SU(m,n) coherent states in the bosonic representation and their generation in optical parametric processes

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The coherent states of SU(m,n) transformations arising in the description of certain multimode nonlinear parametric processes bilinear in boson operators are constructed. The nonclassical properties of the coherent states of SU(m) and those of SU(m, 1) are identified. The dynamics generated by SU(m,n) Hamiltonians is studied.

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I. INTRODUCTION

The Hamiltonians of most of the nonlinear optical parametric processes reduce to a bilinear combination of the bose operators. Those processes are of particular interest in the generation of nonclassical states of radiation, which is a problem of central interest in quantum optics. The Hamiltonian of an optical parametric process involving bilinear combinations of n quantized field modes is a generator of Sp($2n, 4\mathbb{R}$). The nonclassical properties of the states generated by Sp(2,$\mathbb{R}$) $\equiv$SU(1,1) transformation in a single-mode interaction have been widely discussed [1–3]. Those states exhibit squeezing. The nonclassical aspects of the states generated by Sp(4,$\mathbb{R}$) transformations in a two-mode interaction have been reported recently [4]. There is also considerable interest in the nonclassical properties of multimode field states [5–7].

The Sp($2n,\mathbb{R}$) transformations are generated by $n(2n+1)$ bilinear combinations of n modes which include self or degenerate as well as intermode or nondegenerate combinations [5,8]. There are clearly two types of nondegenerate Hermitian bilinear combinations possible with two quantized modes $(a, a^\dagger)$ and $(b, b^\dagger)$. One is the combination $(ga^\dagger\, b^\dagger + g^*\, ab)$ that conserves $(a^\dagger a - b^\dagger b)$. It describes the process of parametric amplification of the two modes. The other combination $(ga^\dagger b + g^* b^\dagger a)$ conserving $(a^\dagger a + b^\dagger b)$ describes the process of frequency conversion. Two modes in a multimode parametric wave-mixing interaction can contribute to both of those processes. However, a class of interactions of considerable interest is one of nondegenerate parametric interactions in which two given modes contribute either to the process of frequency conversion or to the process of amplification. Those interactions involve bilinear combinations of bose operators $\{a_k, a_k^\dagger; b_p, b_p^\dagger\}$ ($k = 1,2,...,m;\ p = 1,2,...,n$) such that two A or two B modes give rise to only the process of frequency conversion whereas the combinations of A with B modes generate only the process of parametric amplification. The Hamiltonians of those interactions are the generators of SU(m,n) [SU(m, 0) $\equiv$ SU(m)]. The process of parametric amplification involving two modes is thus an example of SU(1,1) and that of frequency conversion realizes SU(2) transformation. The SU(m) transformations arise also in the description of the interaction of an m-level atom with classical fields.

The two-mode SU(1,1) and SU(2) Hamiltonians have been extensively studied [2,9–11]. A three-mode Hamiltonian describing competing SU(1,1) and SU(2) processes has also been investigated [12]. That Hamiltonian is, in fact, an example of SU(2,1). Some applications of the bosonic realization of SU(3) have been discussed by Moshinsky [13]. Here we discuss the processes generated by SU(m,n) transformations for arbitarary m and n. Those can be studied in terms of the action of a group element on a state vector in the space of the states of the group or equivalentaly in terms of its action on the bose operators. An SU(m,n) group element acting on a state $|\psi_0\rangle$ in the space of its states generates, in the sense of Perelomov [14], an SU(m,n) coherent state corresponding to the state $|\psi_0\rangle$ as a fiducial state. The coherent states of SU(1,1) and SU(2) have been studied [2,3,14–17] for various fiducial states. The coherent state of a collectively interacting three-level atomic system constructed in Ref. [18] in analogy with the two-level atomic coherent state [16] is an example of the SU(3) coherent state. Here we construct the coherent states of an arbitarary SU(m,n) for some particular fiducial states. We investigate, in particular, the nonclassical properties such as sub-Poisson photon number statistics and single- and two-mode squeezing of the coherent states of SU(m) and SU(m, 1). Those aspects are of fundamental interest in quantum optics. We also discuss the dynamics of the modes under the action of an element of SU(m,n).

The paper is organized as follows. In Sec. I we introduce the bilinear combinations of bose operators that generate the SU(m,n) group and construct its coherent states. The nonclassical aspects of the coherent states of SU(m) and SU(m, 1) are discussed in Sec. II. The physical processes giving rise to SU(m,n) transformations are described in Sec. III. Section IV discusses the evolution of the operators under the group SU(m,n). The main conclusions are summarized in Sec. V.

II. THE COHERENT STATES OF SU(m,n)

Consider the set of operators $\{a_k, a_k^\dagger; b_p, b_p^\dagger\}$ ($k = 1,2,...,m;\ p = 1,2,...,n$) obeying the canonical boson commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [b_p, b_{p'}^\dagger] = \delta_{pp'}, \quad [a_k, b_p] = [a_k, b_p^\dagger] = 0. \tag{2.1}$$

Consider the unitary unimodular transformations generated by those bilinear combinations of the bose operators (2.1) which commute with

$$\hat{N} = \sum_{i=1}^{m} a_i^\dagger a_i - \sum_{p=1}^{n} b_p^\dagger b_p. \tag{2.2}$$

Those transformations, generated by $(m+n)^2-1$ bilinear combinations

$$X_{jk}^{(1)} = a_j^\dagger a_k + \text{H.c.}, \qquad X_{jk}^{(2)} = i(a_j^\dagger a_k - \text{H.c.}), \qquad X_{lk}^{(3)} = a_k^\dagger a_k - a_l^\dagger a_l,$$

$$Y_{pq}^{(1)} = b_p^\dagger b_q + \text{H.c.}, \qquad Y_{pq}^{(2)} = i(b_p^\dagger b_q - \text{H.c.}), \qquad Y_{pq}^{(3)} = b_q^\dagger b_q - b_p^\dagger b_p, \tag{2.3}$$

$$Z_{jp}^{(1)} = a_j b_p + \text{H.c.}, \qquad Z_{jp}^{(2)} = i(a_j b_p - \text{H.c.}), \qquad Z_{ll}^{(3)} = a_l^\dagger a_l + b_l^\dagger b_l + 1, \quad j < k = 1,2,...,m;\ p < q = 1,...,n,$$

constitute the group SU(m,n) [SU$(m,0)$ $\equiv$SU(m)]. An element U of that group is given by

$$U = \exp\left[i \sum_{r=1}^{3} \left(\sum_{i<j=1}^{m} a_{ij}^{(r)} X_{ij}^{(r)} + \sum_{p<q=1}^{n} b_{pq}^{(r)} Y_{pq}^{(r)} + \sum_{i,p} c_{ip}^{(r)} Z_{ip}^{(r)} \right) \right], \tag{2.4}$$

where $a_{ij}^{(r)}$, $b_{pq}^{(r)}$ and $c_{ip}^{(r)}$ are real.

The constraint (2.2) reduces the Hilbert space of the group SU(m,n) into subspaces, each characterized by an eigenvalue N of $\hat{N}$. The Fock states $|\{j_k\},\{l_p\}\rangle$ ($a_k^\dagger a_k|\{j_k\},\{l_p\}\rangle = j_k|\{j_k\},\{l_p\}\rangle$, $b_p^\dagger b_p|\{j_k\},\{l_p\}\rangle = l_p|\{j_k\},\{l_p\}\rangle$ obeying the condition

$$\sum_{k=1}^{m} j_k - \sum_{p=1}^{n} l_p = N \tag{2.5}$$

can be chosen as a basis for representing the vectors $|\psi(N)\rangle$ in the subspace of fixed N. Note that for $n=0$, i.e., for the group SU(m), the states $|\{j_k\}\rangle$, $k = 1,2,...,m$ span an N-dimensional space. The space of the states of SU(m,n) for $n \neq 0$, on the other hand, is always infinite dimensional.

Next, following the definition of Perelomov [14], we construct the coherent states of SU(m,n). To that end one first selects a state $|\psi_0\rangle$, called the fiducial state, from the space of the states of an irreducible representation T of the group G. Let H be the stationary subgroup, i.e., the subgroup of those elements $\{h\}$ of the group which are such that $T(h)|\psi_0\rangle = \exp(i\alpha)|\psi_0\rangle$, where α is a real constant. The coherent state $|\psi_{\text{CS}}\rangle$ with respect to the state $|\psi_0\rangle$ is determined by the point $x = x(g)$ of the factor space G/H corresponding to the element g of the group G. The set of those states is complete [14]. The coherent states for the Heisenberg-Weyl group, i.e., the bosonic coherent states and those for SU(2) and SU(1,1), are well known [2,3,14–17]. Here we construct the coherent states of SU(m,n) for arbitarary m and n.

We construct first the coherent states $|\psi_{\text{CS}}^{(m)}\rangle$ of SU(m) for the fiducial state $|\psi_0\rangle = |\{0\},N\rangle$, where $a_m^\dagger a_m|\{0\},N\rangle = N|\{0\},N\rangle$, $a_k|\{0\},N\rangle = 0$, and $k \neq m$. By definition

$$|\psi_{\text{CS}}^{(m)}(N)\rangle = \exp\left(\sum_{i=1}^{m-1} \{\alpha_i^* a_i^\dagger a_m - \alpha_i a_m^\dagger a_i\} \right) |\{0\},N\rangle. \tag{2.6}$$

In terms of the operator

$$B = \frac{1}{\Gamma} \sum_{i=1}^{m-1} \alpha_i a_i, \tag{2.7}$$

where

$$\Gamma^2 = \sum_{i=1}^{m-1} |\alpha_i|^2, \tag{2.8}$$

the expression (2.6) can be rewritten as

$$|\psi_{\text{CS}}^{(m)}(N)\rangle = \exp\left(\sum_{i=1}^{m-1} \Gamma[B^\dagger a_m - a_m^\dagger B] \right) |\{0\},N\rangle. \tag{2.9}$$

Since $[B,B^\dagger]=1$ it follows that $B^\dagger a_m, a_m^\dagger B$ and $(B^\dagger B - a_m^\dagger a_m)/2$ obey the angular momentum commutation relations. The exponential in (2.9) can therefore be disentangled using the disentangling theorem for angular momentum operators [16] to obtain

$$|\psi_{\text{CS}}^{(m)}(N)\rangle = A_m \sum_{\{j\}} \frac{\sqrt{N!}\mu_1^{j_1}\mu_2^{j_2}\cdots\mu_{m-1}^{j_{m-1}}}{\left[j_1!j_2!\cdots j_{m-1}!\left(N - \sum_{k=1}^{m-1} j_k \right)! \right]^{1/2}} \times \left| j_1, j_2, ..., j_{m-1}, N - \sum_{k=1}^{m-1} j_k \right\rangle, \tag{2.10}$$

where

$$\mu_i = \tan(\Gamma)\alpha_i/\Gamma, \qquad A_m = \left(1+\sum_{i=1}^{m-1}|\mu_i|^2\right)^{-1/2}. \tag{2.11}$$

The coherent state (2.10) of SU(m) corresponding to the fiducial state $|\{0\},N\rangle$ is thus characterized by $m-1$ complex parameters. The general expression (2.10) for an SU(m) coherent state evidently reproduces the known expressions for the coherent states of SU(2) and SU(3) for $m=2$ and $m=3$, respectively.

Next, we construct the coherent state of SU(m,n) corresponding to the fiducial state $|\{0\}\rangle$, i.e., the state of vacuum for all modes. It is defined as

$$|\psi_{\rm CS}^{(m,n)}\rangle = \exp\left(\sum_{i=1}^{m}\sum_{p=1}^{n}\{\alpha_{ip}a_i^\dagger b_p^\dagger - \alpha_{ip}^* a_i b_p\}\right)|\{0\}\rangle. \tag{2.12}$$

The expression (2.12) can be rewritten by defining the operator

$$C_p = \frac{1}{\Gamma_p}\sum_{i=1}^{m}\alpha_{ip}a_i, \tag{2.13}$$

where

$$\Gamma_p^2 = \sum_{i=1}^{m}|\alpha_{ip}|^2 \tag{2.14}$$

as

$$|\psi_{\rm CS}^{(m,n)}\rangle = \exp\left(\sum_{p=1}^{n}\Gamma_p[C_p^\dagger b_p^\dagger - \text{H.c.}]\right)|\{0\}\rangle. \tag{2.15}$$

Since $[C_p, C_p^\dagger]$=1 it follows that $C_p^\dagger b_p^\dagger, b_pC_p$ and $(C_p^\dagger C_p + b_p^\dagger b_p+1)/2$ obey the SU(1,1) commutation relations. The exponential in (2.15) can be disentangled using the disentangling theorem of SU(1,1) operators to get

$$|\psi_{\rm CS}^{(m,n)}\rangle = \exp\left(\sum_{p=1}^{n}\tanh(\Gamma_p)C_p^\dagger b_p^\dagger\right)|\{0\}\rangle, \tag{2.16}$$

which in turn gives, for $n\neq 1$,

$$|\psi_{\rm CS}^{(m,n)}\rangle = \left\{\sum_{\{n\}}\sum_{\{p\}}\prod_{i=1}^{m}\prod_{q=1}^{n}\left[\mu_{iq}^{p_{iq}}\frac{\sqrt{n_i!\left(\sum_{k=1}^{m}p_{kq}\right)!}}{p_{iq}!}\right]\right\} \times\left|\{n\};\sum_{r=1}^{m}p_{r1},\sum_{r=1}^{m}p_{r2},\ldots,\sum_{r=1}^{m}p_{rn}\right\rangle, \tag{2.17}$$

where

$$p_{in} = n_i - \sum_{q=1}^{n-1}p_{iq} \tag{2.18}$$

and

$$\mu_{ip} = \tanh(\Gamma_p)\alpha_{ip}/\Gamma_p. \tag{2.19}$$

The coherent state of SU(m,n) is thus characterized by mn complex variables $\{\mu_{ip}\}$. The expression (2.16) for the coherent state of SU(m,n) assumes a simple form for $n=1$ given by

$$|\psi_{\rm CS}^{(m,1)}\rangle = B_m\sum_{\{n\}}\frac{\mu_1^{n_1}\mu_2^{n_2}\cdots\mu_m^{n_m}\sqrt{(n_1+n_2+\cdots+n_m)!}}{\sqrt{n_1!n_2!\ldots n_m!}} \times\left|n_1,n_2,\ldots,n_m;\sum_{i=1}^{m}n_i\right\rangle, \tag{2.20}$$

where

$$\mu_i = \tanh(\Gamma_1)\alpha_i/\Gamma_1, \quad B_m = \sqrt{1-\sum_{p=1}^{m}|\mu_p|^2}. \tag{2.21}$$

Clearly, $\sum_i|\mu_i|^2<1$. Note also that

$$\sum_{i=1}^{m}[\mu_i^* a_i - |\mu_i|^2 b_1^\dagger]|\psi_{\rm CS}^{(m,1)}\rangle = 0, \tag{2.22}$$

i.e., $|\psi_{\rm CS}^{(m,1)}\rangle$ is an eigenstate of an operator formed by the linear combination of m annihilation and a creation operator.

In the following section we identify some of the nonclassical properties of the coherent states (2.10) and (2.20) of SU(m) and SU(m,1), respectively.

III. NONCLASSICAL PROPERTIES OF SU(m) AND SU(m,1) COHERENT STATES

First we determine the nonclassical properties of the SU(m) coherent states (2.10) corresponding to the fiducial state $|\{0\},N\rangle$, i.e., the state in which the mth mode is in the Fock state $|N\rangle$ and all other modes are in the state of the vacuum. The properties of, say, the ith mode in the state (2.10) can be studied in terms of its reduced density operator ρ_i, which is found to be given by

$$\rho_i = \left(1+\sum_{i=1}^{m-1}|\mu_i|^2\right)^{-1}\sum_{p=0}^{N}\frac{N!|\mu_i|^{2p}|\nu_i|^{2(N-p)}}{(N-p)!p!}|p\rangle\langle p|, \tag{3.1}$$

where

$$|\nu_i|^2 = 1+\sum_{j\neq i}^{m-1}|\mu_j|^2. \tag{3.2}$$

It is straightforward to show that the average occupation number and its variance in the ith mode are given respectively by

$$\langle a_i^\dagger a_i\rangle = N|\mu_i|^2\Bigg/\left(1+\sum_{j=1}^{m-1}|\mu_j|^2\right) \tag{3.3}$$

and

$$\langle (a_i^\dagger a_i)^2\rangle - \langle a_i^\dagger a_i\rangle^2 = N|\mu_i|^2(1+|\nu_i|^2) \Big/ \left(1+\sum_{j=1}^{m-1}|\mu_j|^2\right). \quad (3.4)$$

It then follows that

$$\langle (a_i^\dagger a_i)^2\rangle - \langle a_i^\dagger a_i\rangle^2 - \langle a_i^\dagger a_i\rangle = -N|\mu_i|^4 \Big/ \left(1+\sum_{j=1}^{m-1}|\mu_j|^2\right), \quad (3.5)$$

i.e., the variance is less than the mean. Hence the number distribution in each of the modes in the SU(m) coherent state is nonclassical. Those states, however, exhibit no single- or two-mode squeezing.

Next, we examine the nonclassical properties of the coherent states (2.20) of SU($m,1$). The reduced density matrices for the modes a_i and b_1 in that state are found to be given respectively by

$$\rho_i = \frac{1-\sum_{k=1}^{m}|\mu_k|^2}{\left(1-\sum_{j\neq i=1}^{m}|\mu_j|^2\right)} \sum_j \frac{|\mu_i|^{2p}}{\left(1-\sum_{j\neq i=1}^{m}|\mu_j|^2\right)} |p\rangle\langle p| \quad (3.6)$$

and

$$\rho_b = 1 - \sum_p \left(\sum_{k=1}^{m}|\mu_k|^2\right)^p |p\rangle\langle p|. \quad (3.7)$$

Each of the modes in SU($m,1$) is thus in the thermal state. The statistical properties of individual modes in the coherent state of SU($m,1$) are therefore classical. The nonclassical aspects of those states are reflected in mode-mode correlations. Those correlations can be examined in terms of the two-mode squeezing parameter $S(a,b;\psi)$ between modes a and b. That parameter is a measure of the variance in the operator $A(\psi) = 1/(2\sqrt{2})[a\exp(i\psi/2) + b\exp(i\psi/2) + \mathrm{H.c.}]$:

$$S(a,b;\psi) = \langle A^2(\psi)\rangle - \langle A(\psi)\rangle^2. \quad (3.8)$$

The states for which $S(\psi) < 1/4$ for some ψ are nonclassical. For the SU($m,1$) coherent state (2.20) we find that

$$S(a_i,b;\psi) = \frac{1}{4\left(1-\sum_{j=1}^{m}|\mu_j|^2\right)} \times \left(2|\mu_i|\cos(\phi_i+\psi) + |\mu_i|^2 + \sum_{j=1}^{m}|\mu_j|^2\right) + \frac{1}{4}, \quad (3.9)$$

where $\mu_i = |\mu_i|\exp(i\phi_i)$. The values of μ_i for which $S(a_i,b;\psi) < 1/4$ can be easily identified. For $m = 1$, for example, $S(a_1,b;\pi-\phi) < 1/4$ for all μ. It is, however, evident from (3.9) that for $m \neq 1$, $S(a_i,b;\psi)$ is less than 1/4 only for certain values of μ for any ψ. Thus SU(1,1) is always two-mode squeezed, but SU($m,1$) for $m > 1$ is squeezed only for certain μ. Note also that there is no squeezing involving two a modes. The two-mode squeezing parameter can be determined experimentally by mixing the two modes with a strong local oscillator field of phase ψ and frequency halfway between the frequencies of the two modes. The number fluctuation of the resulting field is then a measure of S [19].

IV. GENERATION OF SU(m) AND SU(m,n) COHERENT STATES

The nonlinear optical parametric interactions have been shown to be useful in generating a variety of nonclassical states [19,20]. The SU(m,n) transformations can be realized in nondegenerate optical parametric processes describing the mixing of the electromagnetic field modes of different frequencies in an optically nonlinear medium. The lowest-order parametric process in a noncentrosymmetric medium is three-photon mixing and that in a centrosymmetric material is four-photon mixing. Since the quantized field modes are characterized by boson operators, the fully quantized Hamiltonian describing the lowest-order nonlinear interaction is a trilinear or quartic combination of boson operators. In practice, however, the parametric processes often involve one or more intense monochromatic fields. The terms corresponding to those fields can be treated as harmonically varying externally prescribed parameters. The Hamiltonian then reduces to linear or bilinear combinations of boson operators. Consider the case of nondegenerate optical parametric processes described by the Hamiltonians bilinear in the boson operators. Evidently there are two types of nondegenerate processes possible involving bilinear combinations with two boson operators $(a,a^\dagger)$ and $(b,b^\dagger)$ leading to two types of Hamiltonians: One is the Hamiltonian

$$H_1 = (gab + g^* a^\dagger b^\dagger), \quad (4.1)$$

describing parametric amplification of the two modes, and the second is the frequency converter Hamiltonian

$$H_2 = (ga^\dagger b + g^* b^\dagger a). \quad (4.2)$$

The Hamiltonian H_1 is a generator of SU(1,1) and H_2 that of SU(2) transformations. Those interactions have been extensively studied. Two given modes, in general, can participate in both the processes simultaneously. However, the SU(m,n) transformations are realized if two given modes participate only in one type of process. A three-mode Hamiltonian describing competition between parametric amplification and frequency conversion with two given modes involved in only one type of process, as discussed by Mishkin and Walls [12], is an example of SU(2,1) transformation.

In order to realize SU(m,n) consider a nonlinear medium pumped by several coherent fields or several independently pumped nonlinear media inside an optical cavity leading to bilinear coupling between $m+n$ quantized field modes $\{a_k, a_k^\dagger; b_p, b_p^\dagger\}$ ($k = 1,2,...,m$; $p = 1,2,...,n$) such that two A or two B modes participate only in the process of frequency conversion described by the form (4.2) whereas the combination of an A with a B mode occcurs only in the form (4.1) characterizing parametric amplification. The most general Hamiltonian describing that interaction in an appropriate rotating frame is given by

$$H = \sum_{i<j=1}^{m} \alpha_{ij} a_i^\dagger a_j + \sum_{p<q=1}^{n} \beta_{pq} b_p^\dagger b_q + \sum_{i=1}^{m}\sum_{p=1}^{n} \gamma_{ip} a_i b_p + \text{H.c.} \tag{4.3}$$

On comparing with (2.3) it is clear that (4.3) is a generator of SU(m,n).

The group SU(m) describes also an m-level atom driven by classical fields. If $|i\rangle$, $i = 1,2,...,m$, are the atomic levels and α_{ij} the strength of coupling between the external field interacting resonantly with levels $|i\rangle$ and $|j\rangle$, then the Hamiltonian of interaction in an appropriate rotating frame is given by

$$H = \sum_{i<j}^{m} \alpha_{ij} A_{ij} + \text{H.c.}, \tag{4.4}$$

where the operators $A_{ij} \equiv |i\rangle\langle j|$ commute with

$$\hat{N} = \sum_{i=1}^{m} A_{ii}. \tag{4.5}$$

By a straightforward examination of the commutation relations of A_{ij} or by going over to the Schwinger representation in terms of the boson operators $a_i, a_i^\dagger$, namely, $A_{ij} = a_i^\dagger a_j$, it is clear that $\{A_{ij}\}$ constitute the SU(m) algebra and hence (4.4) is a generator of the group SU(m). The coherent state of a collectively interacting two-level atomic system, known as the atomic coherent state [16], is an example of SU(2) coherent states. In analogy to the two-level atomic coherent states, Agarwal and Trivedi [18] constructed the coherent state of a collectively interacting system of N three-level atoms. That coherent state is the same as the SU(3) coherent state obtained from the general expression (2.10) for the coherent state of SU(m).

V. DYNAMICS GENERATED BY SU(m,n) TRANSFORMATIONS

In this section we study the evolution of the operators under the Hamiltonian (4.3) generating the group SU(m,n). Consider first the SU(m) transformations. Those correspond to the Hamiltonian (4.3) for $n = 0$. The Heisenberg equations of motion for the annihilation operators in that case form a closed system given by

$$\dot{\tilde{A}}(t) = -iM\tilde{A}(t), \tag{5.1}$$

where

$$\tilde{A}(t) = \text{col}[a_1(t), a_2(t), ..., a_m(t)] \tag{5.2}$$

and M is a matrix whose elements are given by

$$M_{ij} = \alpha_{ij}, \tag{5.3}$$

with $\alpha_{ji} = \alpha_{ij}^*$. Hence $M(t)$ is Hermitian and its eigenvalues are real. If $|\psi_i\rangle$ are the eigenstates of M corresponding to the eigenvalues λ_i ($i = 1,2,...,m$), then it follows from (5.1) that

$$a_i(t) = \sum_{j=1}^{m} f_{ij}(t) a_j, \tag{5.4}$$

where

$$f_{ij}(t) = \sum_{k=1}^{m} \exp(-i\lambda_k t)\langle i|\psi_k\rangle\langle\psi_k|j\rangle. \tag{5.5}$$

Note that the operators $\{a_i(t), a_i^\dagger(t)\}$, obtained by a unitary transformation on the canonical operators $\{a_i, a_i^\dagger\}$, are also canonical. Hence it follows from (5.4) that

$$\sum_{j=1}^{m} f_{ij} f_{i'j}^* = \delta_{ii'}. \tag{5.6}$$

The dynamics of an arbitarary observable can thus be determined by evaluating the functions $f_{ij}(t)$. Those functions are oscillatory. Hence the evolution generated by the group SU(m) is oscillatory.

The operator dynamics can also be used to study the dynamics of an arbitarary state vector. Any state vector can be expanded in terms of the Fock states $|n_1, n_2, ..., n_m\rangle$, which in turn can be written as

$$|n_1, n_2, ..., n_m\rangle = \frac{a_1^{\dagger n_1} a_2^{\dagger n_2} \cdots a_m^{\dagger n_m}}{\sqrt{n_1! n_2! \cdots n_m!}} |\{0\}\rangle. \tag{5.7}$$

Hence, by virtue of the fact that $H|\{0\}\rangle = 0$ it follows that

$$\exp(-iHt)|n_1, n_2, ..., n_m\rangle = \frac{a_1^{\dagger n_1}(t) a_2^{\dagger n_2}(t) \cdots a_m^{\dagger n_m}(t)}{\sqrt{n_1! n_2! \cdots n_m!}} |\{0\}\rangle, \tag{5.8}$$

where $a_i^\dagger(t)$ are given by the Hermitian conjugate of (5.4). The expression (5.8) can be evaluated by expanding each of the $a_i^\dagger(t)$ in powers of $\{a_j^\dagger(0)\}$. If, for example, the initial state is $|\{0\}, N\rangle$, then it follows from (5.8) that the state at time t is the coherent state (2.10) with

$$\mu_i = f_{mi}^*/f_{mm}^*, \quad i = 1,2,...,m-1. \tag{5.9}$$

It also follows that the state

$$|\psi_0\rangle = |z_1, z_2, ..., z_m\rangle \equiv \exp\left(-\tfrac{1}{2}\sum_i |z_i|^2\right) \exp\left(\sum_{i=1}^{m} z_i a_i^\dagger\right) |\{0\}\rangle, \tag{5.10}$$

which is the coherent state for each of the boson annihilation operators, remains coherent under the SU(m) group transformations. By virtue of (5.4),

$$\exp(-iHt)|\psi_0\rangle = \exp\left(-\tfrac{1}{2}\sum_i |z_i|^2\right)\exp\left(\sum_{i,j=1}^{m} f_{ij}(t) z_i a_j^\dagger\right)|\{0\}\rangle. \tag{5.11}$$

Hence a state that is coherent for each of the modes remains coherent under the action of SU(m).

As an example, consider the three-mode Hamiltonian

$$H = \kappa_1(a^\dagger b + b^\dagger a) + \kappa_2(a^\dagger c + c^\dagger a) + i\kappa_3(b^\dagger c - c^\dagger b) \tag{5.12}$$

generating an SU(3) transformation. The eigenvalues of M, defined in (5.3), in that case are $\lambda_1 = 0$ and $\lambda_2 = -\lambda_3 = \Gamma \equiv \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}$. The corresponding eigenvectors are given by

$$|\psi_1\rangle = \frac{1}{\Gamma}\mathrm{col}(\kappa_3, -i\kappa_2, i\kappa_1),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2(\kappa_2^2 + k_3^2)}\Gamma} \times \mathrm{col}[\Gamma\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 + i\Gamma\kappa_3, (\kappa_2^2 + \kappa_3^2)],$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2(\kappa_2^2 + \kappa_3^2)}\Gamma} \times \mathrm{col}[-\Gamma\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 - i\Gamma\kappa_3, (\kappa_2^2 + \kappa_3^2)]. \tag{5.13}$$

The expressions (5.13) for the eigenvectors can be substituted in (5.5) to evaluate $f_{ij}(t)$. The dynamics of any observable and that of the state vector is thus determined. In particular, the state of the field initially in the state $|0, 0, N\rangle$ is the SU(3) coherent state (2.10) at time t with μ_1 and μ_2 given, using Eq. (5.9), by

$$\mu_1 = \frac{2i[\Gamma\kappa_2\cos(\Gamma t/2) + \kappa_1\kappa_3\sin(\Gamma t/2)]\sin(\Gamma t/2)}{(\kappa_2^2 + \kappa_3^2)\cos(\Gamma t) - \kappa_1^2},$$

$$\mu_2 = \frac{2[\Gamma\kappa_3\sin(\Gamma t/2) + \kappa_1\kappa_2\cos(\Gamma t/2)]\cos(\Gamma t/2)}{(\kappa_2^2 + \kappa_3^2)\cos(\Gamma t) - \kappa_1^2}. \tag{5.14}$$

The nonclassical characteristics of those states have already been discussed in Sec. III.

Consider next the special case $\alpha_{ij} = \beta_{ij} = 0$ of the SU(m,n) transformation (4.3) representing the process of parametric amplification. The Heisenberg equations of motion for the bose operators in that case are

$$\dot{\tilde{C}}(t) = -iR\tilde{C}(t), \tag{5.15}$$

where

$$\tilde{C} = \mathrm{col}(a_1, a_2, ..., a_m; b_1^\dagger, b_2^\dagger, ..., b_n^\dagger) \tag{5.16}$$

and the matrix R is given by

$$R_{i,j} = 0,\ R_{i,m+p} = \gamma_{ip},\ R_{m+p,i} = -\gamma_{ip}^*,\ R_{m+p,m+q} = 0,$$
$$i, j = 1, 2, ..., m;\ p, q = 1, 2, ..., n. \tag{5.17}$$

The matrix R is antisymmetric. Hence its eigenvalues are imaginary. If $|\psi_i\rangle$ are the eigenvectors of R corresponding to the eigenvalues $i\lambda_i$ (λ_i are real and $i = 1, 2, ..., m + n$), then it follows that

$$C_i(t) = \sum_{j,k=1}^{m+n} \exp(\lambda_j t)\langle i|\psi_j\rangle\langle\psi_j|k\rangle C_k(0). \tag{5.18}$$

Since R is a traceless matrix, the sum of its eigenvalues is zero. Hence at least one λ_i is positive, thereby implying that the fields in the case of parametric amplification grow with time.

Finally, the Heisenberg equations for the general SU(m,n) evolution read

$$\dot{\tilde{C}} = -iQ\tilde{C}, \tag{5.19}$$

where Q is defined as

$$\begin{aligned} Q_{i,j} &= \alpha_{ij} = Q_{j,i}^* \quad (j \neq i), \\ Q_{i,m+p} &= \gamma_{ip} = -Q_{m+p,i}^* \\ Q_{m+p,m+q} &= \beta_{pq} = Q_{m+q,m+p}^* \quad (p \neq q), \end{aligned} \tag{5.20}$$

$$i, j = 1, 2, ..., m;\ p, q = 1, 2, ..., n.$$

Note that Q is, in general, not a normal matrix. Hence the solution of (5.19) can be written in terms of the right and the left eigenvectors $\{|\psi_k\rangle\}$ and $\{\langle\phi_k|\}$ of Q as

$$C_i(t) = \sum_{j=1}^{m+n} g_{ij}(t)C_j(0), \tag{5.21}$$

where $\{\lambda_k\}$ are, in general, complex and

$$g_{ij}(t) = \sum_{k=1}^{m+n} \exp(-i\lambda_k t)\langle i|\psi_k\rangle\langle\phi_k|j\rangle. \tag{5.22}$$

The dynamical characteristics of the observables under SU(m,n) transformations is thus governed by the functions $\{g_{ij}\}$. The canonical commutation relations obeyed by $\{C_i(t)\}$ imply that

$$\sum_{j=1}^{m} g_{ij}g_{i'j}^* - \sum_{j=m+1}^{m+n} g_{ij}g_{i'j}^* = \delta_{ii'},$$

$$\sum_{j=1}^{m} g_{pj}g_{p'j}^* - \sum_{j=m+1}^{m+n} g_{pj}g_{p'j}^* = -\delta_{pp'},$$

$$i, i' = 1, 2, ..., m;\ p, p' = m + 1, m + 2, ..., m + n. \tag{5.23}$$

The dynamics of a state vector can also be determined by using the solution (5.21) of the Heisenberg equations by expanding the state vector in terms of the Fock states given in terms of the vacuum state $|\{0\}\rangle$ as in (5.6) so that

$$|n_1,n_2,...,n_m\rangle = \frac{a_1^{\dagger n_1}(t)a_2^{\dagger n_2}(t)\cdots a_m^{\dagger n_m}(t)}{\sqrt{n_1!n_2!\cdots n_m!}}|\psi(t)\rangle, \tag{5.24}$$

where

$$|\psi(t)\rangle = \exp(-iHt)|\{0\}\rangle. \tag{5.25}$$

Note, however, that in this case, unlike the case of SU(m), $H|\{0\}\rangle \neq 0$. The state $|\psi(t)\rangle$ can be determined in the case of SU(m, 1) by invoking the fact that $C_i(0)|\{0\}\rangle = 0$ for $i = 1,2,...,m$ and $C_{m+1}(0)^\dagger|\{0\}\rangle = 0$ so that

$$\begin{aligned} &C_i(t)|\psi(t)\rangle = 0, \quad i = 1,2,...,m \\ &C^\dagger_{m+1}(t)|\psi(t)\rangle = 0. \end{aligned} \tag{5.26}$$

The solution of (5.26) can be found by using the expressions (5.21) for $C_i(t)$ and making use of the fact that $|\psi(t)\rangle$ has the form (2.20) of the coherent state of SU(m, 1). Equations (5.26) then lead to

$$\sum_{j=1}^{m} \mu_j g_{ij} + g_{i,m+1} = 0, \quad i = 1,2,...,m \tag{5.27}$$

$$\mu_j g^*_{m+1,m+1} + g^*_{m+1,j} = 0. \tag{5.28}$$

Equation (5.28) determines $\{\mu_i\}$. Those $\{\mu_i\}$, by virtue of (5.23), also satisfy (5.27).

As an example, consider the SU(2,1) process generated by the Hamiltonian

$$H = \kappa_1(a^\dagger b+b^\dagger a)+\kappa_2(ac+c^\dagger a^\dagger)+i\kappa_3(bc-c^\dagger b^\dagger). \tag{5.29}$$

The eigenvalues λ_i, $i = 1,2,3$, of the evolution operator in that case are found to be given by $\lambda_1 = 0$ and $\lambda_2 = -\lambda_3 = D$, where $D = \sqrt{\kappa_1^2 - \kappa_2^2 - \kappa_3^2}$. The eigenvalues are imaginary if $\kappa_1^2 - \kappa_2^2 - \kappa_3^2 < 0$ and real otherwise. Note that κ_1^2 is a measure of the strength of the process of parametric amplification and $\kappa_2^2 + \kappa_3^2$ is that of the process of frequency conversion. Hence the coupled process is of the parametric amplification type or the type of frequency converter, depending on which of the two is dominant. The eigenvectors of Q and $Q^\dagger$ are given respectively by

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{D}\mathrm{col}(\kappa_3, -i\kappa_2, i\kappa_1), \\ |\psi_2\rangle &= \frac{1}{D\sqrt{2(\kappa_2^2+\kappa_3^2)}} \\ &\quad \times\mathrm{col}[D\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 \\ &\quad +iD\kappa_3, -(\kappa_2^2+\kappa_3^2)], \\ |\psi_3\rangle &= \frac{1}{D\sqrt{2(\kappa_2^2+\kappa_3^2)}} \\ &\quad \times\mathrm{col}[-D\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 - iD\kappa_3, -(\kappa_2^2+\kappa_3^2)], \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{D}\mathrm{col}(\kappa_3, -i\kappa_2, -i\kappa_1), \\ |\phi_2\rangle &= \frac{1}{D\sqrt{2(\kappa_2^2+\kappa_3^2)}} \\ &\quad \times\mathrm{col}[D\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 \\ &\quad +iD\kappa_3, (\kappa_2^2+\kappa_3^2)], \\ |\phi_3\rangle &= \frac{1}{D\sqrt{2(\kappa_2^2+\kappa_3^2)}} \\ &\quad \times\mathrm{col}[-D\kappa_2 + i\kappa_1\kappa_3, \kappa_1\kappa_2 - iD\kappa_3, (\kappa_2^2+\kappa_3^2)]. \end{aligned} \tag{5.31}$$

The functions $g_{ij}(t)$ can now be evaluated by substituting (5.30) and (5.31) in (5.22). The dynamics of the observables as well as of the states is thus completely determined. For example, the state evolving from the vacuum state $|\{0\}\rangle$ is the coherent state (2.20) with $\{\mu_i\}$ given, using (5.28), by

$$\begin{aligned} \mu_1 &= i\frac{[\kappa_1\kappa_3\cos(Dt/2) + D\kappa_2\sin(Dt/2)]\cos(Dt/2)}{\{\kappa_1^2 - [\kappa_2^2+\kappa_3^2]\cos(Dt)\}}, \\ \mu_2 &= \frac{[\kappa_1\kappa_2\sin(Dt/2) + D\kappa_3\cos(Dt/2)]\sin(Dt/2)}{\{\kappa_1^2 - [\kappa_2^2+\kappa_3^2]\cos(Dt)\}}. \end{aligned} \tag{5.32}$$

The Hamiltonian (5.12) for $\kappa_3 = 0$ has been studied by Mishkin and Walls [12].

VI. CONCLUSIONS

The Hamiltonians generating SU(m,n) transformations in multimode nondegenerate optical parametric interactions are discussed. Those Hamiltonians acting on a vector in the space of the states of SU(m,n) generate a coherent state of that group. The coherent states of SU(m,n) for some particular fiducial states have been constructed. The nonclassical properties of the coherent states of SU(m) and those of SU(m, 1) are studied. The photon number distribution of each of the modes in the SU(m) coherent state is found to be sub-Poissonian. Each of the modes in the coherent state of SU(m, 1), on the other hand, is found to be in the thermal state. The SU(1,1) coherent state always exhibits two-mode squeezing, but the two-mode squeezing of SU(m, 1) coherent states for $m \neq 1$ depends on the values of the parameters defining the coherent state. The dynamics generated by SU(m,n) Hamiltonians is discussed.

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