - n\sigma) \Pi_{I_g^- \hat{\chi}_0 (I_g^-)^{-1}} d\mu_{\hat{\chi}_f}(\chi') \times \int \theta_{\alpha'}(\chi'' - n\sigma) \Pi_{I_g^- \hat{\chi}_0 (I_g^-)^{-1}} d\mu_{\hat{\chi}_f}(\chi'') |\psi\rangle, \tag{3.13}$$

where the notation $\lim_{m\to i}$ refers to the limit in which the measurement becomes ideal, the projection operators correspond to the operator $I_g^- \hat{\chi}_0 (I_g^-)^{-1}$, the spectral measure corresponds to the operator $\hat{\chi}_f$, and $\alpha, \alpha' \in \{>, <\}$. Since $\Pi_{\hat{\chi}_0 = \chi'} = \Pi_{\hat{\chi}_f = \chi'} \otimes \mathbb{1}_{0,D}$ and $\mathbb{1} - \Pi_{\hat{\chi}_0 = \chi'} = (\mathbb{1} - \Pi_{\hat{\chi}_f = \chi'}) \otimes \mathbb{1}_{0,D}$ in terms of the projection operator $\Pi_{\hat{\chi}_f = \chi'}$ onto eigenvalues of $\hat{\chi}_f$, the naive decoherence functional (2.6) defined by the state $|\phi_p\rangle$ and the projections $\Pi_{\hat{\chi}_f = \chi'}$ and $\mathbb{1} - \Pi_{\hat{\chi}_f = \chi'}$ agrees with $\mathcal{D}^A$.

We can now contrast this line of reasoning with the corresponding deductions for device B. Again, we found that the desired decoherence functional could be computed using operators associated only with the particle and not with the device. Specifically, in the ideal measurement limit we found that

$$\lim_{m\to i} \langle \psi | \Pi_\alpha \Pi_{\alpha'} | \psi \rangle = \langle \psi | \mathcal{C}_{g,\alpha} \mathcal{C}_{g,\alpha'} | \psi \rangle \tag{3.14}$$

for $\alpha \in \{>, <\}$, where $\mathcal{C}_{g,<}$ is defined by its matrix elements

$$\langle x_2, A_2; T_2 | \mathcal{C}_{g,<} | x_1, A_1; T_1 \rangle = \delta(A_2 - A_1) \int_{paths \,\cap\, R=\emptyset} \prod_t dx e^{iS_{\text{free}}} \tag{3.15}$$

and the sum is over all paths that begin at (x_1, T_1) and end at (x_2, T_2). The complimentary operator $\mathcal{C}_{g,>}$ is then defined as $\mathbb{1} - \mathcal{C}_{g,<}$. Since $|x_1, A_1; t\rangle = |x_1; t\rangle \otimes |A_1; t\rangle$ with respect to the *t-dependent* factorization $\mathcal{H}_g = \mathcal{H}_{g,p:t} \otimes \mathcal{H}_{g,D:t}$, Eq. (3.15) implies not that the operator $\mathcal{C}_{g,>}$ is of the form $\mathcal{C}_f \otimes \mathbb{1}_{0,D}$ for some operator $\mathcal{C}_f$ in $\mathcal{H}_{g,p:T_1}$, but that $\mathcal{C}_{g,<}$ is of the form

$$\mathcal{C}_{g,<} = \mathcal{C}_{g,p,<} \otimes I_D, \tag{3.16}$$

where $\mathcal{C}_{g,p,<}$ maps $\mathcal{H}_{g,p;T_1}$ to $\mathcal{H}_{g,p;T_2}$ and I_D is the isomorphism $I_D|A;T_1\rangle = |A;T_2\rangle$ from $\mathcal{H}_{g,D:T_1}$ to $\mathcal{H}_{g,D:T_2}$.

The relationship to the operators $\hat{C}_{\overline{R}}$ and $\hat{C}_R$ that define the HYT decoherence functional can be made clear through the introduction of two more isomorphisms that relate the free particle Hilbert space $\mathcal{H}_F$ on which $\hat{C}_{\overline{R}}$ and $\hat{C}_R$ are defined to $\mathcal{H}_{g,p;T_1}$ and $\mathcal{H}_{g,p;T_2}$. The isomorphisms $I_{g,t}:\mathcal{H}_F \to \mathcal{H}_{g,p;t}$ for $t \in T_1, T_2$ are defined by $I_{g,t}|x;t\rangle = |x;t\rangle$ so that $\mathcal{C}_{g,p,<} = I_{g,T_2}\hat{C}_{\overline{R}}I^{-1}_{g,T_1}$ and $\mathbb{1} - \mathcal{C}_{g,p,<} = I_{g,T_2}(\mathbb{1} - \hat{C}_{\overline{R}})I^{-1}_{g,T_1}$. The subtle point is that

$$\mathcal{C}_{g,>} = \mathbb{1} - \mathcal{C}_{g,<} \neq I_{g,T_2}(\mathbb{1} - \hat{C}_{\overline{R}})I^{-1}_{g,T_1} \otimes I_D. \tag{3.17}$$

That this is so can be seen by adding the right-hand side above to $\mathcal{C}_{g,<}$:

$$I_{g,T_2}(\mathbb{1} - \hat{C}_{\overline{R}})I^{-1}_{g,T_1} \otimes I_D + \mathcal{C}_{g,<} = I_{g,T_2}I^{-1}_{g,T_1} \otimes I_D, \tag{3.18}$$

but $I_{T_2}I^{-1}_{T_1} \otimes I_D$ is the isomorphism between $\mathcal{H}_{p;T_1} \otimes \mathcal{H}_{D;T_1}$ and $\mathcal{H}_{p;T_2} \otimes \mathcal{H}_{D;T_2}$ induced by the corresponding factorizations of the Hilbert space $\mathcal{H}_0$ of the *uncoupled* system. This isomorphism therefore differs from the identity operator in any $\mathcal{H}_g$ for which the coupling constant is nonzero.

How does this all relate to our characterization of device B as a "disturbing" apparatus? We note that the operators picked out by device A were of the form $\Pi_{I_g^- \hat{\chi}_0 (I_g^-)^{-1}} = (I_g^-)^{-1}\Pi_{\hat{\chi}_0=\chi'}I_g^-$ and hence were "undisturbed." Suppose the same were true of $\mathcal{C}_{g,<}$, i.e., that $\mathcal{C}_{g,<} = I_g^-\mathcal{C}_{0,<}(I_g^-)^{-1}$. In this case, since expression (3.15) holds for all g, it holds in particular for the uncoupled system $g = 0$. Since, for the uncoupled system (3.17) is an equality, it follows that

$$\mathcal{C}_{g,>} = I_g^- \; I_{0,T_2}(\mathbb{1} - \hat{C}_{\overline{R}})I^{-1}_{0,T_1} \otimes I_D(I_g^-)^{-1} \tag{3.19}$$

and therefore that the decoherence functionals $\mathcal{D}^B$ and $\mathcal{D}^{\rm HYT}$ are identical. Since this is exactly the line of reasoning used above to relate $\mathcal{D}^A$ to the "naive" result, we attribute the discrepancy between $\mathcal{D}^B$ and $\mathcal{D}^{\rm HYT}$ to the fact that this reasoning is not applicable here. That is, we attribute the difference to the fact that the "response" of device B is to an operator $\mathcal{C}_{g,<}$, which differs from its uncoupled version $\mathcal{C}_{0,<}$ so that we may refer to it as "disturbed" by device B.

IV. DISCUSSION

We have analyzed two "devices" designed to detect the presence of a particle in a spacetime region R. These two devices lead to quite different interpretations of the question, "What is the probability that a quantum free particle will be detected in a space-time region R?" as is illustrated by the different calculations performed and different results obtained in Secs. II and III. We therefore conclude that this question is not well defined without reference to the kind of detector that will be used and that different prescriptions for the calculation of probabilities are appropriate to different physical situations. In particular, since $\mathcal{D}^{\rm HYT}$ is associated only with device B and not device A, this study indicates the type of situations to which probabilities defined by partitions of configuration space paths in a path integral are and are not appropriate, although the sense in which $\mathcal{D}^{\rm HYT}$ is associated with device B is not quite the usual one.

Which of our models and therefore which calculation would be relevant to an actual experimental setting will depend on the experimental details, although Appendix A suggests that the most likely answer is "neither A nor B." Which model is more relevant to philosophical discussions of "measurements in space-time regions" will be subject to the interpretations of the philosopher. We may, however, make the distinction that device A performs a single von Neumann measurement while device B does not and that device A responds to an undisturbed value of χ while the response of B is related to the disturbed operators $\mathcal{C}_>$ and $\mathcal{C}_<$.

Finally, we would like to point out that the model discussed in Sec. III is *our interpretation* of the HYT decoherence functional and we have given no proof that this interpretation is unique. The author would, however, be willing to conjecture that the only interpretation of $\mathcal{D}^{\rm HYT}$ given by a study of detectors is more or less the one that we have described, provided that the various terms in this statement can be more or less precisely defined. In support of this conjecture, note that [6] describes such decoherence functionals as intuitively related to an infinite product of projection operators and the calculation of such path integrals in [7] uses a skeletonized version of the product $\prod_t \widehat{\theta(x(t))}$. Such a product is naturally related to models similar to B in which a large number of independent interactions take place at successive instants of time. Any further clarification of this issue, either by a formalization of the above conjecture and subsequent proof or by a description of other measurement models that provide an alternate interpretation of the HYT decoherence functional, would be much appreciated.

ACKNOWLEDGMENTS

This work was partially supported by NSF Grants Nos. PHY-9005790 and PHY-9396246 and by research funds provided by Syracuse University and the Pennsylvania State University. The author would like to express his thanks to Guillermo Mena Marugán and Jorma Louko for helpful editorial comments and to Jim Hartle for useful discussions.

APPENDIX A: BUBBLE CHAMBERS

Neither apparatus described in Secs. II and III bears much resemblance to common measuring devices actually used to detect particles in space-time regions. In this appendix also, we will not discuss an *accurate* model of a real such device nor do we analyze the simplistic "model" of a bubble chamber that we do present in the manner of Secs. II and III. We describe this model only to show that neither device (A or B) models an apparatus resembling a bubble chamber and choose the bubble chamber for its historical value as the following discussion applies equally well (and equally poorly) to photographic film, scintillation counters, and other such devices.

Each localized group of molecules in a bubble chamber constitutes a degree of freedom: a group either nucleates a bubble or it does not. Let us assume that the chamber is "cold" and that these molecules do not move significantly during the experiment so that we may associate one degree of freedom with each point of the space inside the chamber. Each of these degrees of freedom is sensitive to the amount of time that the particle spends in its vicinity, since the longer the particle stays nearby, the more likely it is that a bubble will form. In order to relate bubble chambers to the "devices" described above, we might construct a model which has one "pointer" variable A_λ for each point λ in the chamber, such that each of these pointers interacts with the particle through a von Neumann–like term when the particle occupies its position between the times T_1 and T_2. We might choose, for example, the action

$$S_{\rm tot} = S_{\rm free} + \int_{-\infty}^{\infty} dt \int_{\lambda \in \text{ chamber}} d\lambda\ \pi_\lambda \delta(x-\lambda) + \int_{-\infty}^{\infty} dt \int_{\lambda \in \text{ chamber}} d\lambda dt\ \pi_\lambda \dot{A}_\lambda. \tag{A1}$$

Such a model couples pointers to the integrals $\mathcal{I}_\lambda = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau\ \mathcal{P}_{\lambda,\tau}(t)$ over times $\tau \in (T_1, T_2)$ instead of to the integrals $\chi = \int_{-\infty}^{\infty} dt \int_R d\lambda d\tau \mathcal{P}_{\lambda,\tau}(t)$ over all of R as in model A or to the individual $\int_{-\infty}^{\infty} dt\ \mathcal{P}_{\lambda,\tau}$ as in model B. Thus we see that neither of the devices described in Secs. II and III models such a device.

We should comment once again that even the model just presented is far from an accurate description of a real bubble chamber. An important feature of the real device is its discrete nature: the degree of freedom at a given point either "triggers" (nucleates a bubble) or does not so that the bubble nucleation proceeds probablistically. For a classic description of bubble chambers which does capture this discrete feature, see [1].

APPENDIX B: THE IDEAL MEASUREMENT LIMIT

In this appendix, we briefly review how the standard "probability" distribution for measurements is reached in the ideal measurement limit. Specifically, we derive the result (2.5).[5] We begin by assuming that state of the pointer at time T_1 is $|g_\sigma(0); g, T_1\rangle$ peaked around $A'_{\lambda,\tau} = 0$, where

$$|g_\sigma(A'); g, T_1\rangle = \int_{-\infty}^{\infty} dA \frac{e^{-(A-A')^2/2\sigma^2}}{\pi^{1/4}\sqrt{\sigma}} |A; g, T_1\rangle \tag{B1}$$

is a normalized state in $\mathcal{H}_{g,D:T_1}$, though the precise form B1 will not be essential to our discussion. Since the initial state of the pointer has a finite width σ, we would like the pointer to move at least this far to the right before we can confidently say that a particle has been detected. Let us therefore define "detection" of the particle as the presence of the pointer at least $n\sigma$ to the right of the origin at time T_2 so that we will be concerned with the projections $\hat{\Pi}_{A(T_2)<n\sigma}$ and $\hat{\Pi}_{A(T_2)>n\sigma}$ onto the appropriate part of the spectrum of $\int_R d\lambda d\tau\ A_{\lambda,\tau}(T_2)$. Decoherence is then immediate since these projections commute. Note that classically we might say that our "confidence that the pointer has responded to the presence of a particle" is determined by n, while the "absolute inaccuracy" in the measurement is given by $n\sigma$.

Now the pointer values before and after the experiment are related by

$$\hat{A}_g(T_2) = \hat{A}_g(T_1) + g I_g^- \hat{\chi}_0 (I_g^-)^{-1} \tag{B2}$$

so that, if the state of our system is

$$|\psi\rangle = |\phi\rangle \otimes |g_\sigma(0); g, T_1\rangle \tag{B3}$$

for some normalized $|\phi_p\rangle \in \mathcal{H}_{f:T_1}$, it is also of the form

$$|\psi\rangle = \int d\mu_{\hat{\chi}_f}(\chi_1)\ \phi(\chi_1)|\chi_1\rangle \otimes |g_\sigma(g\chi_1); g, T_2\rangle \tag{B4}$$

in terms of a factorization defined by the commuting operators $I_g^- \hat{\chi}_0 (I_g^-)^{-1}$ and $\hat{A}_g(T_2)$. Here $|g_\sigma(\chi_1); g, T_2\rangle$ is a normalized state defined in analogy with $|g_\sigma(\chi); g, T_1\rangle$ of Eq. (B1) and $|\chi_1\rangle$ is the eigenstate of $I_g^- \hat{\chi}_0 (I_g^-)^{-1}$ with eigenvalue χ_1. The measure $d\mu_{\chi_f}(\chi_1)$ is the spectral measure for the operator $\hat{\chi}_f$ and the function $\phi(\chi_1)$ is given by $\phi(\chi_1) = \langle \chi_1; T_1 | \phi\rangle$, where $|\chi_1; T_1\rangle$ is the eigenvector of $\hat{\chi}_f$ with eigenvalue χ_1 in $\mathcal{H}_{g,p:T_1}$. The projection that corresponds to the "avoids R" alternative is then

$$\hat{\Pi}_{A(T_2)<n\sigma}|\psi\rangle = \int d\mu_{\hat{\chi}_f}(\chi_1)\phi(\chi_1)|\chi_1\rangle \otimes |h_\sigma(g\chi_1); 1, T_2\rangle, \tag{B5}$$

where

$$|h_\sigma(g\chi); T_2\rangle = \int_{-\infty}^{n\sigma} dA\ \frac{e^{-(A-g\chi)^2/2\sigma^2}}{\sqrt{2\pi\lambda}} |A; T_2\rangle \tag{B6}$$

and the "probability" for detecting the particle in region R is defined by

$$\mathcal{P}_{R,\sigma} = \langle\psi|\hat{\Pi}_{A(T_2)<n\sigma}|\psi\rangle = \int d\mu_{\hat{\chi}_f}(\chi) \int_{-\infty}^{n\sigma} dA \frac{e^{-(A-g\chi)^2/n\sigma^2}}{\sqrt{2\pi\sigma}} |\phi(\chi)|^2. \tag{B7}$$

Note that for any $n > 0$ this goes to zero in the limit $\sigma \to 0$ since $\hat{\chi}_f$ has no normalizable eigenvectors with eigenvalue $\chi \le 0$. Note also that we keep the coupling constant finite while taking this limit or at least, if the

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[5] Strictly speaking we derive the result (2.5) only near $A' = 0$, though the general result follows in the same way. In addition, we note that Eq. (2.5) is only needed near $A' = 0$ for the discussion of Sec. II.

limit of small coupling is to be taken, we must send g to zero more slowly than σ. In this way, we see that the limits of small coupling and ideal measurement do not commute.

Since $\sigma \to 0$ is the limit in which the measurements become more and more accurate, we say that the probability is *one* for a perfect detector to find the particle in region R with any confidence level n. We may in fact take the limit $n \to \infty$ simultaneously with the limit $\sigma \to 0$ so long as the "absolute inaccuracy" $n\sigma$ becomes zero in this limit as well. We therefore refer to the limit $n \to \infty$, $n\sigma \to 0$ as the "ideal measurement limit."

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