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Enhanced squeezing by periodic frequency modulation under parametric instability conditions

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A general analysis of an oscillator with periodic frequency modulation is given. It is shown that squeezing and excitation energy exponentially grow with the number of modulation cycles under the condition of *parametric instability.* This condition yields a prescription for maximized squeezing when the frequency is periodically swept in an adiabatic fashion, with an abrupt return to the initial frequency at the end of each period. The type of modulation considered is shown to have remarkably broad instability domains near *arbitrarily high ratios* of the oscillator period to the modulation cycle duration. This property stands in striking contrast to the rapid narrowing of the squeezing domains with the ratio of the pump frequency to that of the signal in existing parametric processes. We discuss a possible realization of the proposed scheme, based on frequency modulation of a cavity mode in the microwave domain by a periodic train of optical pulses, and show that extremely strong squeezing is feasible under rather moderate requirements.

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I. INTRODUCTION

Reduction in the uncertainty of a quantum observable below that of its counterpart in the ground (vacuum) state of the system is known as squeezing. This class of phenomena has been extensively studied in several contexts. The focus of attention has been on squeezing of electromagnetic field observables through field-matter interactions in bulk media and cavities, and their potential technological applications [1–3]. In a different context, squeezing of atomic [4] and molecular [5–7] observables has been considered. Our concern here is with quadrature squeezing in a harmonic oscillator with temporally modulated frequency, which has been recently addressed by a number of theoretical studies [8–16].

As shown by Graham [9], an abrupt change of the oscillator frequency can squeeze a coherent state, whereas adiabatic change of the frequency and its subsequent restoration only produces cyclic evolution of the system, ending in the original state, without any squeezing. Agarwal and Kumar [12] have analyzed the explicit time dependence of squeezing on nonadiabaticity, ranging anywhere between the abrupt and adiabatic limits, in the simple case of linearly swept restoring force. Lo has obtained [13] a formal solution for squeezing generation starting from a number state, for arbitrary time-dependent parameters of the oscillator. Most recently, Janszky and Adam [14] and Dodonov *et al.* [15] have studied squeezing in the model of temporal Kronig-Penney modulation, i.e., periodically alternating abrupt frequency jumps separated by a quarter-period constant-frequency oscillation. Squeezing of a coherent state by several types of frequency modulation was considered by Abdalla and Colegrave [16].

The extensive investigations of squeezing in a harmonic oscillator with temporally modulated frequency have still left several important questions open: (a) What are the *general conditions* that must be met by the frequency modulation in order to obtain squeezing? (b) How can this squeezing be optimized? (c) Is there a connection between physical mechanisms underlying squeezing in this model and in parametric optical processes [1–3]? (d) Can this model be implemented in optics?

We purport to answer these questions by showing that quadrature squeezing can grow exponentially with time (until disturbed by dissipation) under the condition of *parametric instability* in an oscillator whose frequency is periodically swept in an arbitrary adiabatic way, with an *abrupt* return to the initial frequency at the end of each period. The same condition leads to *exponential growth* of the oscillator energy. Explicit WKB expressions are derived for squeezing accumulated in N cycles of alternating adiabatic modulation and abrupt frequency restoration. These expressions are compared with the exact results for periodic linear sweeping (Sec. II). We then discuss a possible realization of this scheme based on optically induced Kerr-type modulation of the refractive index at a cavity-mode frequency in the microwave domain (Sec. III). A remarkable property of the proposed scheme, which makes it potentially advantageous for parametric amplification and squeezing, is that the spectral width of the parametric instability domains, wherein squeezing is accumulated, *does not change* significantly with the integer ratio of the modulation cycle duration to the oscillator period. This allows amplification or squeezing using modulation frequencies that are *orders of magnitude* lower than the oscillator frequency, in sharp contrast with currently used two-mode parametric processes (Sec. IV), wherein the pump driving frequency is commonly not lower than one half or one third the oscillator (signal) frequency. However, due to an abrupt turnoff of the modulation cycle, the modulating force spectrum has a greater bandwidth, which greatly exceeds the fundamental frequency. The conclusions are summarized in Sec. V.

II. ANALYSIS OF SQUEEZING BY PERIODIC FREQUENCY MODULATION

A. General treatment

The Hamiltonian of a harmonic oscillator with time-dependent frequency is given by

$$\hat{H}(t) = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2(t)\hat{q}^2, \tag{2.1}$$

where $\hat{p}$ and $\hat{q}$ are the momentum and coordinate operators, respectively, and $\omega(t)$ is the time-dependent frequency. We introduce the scaled dimensionless operators

$$\hat{Q}(t) = \sqrt{\frac{\omega_0}{2\hbar}}\hat{q}(t), \quad \hat{P}(t) = \frac{1}{\sqrt{2\hbar\omega_0}}\hat{p}(t), \tag{2.2}$$

where ω_0 is the *initial* frequency (at the $t=0$).

Their evolution following N modulation cycles of duration T, each cycle ending by an *abrupt* return to the initial frequency ω_0, is given by

$$\begin{pmatrix} \hat{Q}(NT) \\ \hat{P}(NT) \end{pmatrix} = U^N(T) \begin{pmatrix} \hat{Q}(0) \\ \hat{P}(0) \end{pmatrix}, \tag{2.3}$$

where the single-cycle evolution matrix is

$$U(T) = \begin{pmatrix} u(T) & v(T) \\ \dot{u}(T)/\omega_0 & \dot{v}(T)/\omega_0 \end{pmatrix}. \tag{2.4}$$

Here u,v are the c-number solutions of the following equation for the single-cycle modulation:

$$\left[\frac{d^2}{dt^2} + \omega^2(t)\right]\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{2.5}$$

with the initial conditions

$$v(0) = \dot{u}(0)/\omega_0 = 0, \quad u(0) = \dot{v}(0)/\omega_0 = 1. \tag{2.6}$$

Since the Wronskian of u and v, $W = \dot{u}v - \dot{v}u$ does not depend on t, it follows from Eq. (2.6) that

$$\det[U(T)] = 1. \tag{2.7}$$

By raising the single-cycle evolution matrix in Eq. (2.4) to the Nth power, we find

$$U^N(T) = \frac{1}{\sin\varphi}\begin{pmatrix} u(T)\sin N\varphi - \sin(N-1)\varphi & v(T)\sin N\varphi \\ [\dot{u}(T)/\omega_0]\sin N\varphi & [\dot{v}(T)/\omega_0]\sin N\varphi - \sin(N-1)\varphi \end{pmatrix}, \tag{2.8}$$

with

$$\varphi = \cos^{-1}\{\tfrac{1}{2}[u(T) + \dot{v}(T)/\omega_0]\}. \tag{2.9}$$

Equations (2.3)–(2.9) constitute the N-cycle Heisenberg-picture solutions in the $\hat{Q} - \hat{P}$ basis. The significance of the solution parameters is noted on writing, e.g., the coordinate variance σ_q after N cycles, for a coherent initial state,

$$\sigma_q(NT) = \{[u(T)\sin N\varphi - \sin(N-1)\varphi]^2 + v^2(T)\sin^2 N\varphi\}/\sin^2\varphi. \tag{2.10}$$

Alternatively, we may work in the usual basis of annihilation and creation operators

$$\hat{a} = \hat{P} + i\hat{Q}, \ \hat{a}^\dagger = \hat{P} - i\hat{Q} \tag{2.11}$$

The N-cycle solution of Eq. (2.3) is thereby recast in the form of a Bogoliubov transformation [17]

$$\hat{a}(NT) = \mu_{NT}\hat{a}_0 + \nu_{NT}\hat{a}_0^\dagger, \tag{2.12}$$

where

$$\mu_{NT} = [\mu_T \sin N\varphi - \sin(N-1)\varphi]/\sin\varphi, \tag{2.13a}$$

$$\nu_{NT} = \nu_T \sin N\varphi/\sin\varphi \tag{2.13b}$$

are expressed via the single-cycle transformation parameters

$$\mu_T = u(T) + \dot{v}(T)/\omega_0 + i[\dot{u}(T)/\omega_0 - v(T)], \tag{2.14a}$$

$$\nu_T = u(T) - \dot{v}(T)/\omega_0 + i[\dot{u}(T)/\omega_0 + v(T)]. \tag{2.14b}$$

The transformation of Eq. (2.12) is unitary and can thus be written as a product of the stretching (squeezing) of the quadratures, described by operator $\hat{S}$, and their rotation, described by operator $\hat{R}$

$$\hat{a}(NT) = \hat{R}^\dagger(\theta_N)\hat{S}^\dagger(\xi_N)\hat{a}_0\hat{S}(\xi_N)\hat{R}(\theta_N) \tag{2.15}$$

with

$$\hat{R}(\theta_N) = \exp[i\theta_N\hat{a}_0^\dagger\hat{a}_0], \tag{2.16a}$$

$$\hat{S}(\xi_N) = \exp\left[\frac{\xi_N{}^*}{2}\hat{a}_0^2 - \frac{\xi_N}{2}\hat{a}_0^{\dagger 2}\right], \tag{2.16b}$$

$$\theta_N = \arg(\mu_{NT}), \quad |\xi_N| = \tanh^{-1}|\nu_{NT}/\mu_{NT}|,$$

$$\arg(\xi_N) = \tfrac{1}{2}[\arg(\nu_{NT}) + \theta_N]. \tag{2.16c}$$

Here the absolute value of ξ_N is the squeezing parameter, whereas θ_N is the rotation angle of the axes (or quadratures).

B. Progressive accumulation of squeezing

In order to obtain physical insight into the evolution, let us consider the transformation $\mathcal{S}$ that diagonalizes the single-cycle evolution matrix

$$\mathcal{S}U(T)\mathcal{S}^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.17}$$

where $\lambda_{1,2}$ are the corresponding (complex) eigenvalues. This transformation yields the following two linear combinations of the operators $\hat{Q}$ and $\hat{P}$

$$\begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \end{pmatrix} = \mathcal{S}\begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix} \tag{2.18}$$

The N-cycle evolution of the "quadratures" $\hat{A}_{1,2}$ obeys, [according to Eqs. (2.17) and (2.18)] the simple transformation rule

$$\hat{A}_{1,2}(NT) = \lambda_{1,2}^N \hat{A}_{1,2}(0). \tag{2.19}$$

Hence, their variances scale as follows

$$\langle \Delta \hat{A}_{1(2)}^2 (NT) \rangle \sim \lambda_{1(2)}^{2N}. \tag{2.20}$$

By virtue of the unimodularity of $U(T)$ [see Eq. (2.7)], $\lambda_1\lambda_2 = 1$. This property is crucial in determining the scaling, as noted from the eigenvalue equation for $U(T)$ [Eq. (2.4)]

$$\lambda^2 - 2\lambda\cos\varphi + 1 = 0, \tag{2.21}$$

whose solutions are

$$\lambda_{1,2} = e^{\pm i\varphi}. \tag{2.22}$$

Two distinct regimes are apparent from these solutions,

(i) If $\mathrm{Im}\varphi \neq 0$, then the quadrature operators discussed above undergo a scaling transformation. This means that, say, $\hat{A}_1(NT)$ is "stretched" as $\exp(N\mathrm{Im}\varphi) \equiv \exp(\xi_N)$, at the expense of "compression" of the orthogonal quadrature $\hat{A}_2(NT)$ as $\exp(-N\mathrm{Im}\varphi) \equiv \exp(-\xi_N)$. Here ξ_N is the squeezing parameter [same as in Eq. (2.16)]. Thus progressive accumulation of squeezing in consecutive modulation cycles is caused by parametric instability.

(ii) Conversely, if $\varphi(T)$ is purely real ($|\lambda_{1,2}| = 1$), then $\xi_N = 0$ and only rotation of the axes takes place. The boundary between stable (limited squeezing) and unstable (unlimited squeezing) regimes corresponds to [cf. Eq. (2.9)].

$$|\cos\varphi| = \tfrac{1}{2}|u + \dot{v}/\omega_0| = 1. \tag{2.23}$$

For $|\cos\varphi| > 1$ we enter the instability domain, in which $\mathrm{Im}\varphi \neq 0$, and thus the squeezing parameter $|\xi_N| > 0$.

C. The WKB regime: alternating adiabatic and abrupt modulation

In what follows, we study the dependence of squeezing on the form of single-cycle modulation by examining the WKB regime, which is amenable to an analytical treatment. The WKB approximation is applicable when the modulation throughout a single cycle is adiabatic [18]. This means that the frequency changes only by a small fraction during one period of vibration, $2\pi|\dot{\omega}|/\omega^2 \ll 1$.

The WKB solutions [18] of the Eq. (2.5) are linear combinations of $\omega^{-1/2}(t)\exp[\pm i\theta(t)]$ with the adiabatic phase

$$\theta(t) = \int_0^t \omega(t')dt'. \tag{2.24}$$

On using this form with the initial conditions (2.6), the single-cycle evolution matrix (2.6) becomes

$$\begin{pmatrix} u(T) & v(T) \\ \dot{u}(T)/\omega_0 & \dot{v}(T)/\omega_0 \end{pmatrix} \approx \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} \times \begin{pmatrix} \cos\theta(T) & \sin\theta(T) \\ -\sin\theta(T) & \cos\theta(T) \end{pmatrix}, \tag{2.25}$$

which is a factorized product of a stretching transformation with the parameter

$$s = \exp|\xi_1| = \sqrt{\frac{\omega(T)}{\omega_0}}, \tag{2.26}$$

ξ_1 being the single-cycle squeezing parameter, and a rotation by the adiabatic phase angle $\theta(T)$.

As noted in the introduction, the modulation must end abruptly at each cycle, if we wish to retain the squeezing accumulated throughout the cycle. This can be seen by comparing the initial coordinate variance $\sigma_q(0)$, for a coherent state, with its value at $t = T$, following a period of adiabatic frequency modulation. On setting $N = 1$ in Eq. (2.10) and using Eq. (2.25) we have

$$\sigma_q(T) = \frac{\hbar}{2m\omega_0}[u^2(T) + v^2(T)] = [\omega_0/\omega(T)]\sigma_q(0). \tag{2.27}$$

Thus, we obtain squeezing, i.e., $\sigma_q(T) < \sigma_q(0)$, if the instantaneous frequency $\omega(T)$ *exceeds* the initial frequency ω_0. However, we should bear in mind that the oscillator state remains nearly coherent throughout the adiabatic evolution. Hence, at $t = T$ we have a coherent state with frequency $\omega(T)$ which is squeezed only *relative* to the initial coherent state with frequency ω_0. In order to observe this squeezing, we need to restore the frequency back to ω_0. The restoration cannot be adiabatic, because it would simply change $\sigma_q(T)$ back to $\sigma_q(0)$, by the inverse of Eq. (2.25). The squeezing accumulated during $0 \leq t \leq T$ can be preserved only by switching the frequency back to ω_0 rather abruptly, preferably much faster than the oscillation period $2\pi/\omega_0$. Then the sudden approximation holds, under which the shape of the oscillator wave function is not disturbed.

The condition for squeezing accumulation in N cycles becomes much more transparent in the WKB approximation. From (2.25) and (2.9) it follows that the transformation parameter φ assumes the form

$$\varphi(T) \approx \cos^{-1}[\cosh \xi_1 \cos \theta(T)]. \tag{2.28}$$

The N-cycle evolution matrix (2.8) is then observed to become diagonal under the resonance condition

$$\theta(T) = k\pi, \quad k = 1, 2, 3, \ldots, \tag{2.29}$$

which signifies "in-phase" consecutive frequency sweeps. The corresponding coordinate variance *exponentially decreases with N*

$$\sigma_q(NT) = \sigma_q(0) e^{-2N|\xi_1|} = \sigma_q(0) \left(\frac{\omega(T)}{\omega_0}\right)^{-N}. \tag{2.30}$$

Hence, the parametric instability noted above is built up near the resonance given by Eq. (2.29). The oscillator energy accumulated during N cycles, is simply

$$\langle E_N(\omega(T))\rangle = \tfrac{1}{2}\langle (Q^2 + P^2)\rangle_N \approx \cosh 2N|\xi_1|, \tag{2.31}$$

exponentially growing from its initial zero-point value.

The stability boundary (2.23) is expressed in WKB approximation by

$$\frac{1 \pm \sin \theta(T)}{\cos \theta(T)} = \sqrt{\frac{\omega(T)}{\omega_0}} = s. \tag{2.32}$$

The resulting stability diagram is shown in Fig. 1. For the range of parameters corresponding to the stable (unshaded) regions, σ_q exhibits a *quasi-periodic behavior* with time, i.e., squeezing is not accumulated with N.

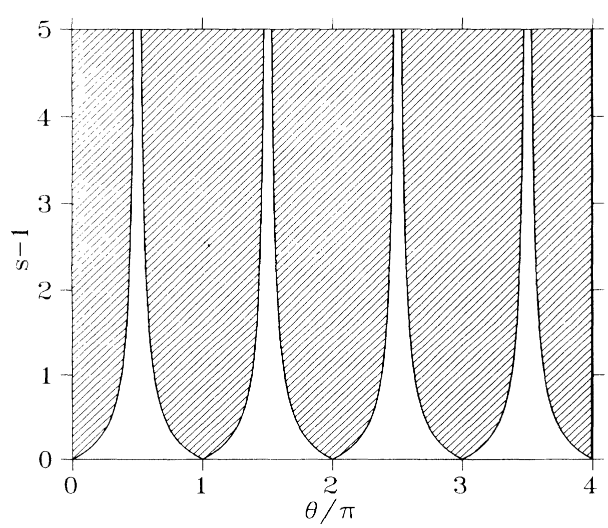

FIG. 1. Stability diagram for periodically modulated oscillator with abrupt frequency restoration. The instability regions are the shaded spaces in the plots of the modulation strength $s-1$ versus the subharmonic ratio θ/π.

In contrast, squeezing grows in an *unlimited fashion* with the number of cycles for parameters corresponding to the shaded (instability) regions.

Two basic limitations on the obtainable squeezing strength must still be reckoned with: (i) Dissipation time limits the number of cycles over which squeezing is accumulated. (ii) Fluctuation of the adiabatic phase about the resonance condition (2.29) reduces the squeezing. (iii) The anharmonicity of the oscillator, caused by nonlinear high-intensity effects may hamper the squeezing degree.

D. Periodic linear sweeping: exact treatment

In order to demonstrate the validity of the foregoing WKB treatment, let us consider the *exactly solvable* case of linear sweeping of the restoring force

$$\omega^2(t) = \omega_0^2(1 + \gamma t), \; 0 \le t \le T. \tag{2.33}$$

In the adiabatic limit, the rate of sweeping γ is much smaller than ω_0, but here we need not impose any such restrictions.

The coefficients u and v of Eqs. (2.3)–(2.5) are expressed in this case in terms of Airy functions [19]

$$u(t) = \frac{1}{\pi}\left[\mathrm{Ai}(z)\mathrm{Bi}'(z_0) - \mathrm{Ai}'(z_0)\mathrm{Bi}(z)\right], \tag{2.34a}$$

$$v(t) = \frac{1}{\pi(\omega_0^2\gamma)^{1/3}}\left[\mathrm{Ai}(z_0)\mathrm{Bi}(z) - \mathrm{Ai}(z)\mathrm{Bi}(z_0)\right], \tag{2.34b}$$

where the prime sign denotes a derivative with respect to the dimensionless variable

$$z(t) = -\left(\frac{\omega_0}{\gamma}\right)^{3/2}(1 + \gamma t), \quad z_0 = z(0). \tag{2.35}$$

The evolution of $\sigma_q(t)$ calculated via Eq. (2.10) for the exact solutions (2.34) is presented in Fig. 2 for various

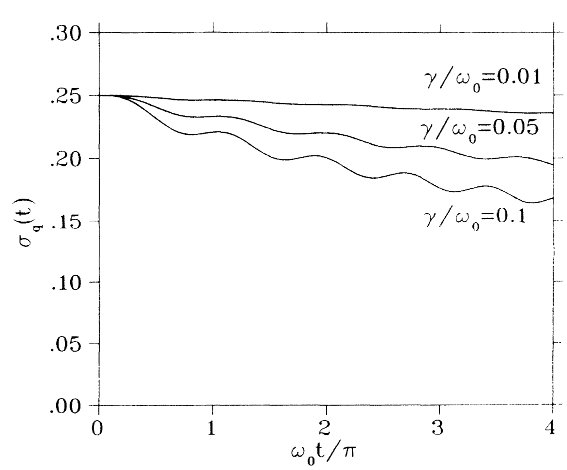

FIG. 2. Coordinate mean variance obtained by periodic linear sweeping for various ratios γ/ω_0.

values of the ratio of γ/ω_0. The overall decline of $\sigma_q(t)$ is in accordance with the adiabatic expression (2.30) and shows the gradual squeezing of the coordinate fluctuations. The small amplitude oscillations with an approximate frequency $2\omega_0$ for $\gamma/\omega_0 = 0.1$ result from a nonadiabatic perturbation of a coherent state. The origin of these oscillations may be understood from the following general relation between quantum variances at two times t and $t+\tau$, valid for the free oscillation with a fixed frequency ω_0 and an arbitrary initial quantum state

$$\sigma_q(t+\tau) = \sigma_q(t)\cos^2(\omega_0\tau) + \frac{\sigma_p(t)}{\omega_0^2}\sin^2(\omega_0\tau) + \frac{\sigma_{pq}(t)}{\omega_0}\sin(2\omega_0\tau), \tag{2.36}$$

where a "mixed" variance σ_{pq} is defined as

$$\sigma_{pq} = \tfrac{1}{2}[\langle(\hat{q}-\langle\hat{q}\rangle)(\hat{p}-\langle\hat{p}\rangle)\rangle + \langle(\hat{p}-\langle\hat{p}\rangle)(\hat{q}-\langle\hat{q}\rangle)\rangle]. \tag{2.37}$$

For coherent and vacuum states one has $\sigma_{pq} = 0, \sigma_p = \omega_0^2\sigma_q$, so that the value of σ_q is time independent. However, for other states σ_q oscillates with the frequency $2\omega_0$. These double-frequency oscillations of the variance are exhibited by the gradually squeezed state of a frequency-modulated oscillator as well, indicating a deformation of an initially coherent state.

Figures 3(a) and 3(b) show the time dependence of the coordinate variance for linear frequency sweeping consisting of three cycles with $\gamma/\omega_0 = 0.01$. After an abrupt switch of the frequency at $t = T$ to its initial value ω_0, the width of the wave packet σ_q starts to oscillate due to the reasons discussed above [see Eq. (2.36)]. Nevertheless, the overall decrease of the σ_q minima is seen throughout the second cycle $T < t < 2T$. The critical role of the duration T of the adiabatic stage becomes clear after the application of the third cycle of modulation at $2T < t < 3T$. If at the end of the second cycle

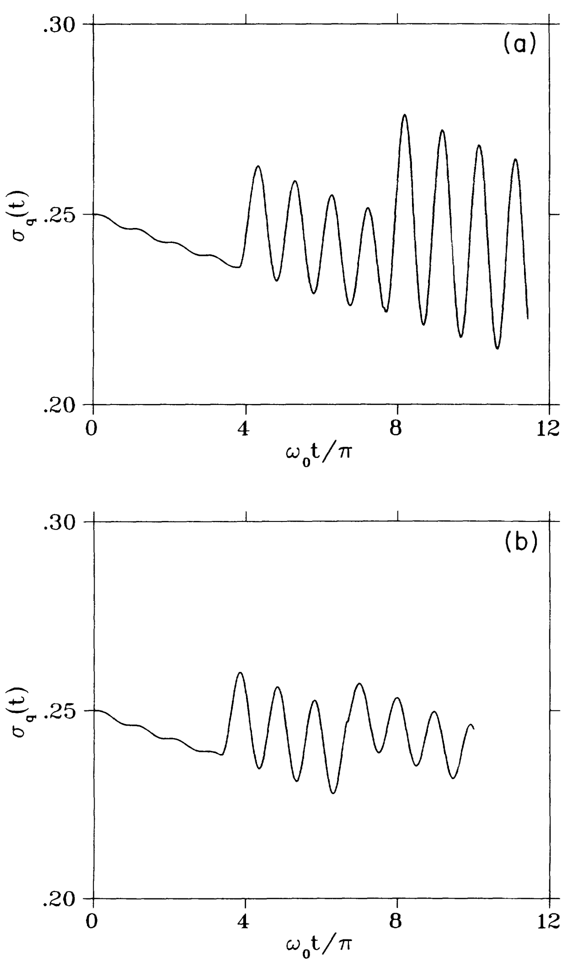

FIG. 3. Time dependence of the coordinate mean variance for three cycles of linear frequency sweeping with $\gamma/\omega_0 = 0.01$: (a) successful choice of T and (b) unsuccessful choice of T.

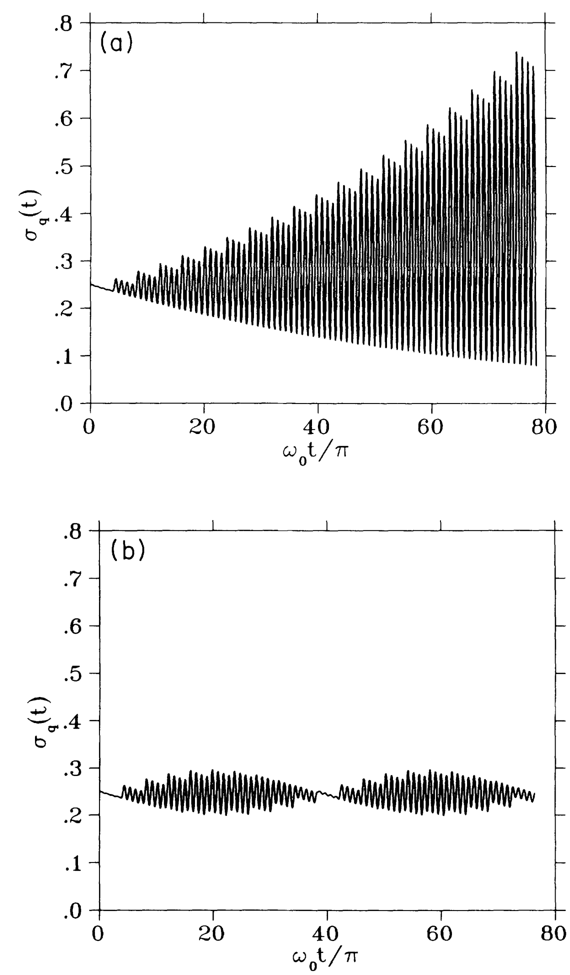

FIG. 4. The coordinate mean variance evolution for many cycles of modulation. The parameters are (a) $\gamma/\omega_0 = 0.01$, $\omega_0 T = 12.3$ and (b) $\gamma/\omega_0 = 0.01, \omega_0 T = 12$. Unlimited squeezing is obtained inside the instability region (a), whereas outside the instability region the envelope of σ_q undergoes oscillations (b).

the oscillating variance is near its minimum, then the next cycle can cause stronger squeezing [see Fig. 3(a)]. For an "unsuccessful" choice of T (outside an instability region) [Fig. 3(b)] it is impossible to have unlimited accumulation of the squeezing. Rather, the envelope of σ_q undergoes long-time oscillations (Fig. 4).

III. POSSIBLE REALIZATION IN MICROWAVE CAVITIES

The foregoing analysis has demonstrated the remarkable properties of parametric amplification and squeezing, achieved by repeated abruptly terminated modulation cycles. The question to be still resolved is whether the realization of such a process is experimentally feasible with significant modulation strength γ, on the one hand, and a short restoration time of the initial oscillation frequency (much faster than the oscillation period $2\pi/\omega_0$) on the other hand. These requirements can be met for a microwave mode of a cavity filled with a dilute gas. A train of short optical pump pulses modulates the frequency of the cavity mode. Each pulse produces Stark shifts of the atomic levels which modify, in turn, the refractive index experienced by the cavity mode, thereby causing its frequency to change. In what follows we analyze this process.

The gas atoms can be viewed as three-level systems, in which the transition $|1\rangle \to |2\rangle$ is at a microwave frequency ω_{12}, whereas $|2\rangle \to |3\rangle$ is at an optical frequency ω_{23} (Fig. 5). The levels $|1\rangle$ and $|3\rangle$ are uncoupled in the dipole approximation. The cavity-mode frequency ω_0 (microwave) is detuned from ω_{12} by $\delta \gg \Gamma$, Γ being the $|1\rangle \to |2\rangle$ transition linewidth. An optical pump pulse centered at frequency $\omega_p = \omega_{23} - \Delta$ shifts the level $|2\rangle$ relative to that of $|1\rangle$ by [20]

$$\omega'_{12} \approx \omega_{12} + \frac{\Omega_p^2}{4\Delta}, \tag{3.1}$$

where Ω_p is the pump-field Rabi frequency. This level shift alters the detuning from $\delta = \omega_0 - \omega_{12}$ to $\delta' = \omega_0 - \omega'_{12}$, thereby changing the refractive index of the gas from n_0 to n'_0. The well-known rule of Lorentzian-line dispersion yields

$$n_0 = 1 + A\frac{\Gamma\delta}{(\Gamma^2 + \delta^2)} \to n'_0 = 1 + A\frac{\Gamma\delta'}{(\Gamma^2 + \delta'^2)}. \tag{3.2}$$

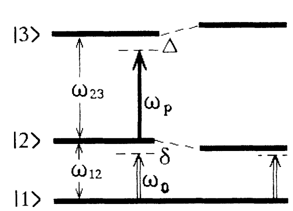

FIG. 5. Frequency modulation scheme for a three-level system.

Here the dimensionless constant $A \propto \rho\mu_{12}^2/\Gamma$ with ρ denoting the gas density, and μ_{12} is the $|1\rangle \to |2\rangle$ transition dipole moment. The difference $\delta - \delta' = \omega'_{12} - \omega_{12}$ is given by Eq. (3.1), so that the single-cycle squeezing (2.26), which is determined by the relative frequency change of the cavity mode (provided $\delta, \delta' \gg \Gamma$) satisfies

$$\exp|\xi_1| = \sqrt{(\omega'_0 - \omega_0)/\omega_0} = \sqrt{1 - n_0/n'_0} \approx \sqrt{A\Gamma(\delta^{-1} - \delta'^{-1})}, \tag{3.3}$$

which is the desired expression for the modulation strength. If the pump field is strong enough, so that the shift Ω_p^2/Δ is of order δ, then the single-cycle squeezing (3.3) is estimated to be $\sim (A\Gamma/\delta)^{1/2}$.

The switching-off time must satisfy

$$\omega_{12} \ll 2\pi/\tau \ll \Delta. \tag{3.4}$$

We can choose Rydberg-level transitions with parameters $\omega_{12} \geq 10$ GHz, $\Gamma \geq 10^5$ Hz, $\Gamma/\delta \approx 0.02$, and $A \approx 0.1$. The pump pulses satisfying (3.4) must then have a tail falling off within a few picoseconds, although the overall pulse duration may be much longer. The exponent in (2.30) that determines the squeezing strength becomes large for $|\xi_N| = N|\xi_1| = N(A\Gamma/\delta)^{1/2} \gg 1$, which requires in this case $N \gg 10^{3/2}$, N being the number of periodically recurring pump pulses. Such a value of N is consistent with the number of cycles, each lasting $\sim 10^{-9}$ sec, that can fit well within the off-resonant absorption lifetime of the gas (in order to avoid dissipation).

IV. COMPARISON WITH TWO-MODE PARAMETRIC COUPLING IN CAVITIES

We are now in the position to compare the merits of the present scheme, discussed in Secs. II and III, with those of the following existing schemes for the generation and squeezing of a signal at frequency ω_s using a pump at a frequency ω_p in a cavity that supports only these two modes:

(a) *Harmonic generation* [21,22]: In the process of nth harmonic upconversion $\omega_s = n\omega_p$, the signal field E_s is driven by the nonlinear polarization $P^{(n)} = \chi^{(n)}E_p^n$, where E_p is the pump field. Classically, E_s then evolves as a driven harmonic oscillator, whose amplitude scales with $\chi^{(n)}E_p^n$. This scaling is in sharp contrast with the growth of the signal field in our scheme, which is governed by the pump-dependent squeezing parameter ξ_1. As seen from Eqs. (2.26), (2.32), and (3.3), this parameter is *independent* of the ratio of the signal frequency ω_0 to the modulation cycle $2\pi/T$, nor is it related to the pump carrier frequency. This remarkable property reflects the abrupt termination of the cycle $U(T)$, which gives rise to a broad excitation spectrum, and yields effective upconversion of the modulation cycle even for high harmonic ratios, $n = \omega_0 T/2\pi \gg 1$.

(b) *Intensity-dependent refractive-index processes*, which have the following two origins:

(i) *Self-phase-modulation* [22,23]: This process, which is induced by the nonlinear refractive index $\chi^{(n)}|E_s|^{n-1}$ (for odd n) is commonly encountered in Kerr media, where $n = 3$. In such processes, the dynamics of E_s is determined by *anharmonic* oscillator equations. This entails a fundamental limitation on squeezing [23]: Anharmonic dispersion will transform an initially coherent state first into a squeezed state (self-squeezing), but subsequently into a "Schrödinger cat," i.e., superposition of coherent components with distinct mean phases that counter-rotate in the phase plane. A large self-squeezing parameter will thus result in rapid quenching of the squeezing. By contrast, in our scheme *harmonic-oscillator dispersion* is maintained (only the oscillation frequency is changed), whence no intrinsic limitation on squeezing exists in time.

(ii) *Pump-induced refractive-index processes governed by Hill's equation*: When the pump is much stronger than the signal, we can make the nondepleted pump approximation [21,24], whereby

$$E_p \approx A_p \cos \kappa_p z, \tag{4.1}$$

assuming that the cavity supports a standing-wave mode of the pump, with wave vector $\kappa_p = \sqrt{\epsilon_p}\omega_p/c$. Then, the nonlinear correction to the refractive index of the signal field is governed by Hill's equation for a periodically modulated oscillator [24], which has the following form for the most common case $n = 3$:

$$\frac{\partial^2 E_s}{\partial^2 z} + \kappa_s^2 \left[1 + \frac{\beta_3 A_p^2}{\kappa_s^2} \cos^2 \kappa_p z\right] E_s = 0, \tag{4.2}$$

where $\beta_3 \sim \chi^{(3)}$, and $\kappa_s = \sqrt{\epsilon_s}\omega_s/c$. The properties of Hill's equation [which in the case of (4.2) becomes Mathieu's equation] may, therefore, elucidate the onset of harmonic or subharmonic amplification in this system. These properties imply that parametric amplification occurs near the resonances (corresponding to *perfect phase matching*) $2\kappa_s/\kappa_p \approx m$, where $m = 1, 2, 3, \ldots$, only *within the instability regions* of the solutions. The width of the instability zone is proportional to $(\beta_3 A_p/\kappa_s^2)^{2m}$ [26] (see also [25]) and, therefore, becomes negligible for $m \gg 1$.

Comparison of the foregoing results with those obtained in Sec. II underscores once more the advantageous property of abrupt switching off of the parametric modulation, for which the instability region width is nearly independent of $m \approx \omega_0 T/\pi$. In Fig. 6 we show the corresponding instability regions of Eq. (4.2) by plotting the modulation amplitude $\beta A_p^2/\kappa_s^2$ versus the wave-vector ratio $2\kappa_s/\kappa_p$. These instability regions should be contrasted with those shown in Fig. 1 for the same values of the modulation strength $s-1$ and frequency ratio θ/π.

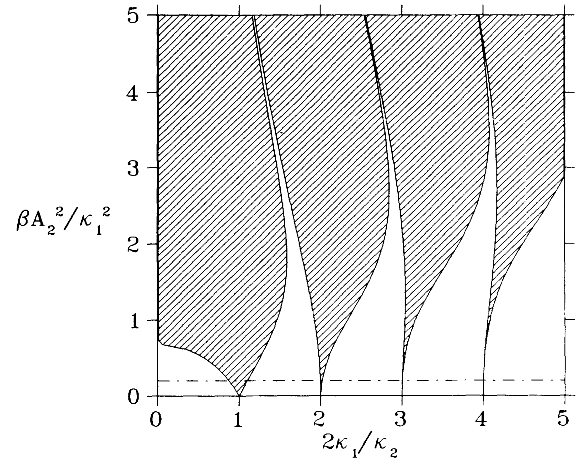

FIG. 6. Stability diagram for a signal field parametrically coupled to an intense pump field (Hill's equation modulation). Compare the width of shaded spaces (instability regions) with those in Fig. 1 for same values on the axes.

V. CONCLUSIONS

Our analysis of periodic frequency sweeping in harmonic oscillators (Sec. II) has elucidated the general principles underlying the occurrence of squeezing and parametric amplification in the various models that have been considered thus far [6,7,9,12–16]. The parametrically unstable solutions, which have been shown to be the key to the desired dynamical evolution, are the temporal counterparts of evanescent waves (band-gap solutions) in structures with spatially periodic refractivity.

The experimental realization of the present scheme in the microwave domain suggested in Sec. III has been shown to present rather modest requirements: Strong squeezing (more than 99%) requires a resonator with a Q value exceeding 10^6, filled with moderately dense gas that should be driven off resonance by a train of several hundred nearly-identical equally spaced optical pulses with picosecond switch-off times. Nevertheless, the existing detection efficiency would limit the observation of much stronger squeezing than in a parametric amplifier (50% squeezing at microwave frequencies [27]). By contrast with currently known pump-upconversion methods (Sec. IV) in our scheme the pump-dependent squeezing parameter and the modulation spectral bands (instability regions) are independent of the frequency ratio between the signal and the modulation.

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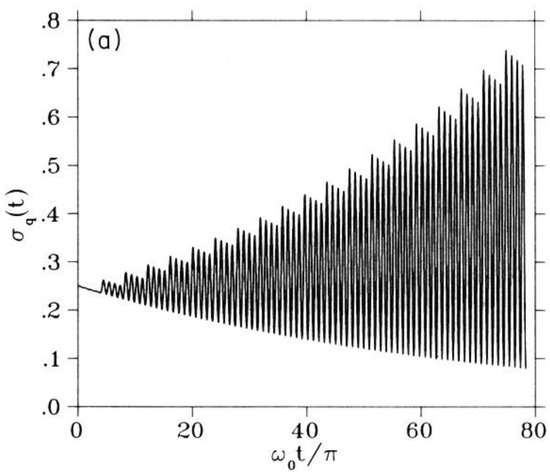

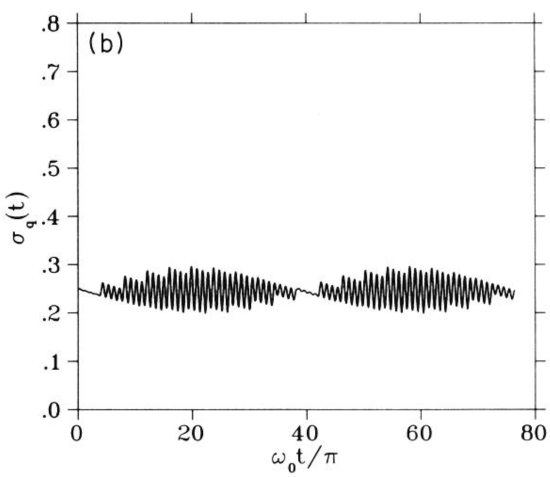

FIG. 4. The coordinate mean variance evolution for many cycles of modulation. The parameters are (a) $\gamma/\omega_0 = 0.01$, $\omega_0 T = 12.3$ and (b) $\gamma/\omega_0 = 0.01, \omega_0 T = 12$. Unlimited squeezing is obtained inside the instability region (a), whereas outside the instability region the envelope of σ_q undergoes oscillations (b).