Anomalous commutation relation and modified spontaneous emission inside a microcavity

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Usual quantum-optical operator relations for a beam splitter are shown to lead to an anomalous commutation relation inside a microcavity. The physical origin of this anomaly is identified as self-interference of the mode whose coherence length is longer than the round-trip length of the cavity. Altered spontaneous emission of an excited atom is found to be a direct manifestation of this anomalous commutation relation. The anomalous Heisenberg uncertainty relations, which are derived from the commutation relation according to the Schwartz inequality, cannot be detected by probing the internal field with a beam splitter. The anomalous commutation relation, however, can be related to the change in the effective reflectivity of the beam splitter. The similarity and difference between an excited atom and a probe beam splitter are discussed.

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Recent interest in quantum optics and cavity quantum electrodynamics has focused on various aspects of manipulating vacuum fluctuations in the electromagnetic field. Squeezing light [1–3] is one form of manipulation, defined as the redistribution of the quantum noises of two noncommutable observables by preserving their uncertainty product. Suppressing or enhancing the spontaneous emission rate of an excited atom in a microcavity is another example, one which is usually ascribed to the change in the density of relevant modes in the cavity [3, 4].

In this paper, we show that conventional quantum-optical operator relations for a beam splitter lead to an anomalous commutation relation inside a microcavity. In addition, we discuss the physical origin and implications of this result. In closing, we discuss a possible way to probe this anomaly.

Let us consider a single-mode monochromatic electromagnetic field which fills the inside and outside of a microcavity, as is shown in Fig. 1. Here, $\hat{a}_\omega$ is the operator for the incoming mode, $\hat{b}_\omega$ represents the internal mode propagating to the left, $\hat{c}_\omega$ represents the one propagating to the right, and $\hat{d}_\omega$ is the outgoing mode operator. We adopt continuum-mode analysis to treat the field [5, 6], in which the commutation relation for the input field is written as

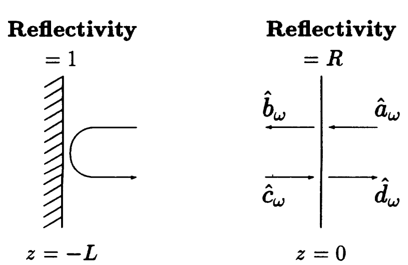

FIG. 1. Microcavity and field operators.

$$[\hat{a}_\omega, \hat{a}^\dagger_{\omega'}] = \delta(\omega' - \omega) . \tag{1}$$

Based on the boundary conditions at the mirror and the half mirror, three inter-relations between the four mode operators are derived. For simplicity, we assume that the mirror at $z = -L$ is made of a perfect conductor in which the electric field amplitude vanishes. This gives

$$\hat{b}_\omega \, e^{ikL} = -\hat{c}_\omega \, e^{-ikL} , \tag{2}$$

where phase shift by propagation is taken into account with wave number k. At $z = 0$, the half mirror gives the operator relations

$$\hat{b}_\omega = \mathbf{t}\hat{a}_\omega + \mathbf{r}\hat{c}_\omega \tag{3}$$

and

$$\hat{d}_\omega = \mathbf{t}\hat{c}_\omega + \mathbf{r}\hat{a}_\omega . \tag{4}$$

These two hold [7, 8] when $\mathbf{t}$ and $\mathbf{r}$ are the amplitude transmittance and reflectivity satisfying the relations

$$|\mathbf{t}|^2 + |\mathbf{r}|^2 = 1 \quad \text{and} \quad \mathbf{t}^*\mathbf{r} + \mathbf{r}^*\mathbf{t} = 0 . \tag{5}$$

For simplicity later in the process, one can choose the phase shift of transmission and reflection without losing generality:

$$\mathbf{t} = i\sqrt{1-R} \quad \text{and} \quad \mathbf{r} = -\sqrt{R} , \tag{6}$$

where R ($0 \le R \le 1$) is the power reflectivity of the half mirror. From Eqs. (2)–(6), we can express $\hat{b}_\omega$, $\hat{c}_\omega$, and $\hat{d}_\omega$ by incident mode operator $\hat{a}_\omega$, resulting in

$$\hat{b}_\omega = \frac{i\sqrt{1-R}}{1-\sqrt{R}e^{2ikL}} \, \hat{a}_\omega , \tag{7}$$

$$\hat{c}_\omega = \frac{-i\sqrt{1-R}}{e^{-2ikL}-\sqrt{R}} \, \hat{a}_\omega , \tag{8}$$

and

$$\hat{d}_\omega = \frac{1-\sqrt{R}e^{-2ikL}}{e^{-2ikL}-\sqrt{R}}\,\hat{a}_\omega \ . \tag{9}$$

Similar mode operator relations can also be found in previous papers [9, 10], where the relations are used only for obtaining the input-output relation of the cavity. In this paper, however, we pay attention to the commutation relation *inside* the cavity, and the importance of mode definition time will be emphasized later. From Eq. (9) and its Hermitian conjugate one can derive

$$[\hat{d}_\omega, \hat{d}^\dagger_{\omega'}] = [\hat{a}_\omega, \hat{a}^\dagger_{\omega'}] \ . \tag{10}$$

Assuming that the incident field satisfies the normal commutation relation (1), it is concluded that the outgoing field also satisfies it. On the other hand, Eqs. (7) and (8) lead to

$$[\hat{b}_\omega, \hat{b}^\dagger_{\omega'}] = [\hat{c}_\omega, \hat{c}^\dagger_{\omega'}] = \frac{1-R}{1-2\sqrt{R}\cos(2\omega L/c)+R}\,\delta(\omega'-\omega) \ , \tag{11}$$

using Eq. (1). Here, wave number k is replaced by ω/c. This equation indicates that the commutation relation is anomalous inside the cavity. The coefficient of the δ function is plotted in Fig. 2 as a function of ω for several values of reflectivity R. This figure clearly shows the crossover between the continuum spectrum in an open space ($R = 0$) and the almost discrete spectrum in a high-Q cavity ($R = 0.99$).

Let us examine the physical origin of this anomaly. It is easily understood that simultaneous equations (2) and (3) are related to the requirement of self-consistency for the internal field operator after one round-trip in the cavity. This means that we have implicitly assumed a long-wave-packet mode having duration longer than one round-trip. Such a long-wave-packet mode interferes with itself during the multifolded round-trip in a cavity. As a consequence, the amplitude of the internal mode is either enhanced or suppressed in comparison with the modes in the free space. This causes the change in the commutator-bracket value.

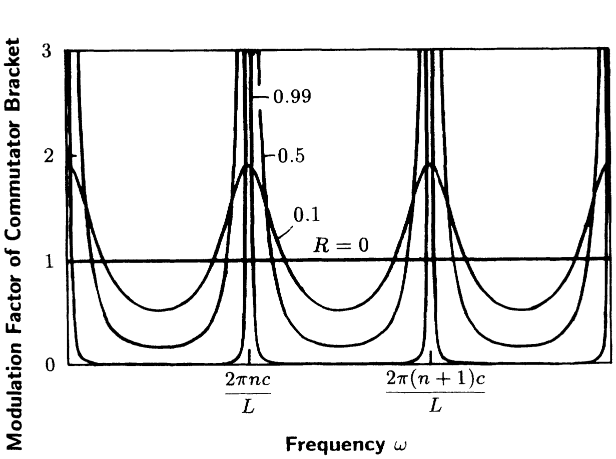

FIG. 2. The commutator-bracket value inside a microcavity as a function of the wave number k for several values of the reflectivity R.

Conversely, if we assume localized pulse modes, the anomaly disappears because there is no self-interference. To see this, let us consider a localized pulse mode by Fourier transform of the continuum suffix ω into time t [11]:

$$\hat{a}_t \equiv \sqrt{\frac{1}{2\pi}}\int d\omega e^{-i\omega t}\hat{a}_\omega \ . \tag{12}$$

This indicates a pulse mode which is localized to time t at $z = 0$ and satisfies the commutation relation

$$[\hat{a}_t, \hat{a}^\dagger_{t'}] = \delta(t'-t) \ , \tag{13}$$

according to Eq. (1). The input-output relationship of a beam splitter is complicated when $\mathbf{t}$ and $\mathbf{r}$ depend on frequency. Here, however, we assume that there is no frequency dependence — no response delay — in $\mathbf{t}$ and $\mathbf{r}$. In this case, index ω may simply be changed into t in Eqs. (3) and (4). Thus, from Eqs. (10) and (12), one can easily obtain

$$[\hat{d}_t, \hat{d}^\dagger_{t'}] = [\hat{a}_t, \hat{a}^\dagger_{t'}] = \delta(t'-t) \ . \tag{14}$$

This means, as is expected, that the outgoing pulse mode also satisfies the normal commutation relation. As for the internal pulse modes, using Eq. (11), one can obtain

$$\begin{aligned}[\hat{b}_t, \hat{b}^\dagger_{t'}] &= [\hat{c}_t, \hat{c}^\dagger_{t'}] \\ &= \delta(t'-t) + \sum_{n=1}^{\infty}\sqrt{R^n}\,\delta\left(t'-t-\frac{2nL}{c}\right) \\ &\quad + \sum_{n=1}^{\infty}\sqrt{R^n}\,\delta\left(t'-t+\frac{2nL}{c}\right) \ , \end{aligned} \tag{15}$$

where $2L/c$ is the round-trip time. The first δ function corresponds to the normal commutation relation due to the fact that there is no pulse-mode self-interference. In other words, the pulse mode does not know whether it is in a cavity or free space. The second summation term of the δ functions indicates the memory of past pulses bounced in the cavity with a decay factor $\sqrt{R^n}$. The third term indicates the influence of the pulse on the future pulse series. This means that the effect of the cavity round-trip appears as an intermode correlation without self-interference for the localized pulse modes. The commutation relation for each Fourier component, however, is not necessarily normal, which is the main argument of this paper.

A detector sensing only a single Fourier component might detect some trace of the anomalous commutation relation. An example of such a detector is an excited two-level atom having a sharp linewidth. The emission probability, including spontaneous and stimulated emission, is proportional to the square of $\hat{E}$ [12], which is expressed, using continuum-mode analysis, as

$$\hat{E}^2 = \left\{ i\int d\omega\sqrt{\frac{\hbar\omega}{4\pi\varepsilon_0 cA}}\left[\hat{b}_\omega e^{-i\omega(t+z/c)} + \hat{c}_\omega e^{-i\omega(t-z/c)}\right] + \text{H.c.}\right\}^2 , \tag{16}$$

where A is the beam's cross-sectional area. With Eqs. (2) and (16) one obtains

$$\hat{E}^2 = \frac{\hbar}{2\pi\varepsilon_0 cA}\int d\omega d\omega'\sqrt{\omega\omega'}\left\{\hat{b}^\dagger_\omega\hat{b}_{\omega'} + \frac{1}{2}[\hat{b}_\omega,\hat{b}^\dagger_{\omega'}]\right\} \times[1-e^{2i\omega(z+L)/c}] \times[1-e^{-2i\omega'(z+L)/c}]\,, \quad (17)$$

when Eq. (1) and the rotating wave approximation are used. On the right-hand side of Eq. (17), $\hat{b}^\dagger_\omega\hat{b}_{\omega'}$ corresponds to the stimulated emission rate, while the commutator-bracket part corresponds to the spontaneous-emission rate. With Eq. (16), spontaneous emission is given by

$$F = \frac{1-R}{1-2\sqrt{R}\cos(2\omega L/c)+R}\sin^2[\omega(z+L)/c]\,, \quad (18)$$

where F is the spontaneous-emission-modification factor, which must be unity in free space. Equation (18) indicates that the spontaneous-emission rate is modulated both in terms of resonance frequency ω and atom position z. As a consequence, the spontaneous-emission probability per excited atom is either suppressed or enhanced inside a microcavity, based on the commutator-bracket value. The right-hand side of this equation reduces to the well-known spontaneous-emission enhancement factor in one special case [4, 13]. With a resonant wave number, F becomes

$$F = \frac{1+\sqrt{R}}{1-\sqrt{R}}\frac{1}{2} \cong \frac{1}{1-\sqrt{R}}, \quad (19)$$

where $\sin^2[\omega(z+L)/c]$ is spatially averaged, and a high-Q cavity condition ($R \sim 1$) is assumed. By using the cavity-quality factor $Q \equiv \frac{\omega\tau}{2\pi}$, where τ is the photon lifetime in the cavity, Eq. (19) reduces to $F = \frac{Q\lambda}{L}$. The three-dimensional version of this result is $F = \frac{Q\lambda^3}{V}$, which agrees with conventional results [4].

The suppression/enhancement of the spontaneous emission can be understood from the viewpoint of self-interference. When a two-level atom has a narrow spectral linewidth $\Delta\omega$, that is, when the atom exhibits a sharp frequency selectivity, the time interval the atom needs to specify the frequency is of the order of $T \equiv 2\pi/\Delta\omega$. The mode-definition time for the atom is thus T. When T is much longer than the time required for one round-trip in the microcavity, an optical wave packet having time duration T interferes with itself, and the effective field amplitude at the position of the atom differs from that in free space. The spontaneous-emission rate is thus either suppressed or enhanced. When the two-level atom has a wide linewidth, making its mode-definition time shorter than the round-trip time of the cavity, the atom either emits or absorbs a photon before the photon interferes with itself in the cavity. The space inside the cavity is equivalent to open space for the atom, and there is no suppression or enhancement of spontaneous emission. Thus the origin of the modification of spontaneous emission is the same as that of the anomalous commutation relation.

Heisenberg uncertainty relation for a set of two noncommuting observables is derived from the corresponding commuation relations using the Schwartz inequality. Because we have shown that the commutation relation between the annihilation and creation operators is anomalous, we know that commutation relation between any two noncommuting observables, say, field quadrature-phase components and number-phase operators, is anomalous. The corresponding uncertainty relations should, therefore, exhibit anomaly inside the cavity. One may expect that the anomalous uncertainty relation can be detected by measuring two conjugate observables in the inside field. If, however, one puts a probe beam splitter of reflectivity R' in the cavity to extract the inside field, as is shown in Fig. 3, one can easily find, by straightforward calculation, that the outcoming beam exhibits the normal commutation relation. This is due to the vacuum modes coupled into the cavity with the probe beam splitter. It is not possible to extract the internal field as it is. It is therefore not possible to measure two conjugate observables in the internal field which violate Heisenberg's uncertainty principle.

It is, however, possible to detect the modified commutator-bracket value as the change in effective reflectivity R'_{eff} of the probe beam splitter. Defining R'_{eff} as the differential ratio of output probe intensity over input probe intensity, it is easy to show that

$$R'_{\text{eff}} = \frac{R'(1-R+RR')}{1+R(1-R')^2-2\sqrt{R}(1-R')\cos(2\omega L/c)}\,. \quad (20)$$

We now assume $R' \ll R$ (thus $R' \ll 1$) because the probe beam splitter should not disturb the field under consideration. This allows us to write

$$F' \equiv \frac{R'_{\text{eff}}}{R'} = \frac{1-R}{1+R-2\sqrt{R}\cos(2\omega L/c)}\,, \quad (21)$$

where F' is the modification factor of the reflectivity of the probe beam splitter. This coincides with the modification factor of the commutation relation in Eq. (11). Thus the reflectivity of the probe beam splitter in free space is effectively enhanced or suppressed in the cavity, as is the spontaneous-emission rate of an atom. The origin of this phenomenon is the same as that of the anomalous commutation relation, i.e., the self-interference in the monochromatic field. The modified reflectivity can be observed using a monochromatic probe beam, but not one with a short pulse, for the same reason as given before.

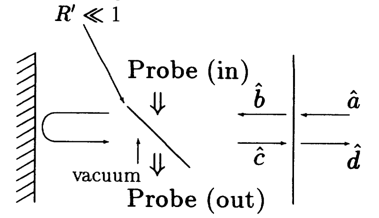

FIG. 3. Microcavity with a probe beam splitter inside.

Equation (21) differs from Eq. (18) by the factor $\sin^2[\omega(z+L)/c]$, which is position dependent. It is worth clarifying the similarity and difference between the two detectors, an excited atom and a beam splitter. First, a detector should be able to couple with the *local* electric (or magnetic) field inside the cavity in probing the alteration of the internal field amplitude. Furthermore, frequency selectivity is required to detect the anomalous commutation relation because the commutation relation appears for (quasi) monochromatic modes. Both an atom and a beam splitter (with a monochromatic probe beam) satisfy the above conditions. Hamiltonians for both schemes are also similar in that excitation of the probe is coupled to deexcitation of the field as

$$\hat{H} = \kappa\hat{b}\hat{a}^\dagger_{\mathrm{probe}} + \kappa^*\hat{b}^\dagger\hat{a}_{\mathrm{probe}} \quad \text{(beam splitter)} \tag{22}$$

and

$$\hat{H} = \kappa\hat{b}\hat{\sigma}^\dagger + \kappa^*\hat{b}^\dagger\hat{\sigma} \quad \text{(atom : Jaynes-Cummings)}, \tag{23}$$

where κ is the coupling constant, $\hat{b}$ the field inside the cavity, $\hat{a}_{\mathrm{probe}}$ the photon creation operator of the probe beam, and $\hat{\sigma}^\dagger$ the transition operator of the probe atom. The modification of the commutator bracket effectively leads to the modification of κ, which physically means the modification of the effective reflectivity for the probe beam splitter or the modification of the spontaneous-emission probability for the probe atom.

The difference between the two detectors is that an atom does not identify the propagation direction while a beam splitter does. An atom responds to the square of the summation of the electric fields such that it senses the interference between the right- and left-propagating waves, which is indicated by the sinusoidal oscillation with respect to z in Eq. (18). A beam splitter identifies these waves and diffracts them in the opposite directions, thereby insensitive to the interference between the counterpropagating waves. It may be worth pointing out here that, in a ring cavity such as the one in Ref. [10], there is no self-interference between counterpropagating beams, and the factor $\sin^2[\omega(z+L)/c]$ is thus wiped out even for a localized two-level atom — an atom and a beam splitter equivalently sense the anomalous commutation relation uniformly in the ring cavity.

In conclusion, we have shown that conventional quantum-optical operator relations for a beam splitter lead to an anomalous commutation relation inside the microcavity. We identified the physical origin of this anomaly as self-interference of the mode inside the microcavity. We also demonstrated that suppression or enhancement of the spontaneous emission of an excited atom inside a microcavity is a direct manifestation of the anomalous commutation relation. However, the anomalous Heisenberg uncertainty relation, which results from this commutation relation by using the Schwartz inequality, cannot be detected. This means that the internal cavity field is not observable if we choose the monochromatic beam mode. For pulse modes, the internal field becomes observable because there is no self-interference and one can extract the modes as they are. The commutation relations for electric and magnetic field quantities are always normal, so long as local-mode description is used. Their Fourier components, however, can exhibit modified commutation relation under some boundary conditions, such as a microcavity.

We have also pointed out that the anomalous magnitude of the commutator bracket can also be detected by a probe beam splitter just as the effective reflectivity is modified. The origin of this is the same as that of the modification of the spontaneous-emission rate. The difference is that the beam splitter does not exhibit the position dependence, whereas an atom does. We have analyzed the requirements for a detection scheme that can sense the anomalous commutation relation inside the cavity, and pointed out the similarity and difference between an atom and a probe beam splitter.

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