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Rotation angle, phases of oscillators with definite circular polarizations, and the composite ideal phase operator

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The exposition of the formalisms of the quantum optical phase has been provided commencing from the quantum mechanics of a massive particle. The clockwise and counterclockwise components of the planar motion are shown to correspond to the signal and idler modes in the heterodyne detection scheme. The appropriate ideal phase proposal has been embodied in this scheme.

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I. INTRODUCTION

In a review paper [1], which was unique for a considerable time, the paradoxical quantum phase was treated alongside the rotation angle operator. The solution of the quantum phase problem devised by Newton [2] resembles the rotation angle operator and the recently developed Ban concept of the phase angle operator [3] is similar to that of the rotation angle operator in spite of the use of thermofield language. The treatment of Ban's operator as an ideal concept and the interpretation of the Shapiro-Wagner model [4] as the feasible phase concept have been performed by Hradil [5].

The rotation angle is characteristic of the polar coordinate system and by reintroducing the Cartesian coordinate operators and their conjugate momenta, we can arrive at the annihilation and creation operators of harmonic oscillators. An analogy between this mechanical system and the optical one enables us to go from the annihilation operators of linear polarizations to those of circular polarizations. We can compare the latter operators with the operators of the signal and idler modes in the Shapiro-Wagner model. Of course, the polarizations used in the exposition of these connections can be distinguished from realistic polarizations occurring in the process of measurement. Recently, the general principle underlying various measuring schemes has been elucidated [6]. Finally, we indicate Ban's phase operator in this physical system and compare it with the original rotation angle operator.

II. TWO-DIMENSIONAL HARMONIC OSCILLATOR

Some papers of fundamental importance consider the quantum rotation angle and the quantum phase (see [7,8]) always in the framework of two different models. In this paper we would like to show that there exists a model enabling us to study the angle of rotation and the phase simultaneously. This model, besides, is related also to the detection schemes (the heterodyne detection). We will consider a two-dimensional harmonic oscillator, i.e., a particle moving freely in the plane and subjected to a quadratic potential. The Hamiltonian reads

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2b}(\hat{x}^2 + \hat{y}^2), \tag{2.1}$$

where m is the mass of the particle, b is the elasticity constant, $\hat{x}$, $\hat{y}$ are position coordinates, and $\hat{p}_x$, $\hat{p}_y$ are the appropriate conjugate momenta, i.e.,

$$\begin{gathered}[\hat{x},\hat{p}_x]= i\hbar\hat{1}, \quad [\hat{y},\hat{p}_y] = i\hbar\hat{1}, \quad [\hat{x},\hat{p}_y] = \hat{0}, \\ [\hat{y},\hat{p}_x]= \hat{0}, \quad [\hat{x},\hat{y}] = \hat{0}, \quad [\hat{p}_x,\hat{p}_y] = \hat{0},\end{gathered} \tag{2.2}$$

with $\hat{0}$ standing for the zero operator.

Let us note that some mathematical difficulties are connected with the relations of canonical conjugation (2.2), which are discussed in a recent paper [9].

Introducing the annihilation operators

$$\begin{aligned}\hat{a}_x &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt[4]{\frac{m}{b}}\hat{x} + i\sqrt[4]{\frac{b}{m}}\hat{p}_x\right), \\ \hat{a}_y &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt[4]{\frac{m}{b}}\hat{y} + i\sqrt[4]{\frac{b}{m}}\hat{p}_y\right)\end{aligned} \tag{2.3}$$

fulfilling the commutation relations ($\hat{a}_x^\dagger$, $\hat{a}_y^\dagger$ are the creation operators)

$$\begin{gathered}[\hat{a}_x,\hat{a}_x^\dagger]= \hat{1}, \quad [\hat{a}_y,\hat{a}_y^\dagger] = \hat{1}, \quad [\hat{a}_x,\hat{a}_y] = \hat{0}, \\ [\hat{a}_x^\dagger,\hat{a}_y^\dagger]= \hat{0}, \quad [\hat{a}_x,\hat{a}_y^\dagger] = \hat{0}, \quad [\hat{a}_y,\hat{a}_x^\dagger] = \hat{0},\end{gathered} \tag{2.4}$$

we can rewrite the Hamiltonian (2.1) in the form

$$\hat{H} = \hbar\omega\left(\hat{a}_x^\dagger \hat{a}_x + \frac{1}{2}\right) + \hbar\omega\left(\hat{a}_y^\dagger \hat{a}_y + \frac{1}{2}\right), \qquad (2.5)$$

where

$$\omega = (mb)^{-\frac{1}{2}}. \qquad (2.6)$$

Hence a two-dimensional harmonic oscillator can be understood as two uncoupled one-dimensional oscillators distinguished by the indices x, y. We can consider the number of energy quanta for each oscillator,

$$\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x, \quad \hat{n}_y = \hat{a}_y^\dagger \hat{a}_y, \qquad (2.7)$$

and, intending to consider appropriate canonically conjugate phases, we arrive at the phase problem. It is true that there exist physically interesting unphysical resolutions of the identity (unnormalizable, not belonging to a Hilbert space) [8]

$$\hat{1} = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} |\varphi_x\rangle\langle\varphi_x| \otimes \hat{1}_{y,p_y}\, d\varphi_x,$$
$$\hat{1} = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} \hat{1}_{x,p_x} \otimes |\varphi_y\rangle\langle\varphi_y|\, d\varphi_y \qquad (2.8)$$

according to the states $|\varphi_x\rangle$, $|\varphi_y\rangle$ with determined phases φ_x, φ_y, respectively,

$$|\varphi_x\rangle = \sum_{n_x=0}^{\infty} \exp(in_x\varphi_x)|n_x\rangle,$$
$$|\varphi_y\rangle = \sum_{n_y=0}^{\infty} \exp(in_y\varphi_y)|n_y\rangle. \qquad (2.9)$$

Here the basic single-mode states $|n_x\rangle$, $|n_y\rangle$,

$$\hat{n}_x|n_x\rangle = n_x|n_x\rangle, \quad \hat{n}_y|n_y\rangle = n_y|n_y\rangle, \qquad (2.10)$$

correspond to the oscillator energies $E_x = \hbar\omega(n_x + \frac{1}{2})$, $E_y = \hbar\omega(n_y + \frac{1}{2})$, and θ is the minimum value of the measured phase. The appropriate basic two-mode states are denoted by $|n_x, n_y\rangle$. However, these resolutions are not orthogonal, not even in a generalized sense, i.e., in the sense of the Dirac δ orthogonality. Several authors have proved that such resolutions of the identity provide the ideal phase concept, i.e., they define phase operators according to the rule [10,4]

$$\hat{M}(\varphi_x) = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} M(\varphi_x)|\varphi_x\rangle\langle\varphi_x| \otimes \hat{1}_{y,p_y}\, d\varphi_x,$$
$$\hat{M}(\varphi_y) = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} M(\varphi_y)\hat{1}_{x,p_x} \otimes |\varphi_y\rangle\langle\varphi_y|\, d\varphi_y, \qquad (2.11)$$

where $M(\varphi)$ can be $\cos\varphi$, $\sin\varphi$, $\exp(i\varphi)$, etc.

Under certain initial conditions it is convenient to regard the two-dimensional oscillator as a plane rotator and in accordance with this we can introduce an exponential operator of the rotation angle Φ,

$$\widehat{\exp}(i\Phi) = \frac{\hat{x} + i\hat{y}}{\sqrt{\hat{x}^2 + \hat{y}^2}}, \qquad (2.12)$$

and an operator of the angular momentum,

$$\hat{M}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x. \qquad (2.13)$$

There exists a Hermitian operator of the rotation angle

$$\hat{\Phi}_\theta = \arg_\theta \widehat{\exp}(i\Phi), \qquad (2.14)$$

whose spectrum fills up the interval $[\theta, \theta + 2\pi)$. The appropriate radius reads

$$\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2} \qquad (2.15)$$

and the following resolution of the identity holds:

$$\hat{1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{1}_{\Phi,M_z} \otimes |p_r\rangle\langle p_r|\, dp_r, \qquad (2.16)$$

where $|p_r\rangle$ is the state with definite radial component p_r of the momentum. For the state $|p_r\rangle$ it holds that

$$|p_r\rangle = \int_0^\infty \exp(irp_r)|r\rangle\, dr, \qquad (2.17)$$

where the state $|r\rangle$ is the position at some radial distance and it is valid that

$$\hat{r}|r\rangle = r|r\rangle. \qquad (2.18)$$

As $r \geq 0$, the spectrum of the operator $\hat{r}$ is bounded from below and we have here the problem of radial momentum, i.e., an analog to the phase problem. Fortunately, this problem is not so acute because the operators $\hat{p}_r$, $\hat{p}_r^2$, etc., which are "local in the r representation" (they are expressed as differential operators), are well defined; for example, it holds that $\hat{p}_r^2 = \widehat{p_r^2}$, and in general

$$\hat{M}(p_r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(p_r)\hat{1}_{\Phi,M_z} \otimes |p_r\rangle\langle p_r|\, dp_r. \qquad (2.19)$$

Nevertheless, we will use the formula

$$\hat{p}_r = \hat{p}_{rA} = \hat{p}_{rS}, \qquad (2.20)$$

where

$$\hat{p}_{rA} = \hat{p}_x\hat{x}\hat{r}^{-1} + \hat{p}_y\hat{y}\hat{r}^{-1} + i\frac{\hbar}{2}\hat{r}^{-1},$$
$$\hat{p}_{rS} = \hat{r}^{-1}\hat{x}\hat{p}_x + \hat{r}^{-1}\hat{y}\hat{p}_y - i\frac{\hbar}{2}\hat{r}^{-1}. \qquad (2.21)$$

An appropriate exponential radial momentum operator $\hat{u}_r = \widehat{\exp}(ip_r)$ is not unitary and it holds that

$$\hat{u}_r|r\rangle = \begin{cases} (r-1)|r-1\rangle & \text{for } r \geq 1 \\ 0_> & \text{for } r \in [0,1), \end{cases} \qquad (2.22)$$

where $0_>$ means the zero ket, in other words, its action is not local.

The operators $\hat{\Phi}_\theta$, $\hat{M}_z$, $\hat{r}$, $\hat{p}_r$ obey the following commutation rules:

$$[\hat{\Phi}_\theta, \hat{M}_z] = i\hbar[\hat{1} - |\theta\rangle\langle\theta| \otimes \hat{1}_{r,p_r}], \quad [\hat{r}, \hat{p}_r] = i\hbar\hat{1}, \tag{2.23}$$
$$[\hat{\Phi}_\theta, \hat{r}] = \hat{0}, \quad [\hat{\Phi}_\theta, \hat{p}_r] = \hat{0}, \quad [\hat{M}_z, \hat{r}] = \hat{0}, \quad [\hat{M}_z, \hat{p}_r] = \hat{0}.$$

We will rewrite the Hamiltonian (2.1) in the form

$$\hat{H} = \hat{H}_{\Phi|r} + \hat{H}_r, \tag{2.24}$$

with the Hamiltonian of the radial oscillator

$$\hat{H}_r = \frac{1}{2m}\hat{p}_r^2 + \frac{1}{2b}\hat{r}^2 \tag{2.25}$$

and the Hamiltonian of the plane rotator

$$\hat{H}_{\Phi|r} = \frac{1}{2m\hat{r}^2}\left(\hat{M}_z^2 - \frac{\hbar^2}{4}\hat{1}\right). \tag{2.26}$$

Let us note that the angular-momentum operator

$$\hat{M}_z = i\hbar(\hat{a}_x\hat{a}_y^\dagger - \hat{a}_x^\dagger\hat{a}_y). \tag{2.27}$$

Let us suppose the physical system to be in the coherent state $|\alpha_x(0), \alpha_y(0)\rangle$ at time $t = 0$, where

$$\alpha_x(0) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt[4]{\frac{m}{b}}x + i\sqrt[4]{\frac{b}{m}}p_x\right),$$
$$\alpha_y(0) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt[4]{\frac{m}{b}}y + i\sqrt[4]{\frac{b}{m}}p_y\right), \tag{2.28}$$

and

$$x = r, \quad y = 0, \quad p_x = 0,$$
$$p_y = \pm p\,(p > 0), \quad \frac{x}{\sqrt{b}} = \frac{p}{\sqrt{m}}. \tag{2.29}$$

The relation (2.29) represents the condition for a circular motion, whereas for $b^{-1/2}x \neq m^{-1/2}p$ we obtain an elliptic motion with the dominating part of the linear motion in the direction of the x or y axis. Under the assumptions (2.29) the system stays in the coherent state for all times, obeying the relations

$$\alpha_x(t) = \alpha_x(0)\exp(-i\omega t),$$
$$\alpha_y(t) = \alpha_y(0)\exp(-i\omega t), \tag{2.30}$$

and according to the classical representation it moves along a circle in a plane. Then we have

$$\alpha_x(0) = \frac{1}{\sqrt{2\hbar}}\sqrt{rp}, \quad \alpha_y(0) = \pm\frac{i}{\sqrt{2\hbar}}\sqrt{rp} \tag{2.31}$$

and

$$\langle\hat{M}_z\rangle = xp_y - yp_x = \pm rp,$$
$$\langle(\Delta\hat{M}_z)^2\rangle = \frac{\hbar}{2\omega}\left(\frac{1}{b}x^2 + \frac{1}{m}p^2\right)$$
$$= \frac{\hbar x^2}{\omega b} = \frac{\hbar p^2}{\omega m} = \hbar xp = \hbar|\langle\hat{M}_z\rangle|. \tag{2.32}$$

The selected coherent state $|\alpha_x(0), \alpha_y(0)\rangle$ is not an eigenstate of the component of the momentum operator $\hat{M}_z$. Further, the rotation angle Φ and the phases φ_x, φ_y fulfill the relations

$$\Phi \equiv \Phi(t) = \pm\omega t, \quad \varphi_x \equiv \varphi_x(t) = -\omega t,$$
$$\varphi_y \equiv \varphi_y(t) = \pm\frac{\pi}{2} - \omega t. \tag{2.33}$$

All these quantities are related to the complex amplitudes $\alpha_x(t)$, $\alpha_y(t)$. If we choose the lower sign, it holds that $\Phi = \varphi_x$ for all times and it means that at least in the quasiclassical limit the phase φ_x (the phase of the harmonic oscillator) presents itself as a rotation angle. Besides $\varphi_y = \varphi_x - \frac{\pi}{2}$, in general, $\varphi_y = \varphi_x \pm \frac{\pi}{2}$. Under these assumptions the rotation angle operators $\hat{M}(\Phi) = M(\hat{\Phi}_\theta)$ represent a feasible phase concept for the ideal phase operators $\hat{M}(\varphi_x)$.

III. MECHANICAL MOTION AND OPTICAL POLARIZATIONS

We will study the model of "a ball on a plate" (see Fig. 1) in greater detail because two one-dimensional oscillators are, without doubt, related also to two x, y polarization states of light and to appropriate x, y modes in quantum optics.

Let us note that the relation (2.33) with the lower sign describes the ball rotating in a clockwise direction and that the upper sign stands for a counterclockwise rotation. For analogy we consider a monochromatic plane wave of frequency ω traveling in the z direction. According to [11] it holds that the components of the electric vector

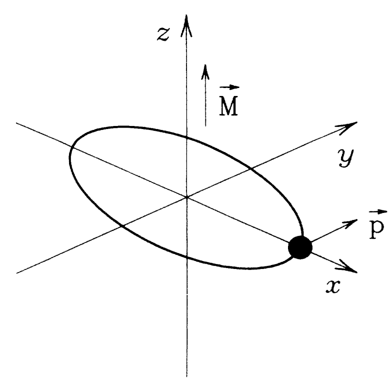

FIG. 1. Counterclockwise motion of a ball in the (x, y) plane.

$$\hat{E}_{x|_{z=0}} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(i\hat{a}_x - i\hat{a}_x^\dagger),$$
$$\hat{E}_{y|_{z=0}} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(i\hat{a}_y - i\hat{a}_y^\dagger), \quad (3.1)$$

where ϵ_0 is the permeability of vacuum, ω, which is the wave frequency, may be held for the oscillator frequency ω, V is the cavity volume, and the electromagnetic operators $\hat{a}_x$, $\hat{a}_y$ may easily be identified with the mechanical operators $\hat{a}_x$, $\hat{a}_y$ from (2.3).

On substituting the mechanical quantities and operators from (2.3) into (3.1), we obtain

$$\hat{E}_{x|_{z=0}} = -\sqrt{\frac{1}{\epsilon_0 mV}}\,\hat{p}_x,$$
$$\hat{E}_{y|_{z=0}} = -\sqrt{\frac{1}{\epsilon_0 mV}}\,\hat{p}_y. \quad (3.2)$$

Although the correspondence (3.2) is complicated for an elliptically polarized light, for the circular motions the direction of the (x, y) rotation is the same as that of (p_x, p_y) or $(-p_x, -p_y)$ rotation.

We introduce the operators

$$\hat{a}_{\mp} = \frac{1}{\sqrt{2}}(\hat{a}_x \pm i\hat{a}_y), \quad (3.3)$$

where the operators $\hat{a}_x$, $\hat{a}_y$ are given in (2.3). As usual the appropriate coherent states $|\alpha_-, \alpha_+\rangle$ are defined as eigenstates with the property

$$\hat{a}_-|\alpha_-, \alpha_+\rangle = \alpha_-|\alpha_-, \alpha_+\rangle, \quad \hat{a}_+|\alpha_-, \alpha_+\rangle = \alpha_+|\alpha_-, \alpha_+\rangle \quad (3.4)$$

and it holds that $|\alpha_-, \alpha_+\rangle \equiv |\alpha_x, \alpha_y\rangle$ for (3.3). By the formulas

$$\alpha_- = \frac{1}{\sqrt{2}}(\alpha_x + i\alpha_y), \quad \alpha_+ = \frac{1}{\sqrt{2}}(\alpha_x - i\alpha_y), \quad (3.5)$$

we obtain from (2.31) that $\alpha_-(0) = 0$, $\alpha_+(0) = \frac{1}{\sqrt{\hbar}}\sqrt{rp}$ for the upper sign in (2.31) and that $\alpha_-(0) = \frac{1}{\sqrt{\hbar}}\sqrt{rp}$, $\alpha_+(0) = 0$ for the lower sign in (2.31). So, in the mechanical model, the complex amplitude $\alpha_+(t)$ $[\alpha_-(t)]$ is related to the counterclockwise (clockwise) rotation.

To investigate the meaning of the subscript minus for the electric vector, let us consider the coherent state $|\alpha_-, \alpha_+\rangle$, where $\alpha_-(0) = \frac{1}{\sqrt{\hbar}}\sqrt{rp}$, $\alpha_+(0) = 0$, so that

$$\hat{a}_x(t)|\alpha_-, \alpha_+\rangle = \frac{1}{\sqrt{2\hbar}}\sqrt{rp}\exp(-i\omega t)|\alpha_-, \alpha_+\rangle,$$
$$\hat{a}_y(t)|\alpha_-, \alpha_+\rangle = -\frac{i}{\sqrt{2\hbar}}\sqrt{rp}\exp(-i\omega t)|\alpha_-, \alpha_+\rangle. \quad (3.6)$$

With respect to (3.2)

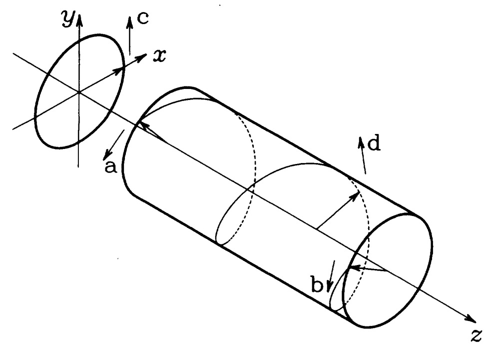

FIG. 2. The electric field of the right circularly polarized plane wave propagating in the direction of the z axis and restricted to this axis with the indication of a rotation.

$$\hat{E}_{x|_{z=0}}|\alpha_-, \alpha_+\rangle = \sqrt{\frac{1}{\epsilon_0 mV}}p\sin(\omega t)|\alpha_-, \alpha_+\rangle,$$
$$\hat{E}_{y|_{z=0}}|\alpha_-, \alpha_+\rangle = \sqrt{\frac{1}{\epsilon_0 mV}}p\cos(\omega t)|\alpha_-, \alpha_+\rangle, \quad (3.7)$$

which describes a clockwise rotation of the electric vector. According to the serious comments [12,13] there are two conventions and this rotation corresponds to the right circular polarization by the textbook (traditional) convention and to the left circular polarization by the engineering (natural) convention.

Similar reasoning exists for the use of the subscript plus. Here $\alpha_-(0) = 0$, $\alpha_+(0) = \frac{1}{\sqrt{\hbar}}\sqrt{rp}$ so that

$$\hat{a}_x(t)|\alpha_-, \alpha_+\rangle = \frac{1}{\sqrt{2\hbar}}\sqrt{rp}\exp(-i\omega t)|\alpha_-, \alpha_+\rangle,$$
$$\hat{a}_y(t)|\alpha_-, \alpha_+\rangle = \frac{i}{\sqrt{2\hbar}}\sqrt{rp}\exp(-i\omega t)|\alpha_-, \alpha_+\rangle. \quad (3.8)$$

We obtain that

$$\hat{E}_{x|_{z=0}}|\alpha_-, \alpha_+\rangle = \sqrt{\frac{1}{\epsilon_0 mV}}p\sin(\omega t)|\alpha_-, \alpha_+\rangle,$$
$$\hat{E}_{y|_{z=0}}|\alpha_-, \alpha_+\rangle = -\sqrt{\frac{1}{\epsilon_0 mV}}p\cos(\omega t)|\alpha_-, \alpha_+\rangle, \quad (3.9)$$

which represents the counterclockwise direction. It corresponds to the left circular polarization by the textbook (traditional) convention and to the right circular polarization by the engineering (natural) convention. See Fig. 2, where the vector field for the right circularly polarized light according to the engineering convention [12] is plotted. In this figure the electric field restricted to the z axis can be considered rotating without displacement as well as gliding without rotation. The tip of this electric field vector with components $E_{x|_{z=0}}$, $E_{y|_{z=0}}$ exerts the same rotation as the ball in Fig. 1.

IV. ANGLE AND ANGULAR MOMENTUM

In terms of the annihilation and creation operators $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_x^\dagger$, $\hat{a}_y^\dagger$ we can rewrite the exponential rotation angle operator (2.12) as

$$\widehat{\exp}(i\Phi) = \frac{\mathrm{Re}\,\hat{a}_x + i\mathrm{Re}\,\hat{a}_y}{\sqrt{(\mathrm{Re}\,\hat{a}_x)^2 + (\mathrm{Re}\,\hat{a}_y)^2}}. \tag{4.1}$$

Substituting the operators $\hat{a}_-$, $\hat{a}_+$ according to (3.3),

$$\hat{a}_x = \frac{1}{\sqrt{2}}(\hat{a}_- + \hat{a}_+), \quad \hat{a}_y = \frac{1}{\sqrt{2}}(-i\hat{a}_- + i\hat{a}_+), \tag{4.2}$$

and introducing the complex amplitude operator

$$\hat{\alpha} = \frac{1}{2}(\hat{a}_- + \hat{a}_+^\dagger), \tag{4.3}$$

we obtain

$$\widehat{\exp}(i\Phi) = \frac{\hat{\alpha}}{|\hat{\alpha}|}, \tag{4.4}$$

where

$$|\hat{\alpha}| = (\hat{\alpha}\hat{\alpha}^\dagger)^{1/2} = (\hat{\alpha}^\dagger\hat{\alpha})^{1/2}. \tag{4.5}$$

As in (2.7) we introduce the number operators

$$\hat{n}_- = \hat{a}_-^\dagger \hat{a}_-, \quad \hat{n}_+ = \hat{a}_+^\dagger \hat{a}_+, \tag{4.6}$$

and we note that their two-mode eigenstates $|n_-, n_+\rangle = |n_-\rangle \otimes |n_+\rangle$ have the property

$$\begin{aligned} \hat{n}_-|n_-, n_+\rangle &= n_-|n_-, n_+\rangle, \\ \hat{n}_+|n_-, n_+\rangle &= n_+|n_-, n_+\rangle. \end{aligned} \tag{4.7}$$

In Sec. III we have shown that the upper sign in (2.31) leads to the counterclockwise rotation of the ball and the lower sign corresponds to the clockwise rotation of this particle. Let us assume more generally that the massive particle is in the state

$$\hat{\rho} = \int \Phi_{\mathcal{N}}(\alpha_-)|\alpha_-\rangle\langle\alpha_-|\, d^2\alpha_- \otimes |0\rangle_{++}\langle 0|, \tag{4.8}$$

where $\Phi_{\mathcal{N}}(\alpha_-)$ is interpreted as the quasidistribution related to the normal ordering of the reduced operators $\hat{a}_-$, $\hat{a}_-^\dagger$ [14]. Since the plus mode is in the vacuum state, it holds approximately that $\hat{a}_+ \approx \hat{0}$, i.e., the operator $\hat{a}_+$ behaves approximately like the operator $\hat{0}$.

Let us note that the angular momentum operator

$$\hat{M}_z = -\hbar(\hat{n}_- - \hat{n}_+). \tag{4.9}$$

Under the given assumption on $\hat{a}_+$ it holds that $\hat{M}_z \approx -\hbar\hat{n}_-$. Quite analogously to the couple of oscillators $\hat{a}_x$, $\hat{a}_y$, also for the oscillator $\hat{a}_-$, which is complementary to the oscillator $\hat{a}_+$, the resolution of identity reads

$$\hat{1} = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} |\varphi\rangle_{--}\langle\varphi| \otimes \hat{1}_{a_+}\, d\varphi. \tag{4.10}$$

From the definition (2.12) it follows that

$$\widehat{\exp}(i\Phi) = \left(\frac{\hat{a}_- + \hat{a}_+^\dagger}{\hat{a}_-^\dagger + \hat{a}_+}\right)^{1/2}, \tag{4.11}$$

where the "fraction"

$$\begin{aligned} \left(\frac{\hat{a}_- + \hat{a}_+^\dagger}{\hat{a}_-^\dagger + \hat{a}_+}\right)^{1/2} &= (\hat{a}_-^\dagger + \hat{a}_+)^{-1/2}(\hat{a}_- + \hat{a}_+^\dagger)^{1/2} \\ &= (\hat{a}_- + \hat{a}_+^\dagger)^{1/2}(\hat{a}_-^\dagger + \hat{a}_+)^{-1/2}. \end{aligned} \tag{4.12}$$

Because $\hat{a}_+ \approx \hat{0}$, we use the second possibility in (4.12), antinormally ordered in minus mode operators and normally ordered in plus mode operators. This suggests the formal expansion

$$\begin{aligned} (\hat{a}_+^\dagger + \hat{a}_-)^{1/2}(\hat{a}_+ + \hat{a}_-^\dagger)^{-1/2} &= \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} \binom{\frac{1}{2}}{m}\binom{-\frac{1}{2}}{m'} \hat{a}_+^{\dagger m} \\ &\quad \times \hat{a}_-^{\frac{1}{2}-m}\hat{a}_+^{m'}\hat{a}_-^{\dagger(-\frac{1}{2}-m')}. \end{aligned} \tag{4.13}$$

As $\hat{a}_+ \approx \hat{0}$, we retain the term with $m = m' = 0$ and observe that

$$(\hat{a}_+^\dagger + \hat{a}_-)^{1/2}(\hat{a}_+ + \hat{a}_-^\dagger)^{-1/2} \approx \hat{a}_-^{\frac{1}{2}}\hat{a}_-^{\dagger(-\frac{1}{2})}. \tag{4.14}$$

Confining ourselves to the minus mode, we obtain that

$$\langle n_-|\hat{a}_-^{\frac{1}{2}}\hat{a}_-^{\dagger(-\frac{1}{2})}|n_-'\rangle = \delta_{n_-+1,n_-'}\sqrt{(n_- + 1)_{\frac{1}{2}}(n_-' + 1)_{-\frac{1}{2}}}, \tag{4.15}$$

where

$$(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}. \tag{4.16}$$

This proves the identity with the Paul exponential phase operator [15]

$$\hat{E}_-^{(1)} \equiv \hat{a}_-^{\frac{1}{2}}\hat{a}_-^{\dagger(-\frac{1}{2})}. \tag{4.17}$$

More generally, other Paul's exponential operators read

$$\hat{E}_-^{(k)} \equiv \hat{a}_-^{\frac{k}{2}}\hat{a}_-^{\dagger(-\frac{k}{2})}. \tag{4.18}$$

The role of the antinormal ordering is revealed by the mapping theorem for the reduced state $\hat{\rho}_- = {}_+\langle 0|\hat{\rho}|0\rangle_+$ and Paul's exponential operators,

$$\mathrm{Tr}\left\{\hat{\rho}_-\hat{E}_-^{(k)}\right\} = \int \alpha_-^{\frac{k}{2}}\alpha_-^{*(-\frac{k}{2})}\Phi_{\mathcal{A}}(\alpha_-)\, d^2\alpha_-, \tag{4.19}$$

where the quasidistribution related to the antinormal ordering of the reduced operators $\hat{a}_-$, $\hat{a}_-^\dagger$,

$$\Phi_{\mathcal{A}}(\alpha_-) = \frac{1}{\pi}\hat{\rho}_-^{(\mathcal{N})}(\hat{a}_-, \hat{a}_-^\dagger)\big|_{\hat{a}_- \to \alpha_-, \hat{a}_-^\dagger \to \alpha_-^*}. \tag{4.20}$$

We will show below that under the assumption (4.8), the reduced operator

$${}_+\langle 0|\widehat{\exp}(i\Phi)|0\rangle_+ = \hat{E}_-^{(1)}. \tag{4.21}$$

The connection to quantum optics with its detection schemes can be understood by conceding that the two-dimensional harmonic oscillator will have quantum optics features and its definition will use not quantum mechanical but quantum optical notions.

The relation (4.3) defines an analog to the operator $\hat{\alpha} \sim \hat{a}_S + \hat{a}_I^\dagger$ in the Shapiro-Wagner model of the heterodyne detection [4]. The components Re$\hat{\alpha}$, Im$\hat{\alpha}$ commute and their measurement provides the distribution

$$\Phi_{\rm meas}(\alpha) = 2\Phi_{\mathcal{A}}(2\alpha). \tag{4.22}$$

For a pure state and assuming instead of (4.8) that

$$\hat{\rho} = \int \Phi_{\mathcal{N}}(\alpha_-)|\alpha_-\rangle\langle\alpha_-|\, d^2\alpha_- \otimes \int \Phi_{\mathcal{N}}(\alpha_+^*)|\alpha_+\rangle\langle\alpha_+|\, d^2\alpha_+, \tag{4.23}$$

we obtain

$$\Phi_{\rm meas}(\alpha) = \pi\Phi_S^2(\alpha_-)_{|\alpha_-=\alpha}, \tag{4.24}$$

where the Wigner function related to the symmetric ordering of the reduced operators $\hat{a}_-$, $\hat{a}_-^\dagger$ reads

$$\Phi_S(\alpha) = \frac{1}{\pi}\hat{\rho}^{(S)}(\hat{a}_-, \hat{a}_-^\dagger)_{|\hat{a}_-\to\alpha_-, \hat{a}_-^\dagger\to\alpha_-^*}. \tag{4.25}$$

These properties are apart from a scale factor in accordance with [16].

V. REPRESENTATION OF EXPONENTIAL PHASE OPERATOR IN NUMBER-STATE BASIS

The following expansion according to the basis $|n_-, n_+\rangle$ holds:

$$\widehat{\exp}(ik\Phi) = \sum_{n_-=0}^{\infty}\sum_{n_+=0}^{\infty}\sum_{n'_-=0}^{\infty}\sum_{n'_+=0}^{\infty} \langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle |n_-, n_+\rangle\langle n'_-, n'_+|. \tag{5.1}$$

We will express the coefficients in (5.1) explicitly. The matrix elements of the operator $\widehat{\exp}(ik\Phi)$ in the basis $|M_z, r\rangle$ read

$$\langle M_z, r|\widehat{\exp}(ik\Phi)|M'_z, r'\rangle = \delta_{\frac{M_z}{\hbar}, \frac{M'_z}{\hbar}+k}\delta(r - r'). \tag{5.2}$$

The coefficients of the transition from the basis $|M_z, r\rangle$ to the basis $|n_-, n_+\rangle$ can be determined as follows:

$$\langle M_z, r|n_-, n_+\rangle = \int_\theta^{\theta+2\pi} \langle M_z|\Phi\rangle\langle\Phi, r|n_-, n_+\rangle\, d\Phi, \tag{5.3}$$

where

$$\langle M_z|\Phi\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(-i\frac{M_z}{\hbar}\Phi\right), \tag{5.4}$$

and $\langle\Phi, r|n_-, n_+\rangle$ is given in (A15). As a result we obtain

$$\langle M_z, r|n_-, n_+\rangle = \delta_{\frac{M_z}{\hbar}, -(n_- - n_+)}(-1)^{n_+}\sqrt{\frac{n_+!}{n_-!}}(Ar)^{n_- - n_+} \times \mathrm{L}_{n_+}^{n_- - n_+}(A^2r^2)A\sqrt{2r}\exp(-\tfrac{1}{2}A^2r^2), \quad n_- \geq n_+ \tag{5.5}$$

where A and the Laguerre polynomials $\mathrm{L}_n^\gamma(x)$ are defined in (A4) and (A12), respectively. For $n_- < n_+$ the coefficients are obtained from those in (5.5) by interchanging n_- and n_+,

$$\langle M_z, r|n_-, n_+\rangle\, (n_- < n_+) = \langle M_z, r|n_+, n_-\rangle\, (n_- \geq n_+). \tag{5.6}$$

The matrix elements of the operator $\widehat{\exp}(ik\Phi)$ in the basis $|n_-, n_+\rangle$ can be determined as follows:

$$\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \sum_{\frac{M_z}{\hbar}}\sum_{\frac{M'_z}{\hbar}}\int_0^\infty\int_0^\infty \langle n_-, n_+|M_z, r\rangle\langle M_z, r|\widehat{\exp}(ik\Phi)|M'_z, r'\rangle\langle M'_z, r'|n'_-, n'_+\rangle\, dr\,dr'. \tag{5.7}$$

They do not depend on the mechanical quantities m, b and read, for $n_- \geq n_+$, $n'_- \geq n'_+$, $n_+ \geq n'_+$ and for $n_- \leq n_+$, $n'_- \geq n'_+$, $n_- \geq n'_+$,

$$\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \delta_{k, n'_- - n'_+ - (n_- - n_+)}(-1)^{n'_+}\sqrt{\frac{n'_-!n'_+!}{n_-!n_+!}} \times \sum_{j=M_+}^{n'_+} \frac{(-1)^j\Gamma(\frac{k}{2}+j+1)\Gamma(\frac{k}{2}+n_- - n_+ + j + 1)}{j!(n'_+ - j)!(j + n'_- - n'_+)!\Gamma(\frac{k}{2} - n_+ + j + 1)}, \tag{5.8}$$

where $M_+ = \max(0, n_+ - \frac{k}{2})$ for k an even number and $M = 0$ for k an odd number. On letting $n_+ = 0$, we arrive at formula (4.15) again, and we prove the relation (4.21).

For $n_- \le n_+$, $n'_- \le n'_+$, $n_- \ge n'_-$ and for $n_- \ge n_+$, $n'_- \le n'_+$, $n_+ \ge n'_-$ it holds that

$$\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \delta_{k,n'_- - n'_+ - (n_- - n_+)}(-1)^{n'_-}\sqrt{\frac{n'_-!n'_+!}{n_-!n_+!}} \times \sum_{j=M_-}^{n'_-} \frac{(-1)^j\Gamma(-\frac{k}{2}+j+1)\Gamma(-\frac{k}{2}-n_-+n_++j+1)}{j!(n'_- - j)!(j-n'_- + n'_+)!\Gamma(-\frac{k}{2}-n_-+j+1)}, \tag{5.9}$$

where $M_- = \max(0, n_- + \frac{k}{2})$ for k an even number and $M = 0$ for k an odd number.

For $n_- \ge n_+$, $n'_- \ge n'_+$, $n_+ < n'_+$, we use formula (5.8) according to the rule

$$\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \langle n'_-, n'_+|\widehat{\exp}(-ik\Phi)|n_-, n_+\rangle. \tag{5.10}$$

In the case $n_- \le n_+$, $n'_- \le n'_+$, $n_- < n'_-$, formula (5.9) and the rule (5.10) are used. For $n_- \ge n_+$, $n'_- \le n'_+$, $n_+ < n'_-$, when formula (5.9) is not indicated, we use formula (5.8) according to the rule (5.10). Similarly, in the case $n_- \le n_+$, $n'_- \ge n'_+$, $n_- < n'_+$, formula (5.9) and the rule (5.10) are used.

To get a quasiclassical picture of this quantum physical system, it is important to analyze asymptotically the matrix elements (5.7), as the angular momentum M_z tends to $-\infty$ $(+\infty)$ with the energy $\hbar\omega n_+$ $(\hbar\omega n_-)$ moderate. Performing the analysis of the coefficients in (5.8) for n_- and n'_- tending to infinity, we obtain

$$\begin{aligned}\lim_{n_-\to\infty}\lim_{n'_-\to\infty}\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle &= (-1)^{n_+}\sum_{j=M_+}^{n_+}\frac{(-1)^j\Gamma(j+\frac{k}{2}+1)}{j!(n_+-j)!\Gamma(\frac{k}{2}-n_++j+1)}\delta_{n_+,n'_+}\lim_{n_-\to\infty}\lim_{n'_-\to\infty}\delta_{k,n'_- - n_-}\\ &= \delta_{n_+,n'_+}\lim_{n_-\to\infty}\lim_{n'_-\to\infty}\delta_{k,n'_- - n_-},\end{aligned} \tag{5.11}$$

for $n_+ \ge n'_+$. The similar limiting procedure for the coefficients (5.9) with n_+ and n'_+ tending to infinity yields

$$\begin{aligned}\lim_{n_+\to\infty}\lim_{n'_+\to\infty}\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle &= (-1)^{n_-}\sum_{j=M_-}^{n_-}\frac{(-1)^j\Gamma(-\frac{k}{2}+j+1)}{j!(n_- - j)!\Gamma(-\frac{k}{2}+j-n_-+1)}\delta_{n_-,n'_-}\lim_{n_+\to\infty}\lim_{n'_+\to\infty}\delta_{k,n_+-n'_+}\\ &= \delta_{n_-,n'_-}\lim_{n_+\to\infty}\lim_{n'_+\to\infty}\delta_{k,n_+-n'_+},\end{aligned} \tag{5.12}$$

for $n_- \ge n'_-$.

Using the property (5.10), we find that the results (5.11) and (5.12) have a general validity for $n_+ \le n'_+$,

$$\lim_{n_-\to\infty}\lim_{n'_-\to\infty}\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \delta_{n_+,n'_+}\lim_{n_-\to\infty}\lim_{n'_-\to\infty}\delta_{k,n'_- - n_-}, \tag{5.13}$$

and for $n_- \le n'_-$,

$$\lim_{n_+\to\infty}\lim_{n'_+\to\infty}\langle n_-, n_+|\widehat{\exp}(ik\Phi)|n'_-, n'_+\rangle = \delta_{n_-,n'_-}\lim_{n_+\to\infty}\lim_{n'_+\to\infty}\delta_{k,n_+-n'_+}. \tag{5.14}$$

The two limiting procedures performed can be interpreted as an approximate equality for the exponential operator of the rotation angle on the assumption of (a) strong signal mode and weak idler mode of light fields or symmetrically of (b) weak signal mode and strong idler mode of radiation. For $k = 1$ it holds that

$$\widehat{\exp}(i\Phi) \approx (\hat{n}_- \ge \hat{n}_+)\widehat{\exp}(i\varphi_-) + (\hat{n}_- < \hat{n}_+)\widehat{\exp}(-i\varphi_+), \tag{5.15}$$

where the symbols in parentheses are operators diagonal in the minus plus number-state basis and enjoying the property

$$\begin{aligned}(\hat{n}_- \ge \hat{n}_+)|n_-, n_+\rangle &= \begin{cases}|n_-, n_+\rangle & \text{if } n_- \ge n_+\\ 0 & \text{otherwise,}\end{cases}\\ (\hat{n}_- < \hat{n}_+)|n_-, n_+\rangle &= \begin{cases}|n_-, n_+\rangle & \text{if } n_- < n_+\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{5.16}$$

and the Susskind-Glogower operators

$$\widehat{\exp}(i\varphi_\mp) = (\hat{n}_\mp + \hat{1})^{-\frac{1}{2}}\hat{a}_\mp. \tag{5.17}$$

The operator sum on the right-hand side of (5.15) enables us to consider any quantum superposition (Schrödinger cat) of the assumptions (a) and (b). Motivated by the approximate equality (5.15), we define the ideal phase operator for the signal mode

$$\widehat{\exp}(i\Phi_{-+}) = (\hat{n}_- \geq \hat{n}_+)\widehat{\exp}(i\varphi_-) + (\hat{n}_- < \hat{n}_+)\widehat{\exp}(-i\varphi_+). \tag{5.18}$$

From this

$$\langle n_-, n_+|\widehat{\exp}(i\Phi_{-+})|n'_-, n'_+\rangle = \delta_{1,n'_- - n'_+ - (n_- - n_+)} \times \delta_{\min(n_-,n_+),\min(n'_-,n'_+)}. \tag{5.19}$$

Thus we have obtained another definition for the unitary exponential phase operator $\widehat{\exp}(i\Phi_{-+})$ first considered by Ban and expressed by him in terms of the relative number states $\{|n,m\rangle\rangle, -\infty < n < \infty, m \geq 0\}$ [3,17]. The relative number states are connected with the number states $|n_-, n_+\rangle$,

$$|n,m\rangle\rangle = \theta(n)|n_- = m+n, n_+ = m\rangle + \theta(-1-n)|n_- = m, n_+ = m-n\rangle, \tag{5.20}$$

where

$$\theta(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases} \tag{5.21}$$

Rewriting the relation (5.19) in Ban's basis, we easily obtain the more general relation

$$\langle\langle n,m|\widehat{\exp}(ik\Phi_{-+})|n',m'\rangle\rangle = \delta_{k,n'-n}\delta_{mm'}. \tag{5.22}$$

The asymptotic analysis of the ideal phase operator presents no problem and employs the relative number states. From the relation (5.22) it follows that

$$\lim_{n\to\infty}\lim_{n'\to\infty}\langle\langle n,m|\widehat{\exp}(ik\Phi_{-+})|n',m'\rangle\rangle = \delta_{mm'}\lim_{n\to\infty}\lim_{n'\to\infty}\delta_{k,n'-n}, \tag{5.23}$$

$$\lim_{n\to-\infty}\lim_{n'\to-\infty}\langle\langle n,m|\widehat{\exp}(ik\Phi_{-+})|n',m'\rangle\rangle = \delta_{mm'}\lim_{n\to-\infty}\lim_{n'\to-\infty}\delta_{k,n'-n}. \tag{5.24}$$

The operator $\widehat{\exp}(ik\Phi_{-+})$ has the property

$$[\hat{N}, \widehat{\exp}(ik\Phi_{-+})] = -k\widehat{\exp}(ik\Phi_{-+}), \quad \hat{N} = \hat{n}_- - \hat{n}_+, \tag{5.25}$$

which can be derived from the relations

$$[\hat{n}_\mp, \widehat{\exp}(\pm i\varphi_\mp)] = \mp\widehat{\exp}(\pm i\varphi_\mp). \tag{5.26}$$

Analogously, in the quantum mechanics of a massive particle the operator $\widehat{\exp}(ik\Phi)$ obeys the commutation relation

$$[\widehat{\exp}(ik\Phi), \hat{M}_z] = -\hbar k\widehat{\exp}(ik\Phi). \tag{5.27}$$

Considering the resolution of identity in the form

$$\hat{1} = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} \hat{1}_{N,\Phi_{-+}} \otimes |\varphi\rangle\rangle\langle\langle\varphi|\, d\varphi, \tag{5.28}$$

where the phase states

$$|\varphi\rangle\rangle = \sum_{m=0}^{\infty} \exp(im\varphi)|m\rangle\rangle, \tag{5.29}$$

with $|n,m\rangle\rangle = |n\rangle\rangle \otimes |m\rangle\rangle$, we may define the operator

$$\hat{M}(\varphi) = \frac{1}{2\pi}\int_\theta^{\theta+2\pi} M(\varphi)\hat{1}_{N,\Phi_{-+}} \otimes |\varphi\rangle\rangle\langle\langle\varphi|\, d\varphi. \tag{5.30}$$

The eigenstates of the operator $\widehat{\exp}(i\varphi)$ have been determined in [18].

The concept of the operator $(\hat{n}_- \geq \hat{n}_+)$ and of its eigenstates $|\psi_-\rangle$ is intuitively clear,

$$(\hat{n}_- \geq \hat{n}_+)|\psi_-\rangle = |\psi_-\rangle, \tag{5.31}$$

which enables us to restrict the operators $\hat{M}(\Phi_{-+})$ to the operators $\hat{M}(\varphi_-)$ in the expectations,

$$\langle\psi_-|\hat{M}(\Phi_{-+})|\psi_-\rangle = \langle\psi_-|\hat{M}(\varphi_-)|\psi_-\rangle. \tag{5.32}$$

This is a generalization of the result stated by Ban [19]. Similarly, $(\hat{n}_- < \hat{n}_+)$ denotes the operator whose eigenstates $|\psi_+\rangle$,

$$(\hat{n}_- < \hat{n}_+)|\psi_+\rangle = |\psi_+\rangle, \tag{5.33}$$

allow us to restrict the operators $\hat{M}(\Phi_{-+})$ to the operators $\hat{M}(-\varphi_+)$ in the quantum averages

$$\langle\psi_+|\hat{M}(\Phi_{-+})|\psi_+\rangle = \langle\psi_+|\hat{M}(-\varphi_+)|\psi_+\rangle. \tag{5.34}$$

On the assumption (4.8), the reduced operator [19]

$${}_+\langle 0|\widehat{\exp}(i\Phi_{-+})|0\rangle_+ = \widehat{\exp}(i\varphi_-). \tag{5.35}$$

VI. THE IDEAL PHASE CONCEPT AS AN APPROXIMATION OF THE ROTATION ANGLE

The approximate equality (5.15) should be analyzed and we will provide a numerical analysis of some closely related expressions in the following. The properties of the relations (5.15) are expressed using a suitable averaging instead of computing selected matrix elements.

We will consider a special case of the partial phase states [20], which share the property of near number states

$$|t_{n_-,u}\rangle_- = \frac{u^{n_-}}{\sqrt{2}}(|n_-\rangle + u|n_- + 1\rangle),$$
$$|t_{n_+,u}\rangle_+ = \frac{u^{n_+}}{\sqrt{2}}(|n_+\rangle + u|n_+ + 1\rangle); \tag{6.1}$$

here $|n_{\mp}\rangle$ ($|n_{\mp}+1\rangle$) denotes the lower (upper) number state possible and u is a complex unity. It holds that $u=\exp(i\bar{\varphi})$, where $\bar{\varphi}$ is the preferred phase. We assume that the physical system is in the state

$$|\psi\rangle = |t_{n_-,1}, t_{n_+,u}\rangle = |t_{n_-,1}\rangle_- \otimes |t_{n_+,u}\rangle_+. \tag{6.2}$$

Here the preferred phase in the minus mode is $\bar{\varphi}_- = 0$, whereas $\bar{\varphi}_+$ in the plus mode varies.

In the relation $\widehat{\exp}(i\Phi) \approx \widehat{\exp}(i\Phi_{-+})$ we go over to the operators $\widehat{\cos}\Phi$ and $\widehat{\cos}\Phi_{-+}$. The characteristics under study are the expectations

$$\begin{aligned}\langle\widehat{\cos}\,\Phi\rangle &= \langle t_{n_-,1}, t_{n_+,u}|\widehat{\cos}\,\Phi|t_{n_-,1}, t_{n_+,u}\rangle \\ &= \frac{|u|^{2n_+}}{4}\{\langle n_-, n_+|\widehat{\exp}(i\Phi)|n_- + 1, n_+\rangle + |u|^2\langle n_-, n_+ + 1|\widehat{\exp}(i\Phi)|n_- + 1, n_+ + 1\rangle \\ &\quad + \mathrm{Re}\, u[\langle n_-, n_+ + 1|\widehat{\exp}(i\Phi)|n_-, n_+\rangle + \langle n_- + 1, n_+ + 1|\widehat{\exp}(i\Phi)|n_- + 1, n_+\rangle]\},\end{aligned} \tag{6.3}$$

where the matrix elements on the right-hand side are determined by formulas (5.8), (5.9), and the quantum average

$$\langle\widehat{\cos}\Phi_{-+}\rangle = \langle t_{n_-,1}, t_{n_+,u}|\widehat{\cos}\Phi_{-+}|t_{n_-,1}, t_{n_+,u}\rangle, \tag{6.4}$$

which is of the same form but with the matrix elements given in (5.19).

In Figs. 3–12 we see the dependence of $\langle\widehat{\cos}\Phi_{-+}\rangle$ on the lower possible photon numbers in the minus and plus modes and the similarities and differences between this characteristic and its analog $\langle\widehat{\cos}\Phi\rangle$. For $n_+ \gg n_-$ it holds that the state (6.2) exhibits the property (5.33) or $\langle\widehat{\cos}\Phi_{-+}\rangle = \langle\widehat{\cos}\varphi_+\rangle$ and in the chosen state of the field $\langle\widehat{\cos}\varphi_+\rangle = \frac{1}{2}\cos\bar{\varphi}_+$. Both the general and the specific properties are obvious from Figs. 3–7, where the preferred phase $\bar{\varphi}_+ = \pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, 0$, respectively. In Figs. 8–12 appropriate to the preferred phase $\bar{\varphi}_+ = \pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, 0$, respectively, a similar behavior can be observed for $\langle\widehat{\cos}\Phi\rangle \approx \langle\widehat{\cos}\varphi_+\rangle = \frac{1}{2}\cos\bar{\varphi}_+$. When $n_+ \ll n_-$, the state (6.2) has the property (5.31) or $\langle\widehat{\cos}\Phi_{-+}\rangle = \langle\widehat{\cos}\varphi_-\rangle$, and in the state under study $\langle\widehat{\cos}\varphi_-\rangle = \frac{1}{2}\cos\bar{\varphi}_- = \frac{1}{2}$. This is obvious from Figs. 3–7, where the preferred phase $\bar{\varphi}_+$ changes but the preferred phase $\bar{\varphi}_-$ remains constant. In Figs. 8–12 we observe $\langle\widehat{\cos}\Phi\rangle \approx \frac{1}{2}$ with small differences due to the changes in the plus mode. If the photon numbers are approximately equal, $n_- \approx n_+$, $\langle\widehat{\cos}\Phi_{-+}\rangle$ has a jump discontinuity. A particular effect is observed in $\langle\widehat{\cos}\Phi\rangle$, the value of which is greater than expected. If $n_- = n_+$, the situation corresponds to that of the squared Wigner function [16]. The increased value of $\langle\widehat{\cos}\Phi\rangle$ can be seen in Figs. 11 and 12, from which an intuitive conclusion can be drawn that although the

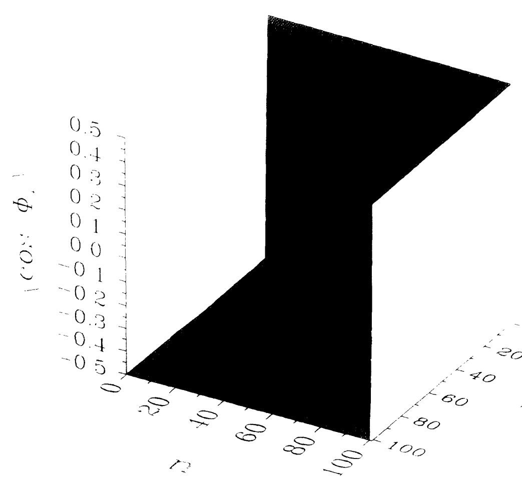

FIG. 3. Expectation values of the Ban operator $\widehat{\cos}\Phi_{-+}$ in the two-mode partial phase states with the preferred phases $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \pi$ and the lower possible photon numbers n_-, n_+.

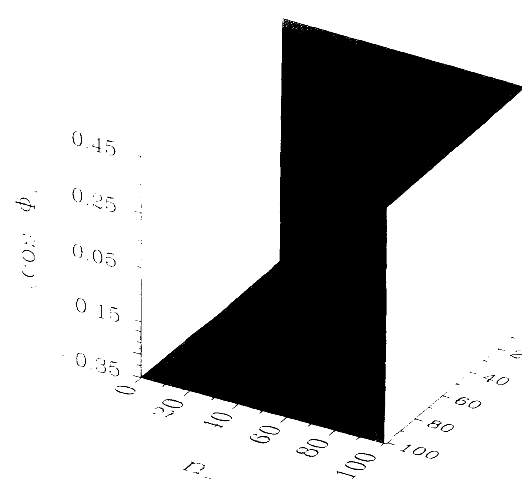

FIG. 4. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{3\pi}{4}$.

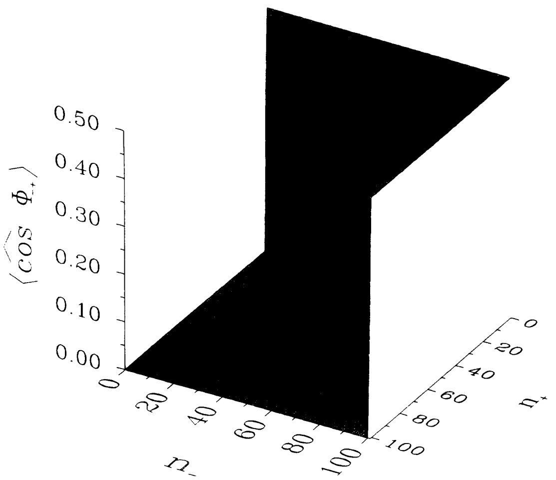

FIG. 5. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{2}$.

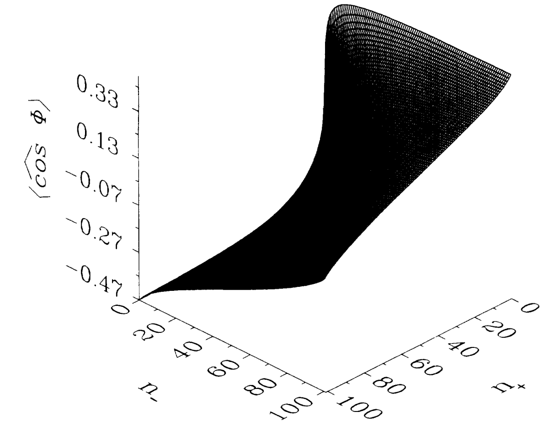

FIG. 8. Expectation values of the feasible phase operator $\widehat{\cos\Phi}$ in the two-mode partial phase states with the preferred phases $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \pi$ and the lower possible photon numbers n_-, n_+.

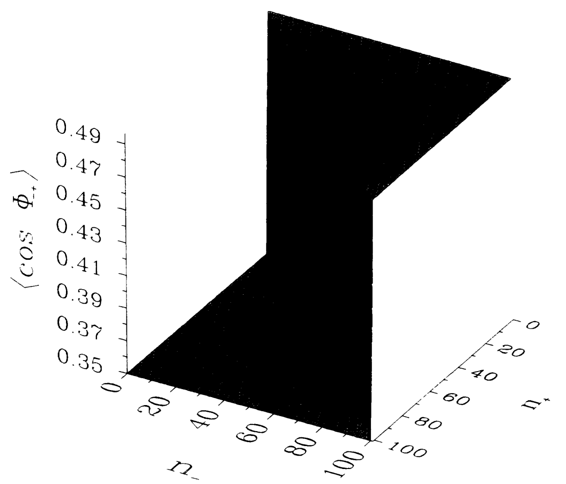

FIG. 6. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{4}$.

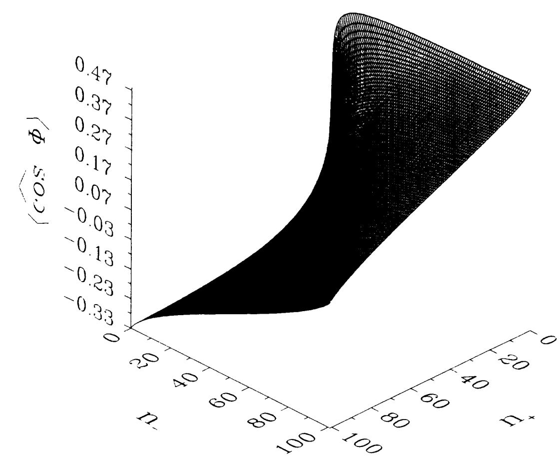

FIG. 9. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{3\pi}{4}$.

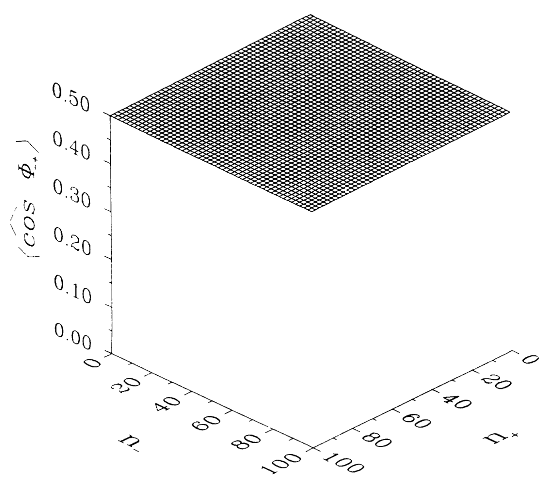

FIG. 7. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = 0$.

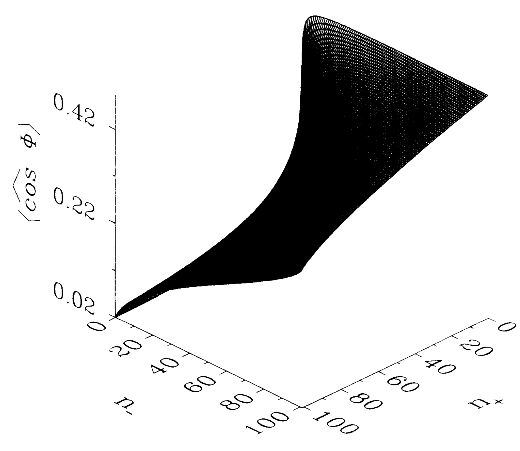

FIG. 10. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{2}$.

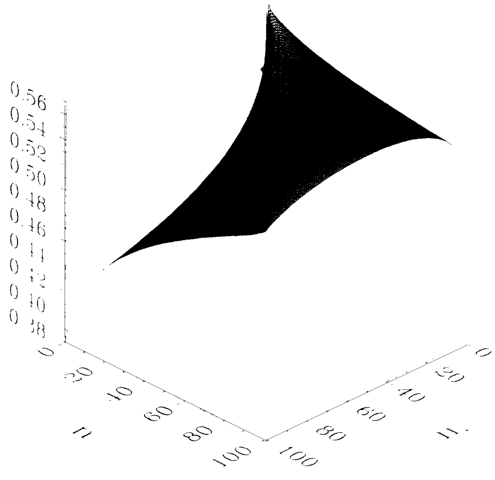

FIG. 11. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{4}$.

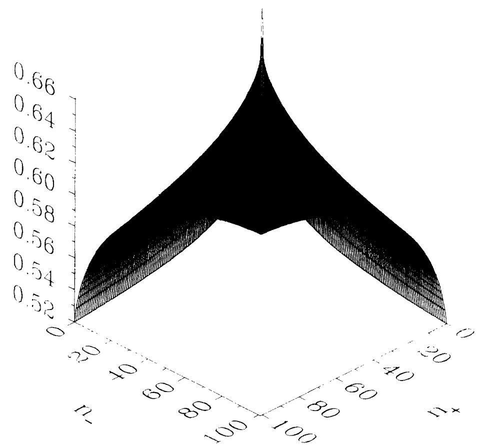

FIG. 12. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = 0$.

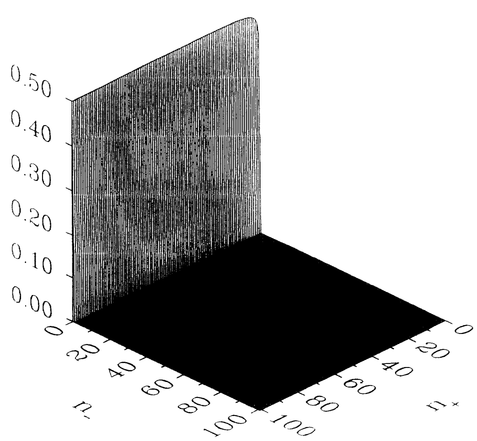

FIG. 13. Expectation values of the feasible phase operator $\widehat{\cos}\Phi$ in the states $|t_{n_-,1}\rangle_- \otimes |0\rangle_+$ with the preferred phase $\bar{\varphi}_- = 0$ and the lower possible photon numbers n_-, n_+.

preferred angle related to the operator $\hat{\Phi}_\theta$ remains unchanged, the dispersion of this angle decreases for n_- and n_+ approaching each other to the squared Wigner function level.

In addition to the foregoing special case in (6.2), $|\psi\rangle = |t_{n_-,1}, t_{n_-,1}\rangle$, we will treat the special case

$$|\psi\rangle = |t_{n_-,1}\rangle \otimes |0\rangle_+. \tag{6.5}$$

Whereas the Susskind-Glogower operator provides the result $\langle\widehat{\cos}\varphi_-\rangle = \frac{1}{2}$ for each lower possible photon number n_-, the feasible phase operator yields the values plotted in Fig. 13 and $\langle\widehat{\cos}\Phi\rangle$ converges monotonously to one half as n_- tends to infinity.

VII. CONCLUSION

We have involved the quantum mechanical approach to investigate the feasible phase concept in quantum optics and to compare it with the ideal ones. Under suitable initial conditions the quantum theory of a two-dimensional harmonic oscillator exhibits circular motions of a Gaussian wave packet. The study of a change of the basis for the position vectors and the conjugate change for the momenta, motivated by an analogy with linear and circular polarizations in the theory of the plane electromagnetic wave, provides all necessary characteristics of the rotation angle as a feasible phase concept. In addition to the linear motion components, circular motion components are considered and distinguished according to the clockwise and counterclockwise directions. The identity between the rotation angle and the measured phase in the detection schemes providing simultaneously measurable quantities is emphasized. For sufficiently strong signal mode the feasible phase concept approximates the ideal phase in this mode. The Paul phase concept is proven to be a projection of the feasible phase concept rather than that of the ideal phase concept. The ideal phase concepts in the separate modes (quantum mechanically the separate degrees of freedom) are unified to yield not only the Ban ideal phase proposal but also a generalization of his recent results related to the projections on separate modes. The foregoing comparisons have been illustrated using partial phase states and the cosine phase operator. For sufficiently strong signal mode the feasible phase concept approximates the ideal phase in this mode.

ACKNOWLEDGMENT

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APPENDIX: TRANSITION MATRIX FROM THE BASIS $|x, y\rangle$ TO THE BASIS $|n_-, n_+\rangle$.

It is well known that the transition matrix from the basis $|x\rangle$ to $|n_x\rangle$ and that from $|y\rangle$ to $|n_y\rangle$ read [21]

$$\langle x|n_x\rangle = \mathrm{H}_{n_x}(Ax)\sqrt{\frac{A}{2^{n_x}n_x!\sqrt{\pi}}}\exp(-A^2x^2) \quad \text{(A1)}$$

and

$$\langle y|n_y\rangle = \mathrm{H}_{n_y}(Ay)\sqrt{\frac{A}{2^{n_y}n_y!\sqrt{\pi}}}\exp(-A^2y^2), \quad \text{(A2)}$$

where the Hermite polynomials

$$\mathrm{H}_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!(n-2k)!}(-1)^k(2x)^{n-2k},$$

$$\left[\frac{n}{2}\right]=\begin{cases}\frac{n}{2} & \text{for } n \text{ even}\\ \frac{n-1}{2} & \text{for } n \text{ odd,}\end{cases} \quad \text{(A3)}$$

and

$$A=\frac{1}{\sqrt{\hbar}}\sqrt[4]{\frac{m}{b}}. \quad \text{(A4)}$$

As a consequence of tensorial products, we obtain that

$$\langle x,y|n_x,n_y\rangle = \mathrm{H}_{n_x}(Ax)\mathrm{H}_{n_y}(Ay)\times\sqrt{\frac{A^2}{2^{n_x+n_y}n_x!\,n_y!\,\pi}}\exp[-A^2(x^2+y^2)]. \quad \text{(A5)}$$

Using the Rodrigues type formula for the Hermite polynomials, we arrive at

$$\langle x,y|n_x,n_y\rangle = (-1)^{n_x+n_y}\times\sqrt{\frac{A^2}{2^{n_x+n_y}n_x!\,n_y!\,\pi}}\exp[-A^2(x^2+y^2)]\times\frac{\partial^{n_x+n_y}}{\partial(Ax)^{n_x}\partial(Ay)^{n_y}}\exp[-A^2(x^2+y^2)]. \quad \text{(A6)}$$

On respecting the SU(2) group [22], the definition of the number states, and the property

$$|n_x=0,n_y=0\rangle = |n_-=0,n_+=0\rangle, \quad \text{(A7)}$$

we rederive the expansion

$$|n_-,n_+\rangle = \sqrt{\frac{1}{2^{n_-+n_+}n_-!\,n_+!}}\sum_{j=0}^{n_-}\sum_{k=0}^{n_+}(-i)^{n_--j}i^{n_+-k}\frac{n_-!\,n_+!}{j!\,(n_--j)!\,k!\,(n_+-k)!}\sqrt{(j+k)!\,(n_-+n_+-j-k)!}\times|n_x=j-k,n_y=n_-+n_+-j-k\rangle. \quad \text{(A8)}$$

Hence,

$$\langle x,y|n_-,n_+\rangle = \frac{1}{2^{n_-+n_+}}\sqrt{\frac{A^2}{\pi}\exp[A^2(x^2+y^2)]}\sqrt{\frac{1}{n_-!\,n_+!}}\left(-\frac{\partial}{\partial(Ax)}+i\frac{\partial}{\partial(Ay)}\right)^{n_-}\times\left(-\frac{\partial}{\partial(Ax)}-i\frac{\partial}{\partial(Ay)}\right)^{n_+}\exp[-A^2(x^2+y^2)]. \quad \text{(A9)}$$

Substituting

$$\alpha=\frac{A}{2}(x+iy),\ \frac{\partial}{\partial\alpha}=\frac{1}{A}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \quad \text{(A10)}$$

and using the Rodrigues type formula for the Laguerre polynomials, we obtain

$$\langle x,y|n_-,n_+\rangle = (-1)^{n_+}2^{n_-+n_+}\frac{A}{\sqrt{\pi}}\sqrt{\frac{n_+!}{n_-!}}\left[\frac{A}{2}(x-iy)\right]^{n_--n_+}\exp\left(-\frac{A^2}{2}(x^2+y^2)\right)\mathrm{L}_{n_+}^{n_--n_+}[A^2(x^2+y^2)]$$

$$\text{for } n_-\geq n_+; \quad \text{(A11)}$$

here the Laguerre polynomials read

$$\mathrm{L}_n^{\gamma}(x)=\Gamma(n+\gamma+1)\sum_{j=0}^{n}\frac{(-x)^j}{j!(n-j)!\Gamma(j+\gamma+1)}, \quad \text{(A12)}$$

and they are expected from the theory of the Wigner function. For $n_+\geq n_-$ it holds that

$$\langle x,y|n_-,n_+\rangle = \langle x,y|n_+,n_-\rangle^*. \quad \text{(A13)}$$

Using the property of the wave functions

$$\langle\Phi,r|n_-,n_+\rangle = \langle x=r\cos\Phi,y=r\sin\Phi|n_-,n_+\rangle\times\sqrt{\left|\frac{\partial(x,y)}{\partial(\Phi,r)}\right|}, \quad \text{(A14)}$$

we get the formulas

$$\langle\Phi,r|n_-,n_+\rangle = \frac{(-1)^{n_+}}{\sqrt{\pi}}\sqrt{\frac{n_+!}{n_-!}}[Ar\exp(-i\Phi)]^{n_--n_+} \times \mathrm{L}_{n_+}^{n_--n_+}(A^2r^2)A\sqrt{r}\exp\left(-\frac{1}{2}A^2r^2\right), \quad n_-\geq n_+; \tag{A15}$$

$$\langle\Phi,r|n_-,n_+\rangle = \langle\Phi,r|n_+,n_-\rangle^*, \quad n_-\leq n_+.$$

From the above derivation it is obvious that $\langle\Phi,r|n_-,n_+\rangle$ have the properties

$$\int_0^\infty\int_\theta^{\theta+2\pi}\langle n_-,n_+|\Phi,r\rangle\langle\Phi,r|n'_-,n'_+\rangle d\Phi dr = \langle n_-,n_+|n'_-,n'_+\rangle, \tag{A16}$$

$$\sum_{n_-=0}^{\infty}\sum_{n_+=0}^{\infty}\langle\Phi,r|n_-,n_+\rangle\langle n_-,n_+|\Phi',r'\rangle = \langle\Phi,r|\Phi',r'\rangle. \tag{A17}$$

In terms of the relative number states (5.20), the representation of the position states $|\Phi,r\rangle$ is of a simpler form,

$$\langle\Phi,r|n,m\rangle\rangle = \frac{(-1)^m}{\sqrt{\pi}}\sqrt{\frac{m!}{(m+|n|)!}}(Ar)^{|n|}\exp(-in\Phi) \times \mathrm{L}_m^{|n|}(A^2r^2)A\sqrt{r}\exp\left(-\frac{1}{2}A^2r^2\right). \tag{A18}$$

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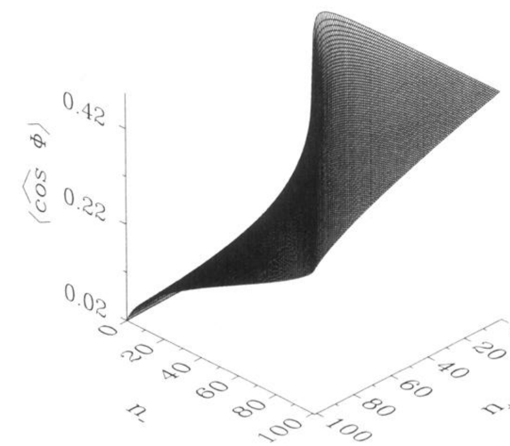

FIG. 10. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{2}$.

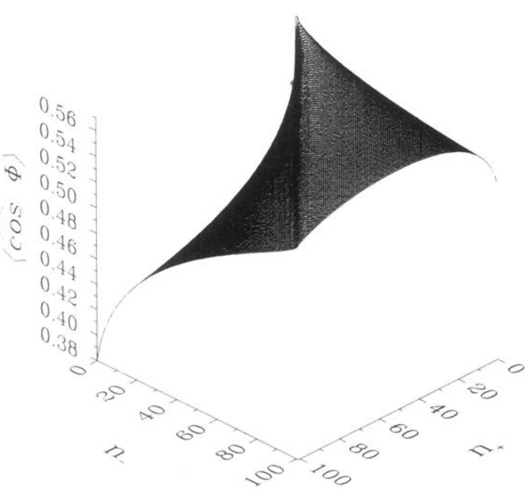

FIG. 11. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{4}$.

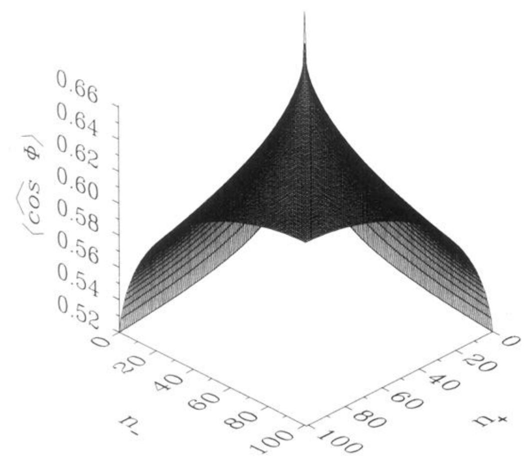

FIG. 12. Same as Fig. 8, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = 0$.

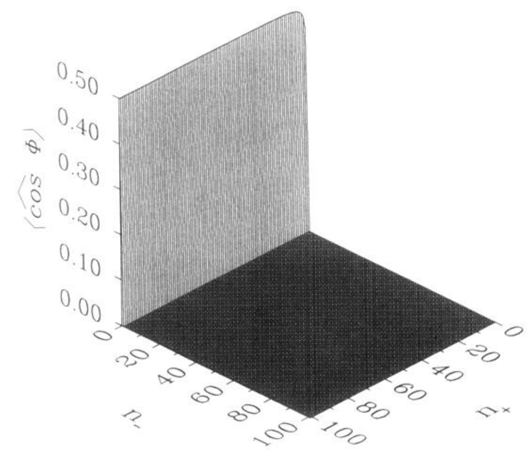

FIG. 13. Expectation values of the feasible phase operator $\widehat{\cos}\Phi$ in the states $|t_{n_-,1}\rangle_- \otimes |0\rangle_+$ with the preferred phase $\bar{\varphi}_- = 0$ and the lower possible photon numbers n_-, n_+.

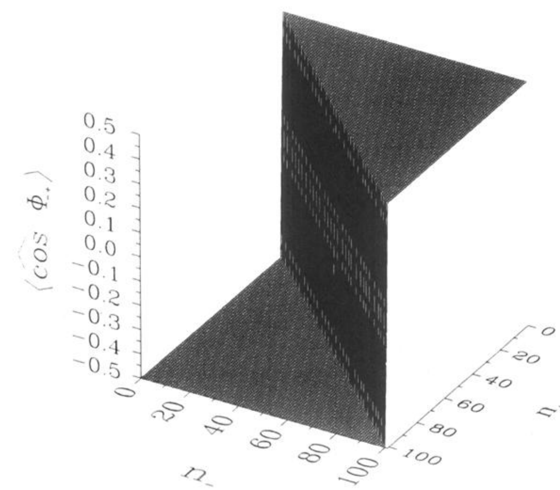

FIG. 3. Expectation values of the Ban operator $\widehat{\cos}\Phi_{-+}$ in the two-mode partial phase states with the preferred phases $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \pi$ and the lower possible photon numbers n_-, n_+.

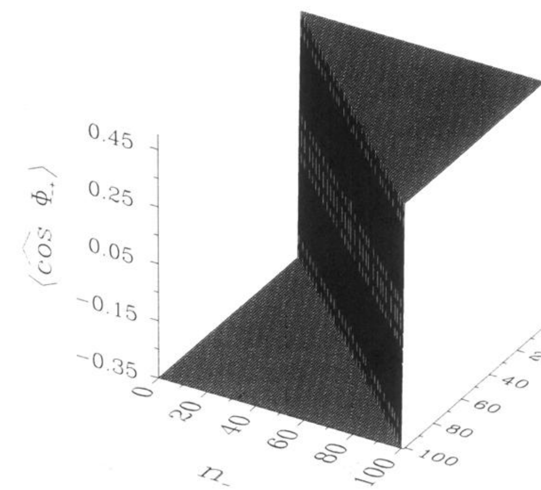

FIG. 4. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{3\pi}{4}$.

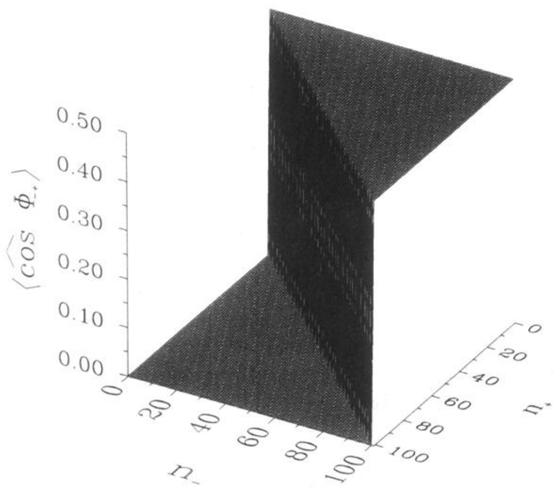

FIG. 5. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{2}$.

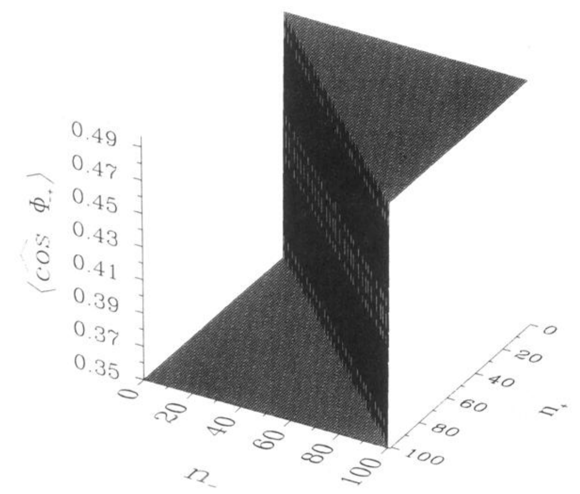

FIG. 6. Same as Fig. 3, but with $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \frac{\pi}{4}$.

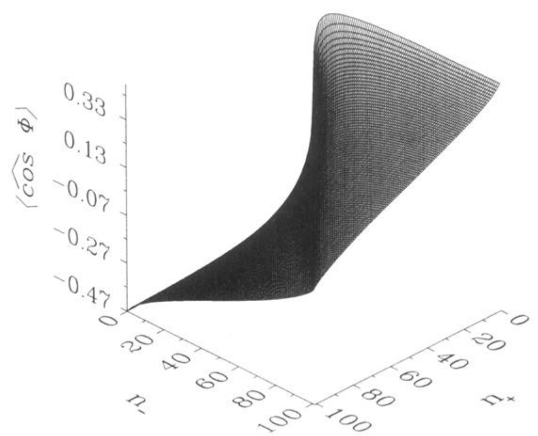

FIG. 8. Expectation values of the feasible phase operator $\widehat{\cos}\Phi$ in the two-mode partial phase states with the preferred phases $\bar{\varphi}_- = 0$, $\bar{\varphi}_+ = \pi$ and the lower possible photon numbers n_-, n_+.

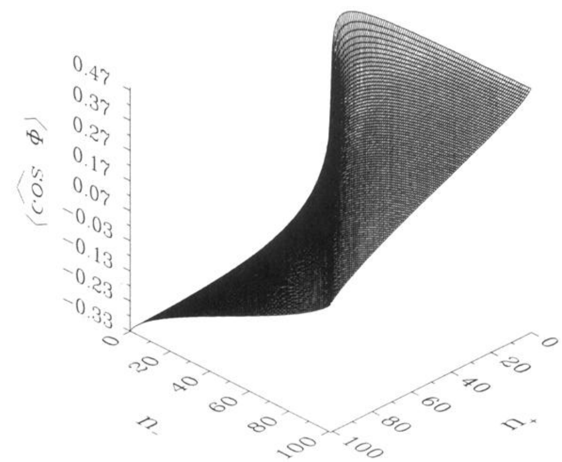

FIG. 9. Same as Fig. 8, but with $\bar{\varphi}_{-} = 0$, $\bar{\varphi}_{+} = \frac{3\pi}{4}$.