Quantum-limited measurements with the atomic force microscope

G. J. Milburn
Department of Physics, University of Queensland, St. Lucia 4072, Australia

K. Jacobs and D. F. Walls
Department of Physics, University of Auckland, Auckland, New Zealand
(Received 27 April 1994)

We consider the quantum and classical noise limits to position measurement and force detection by an atomic force microscope (AFM) with an optical readout of cantilever position. We model this by treating the cantilever as a perfectly reflecting mirror for a highly damped optical cavity. There are three sources of noise: the shot noise in the output laser measurement, the thermal noise in the cantilever, and the measurement back-action noise. This last source of noise becomes large for good measurements, measurements for which there is a high correlation between the output phase of the light and the changing position of the cantilever. The back action simultaneously drives a diffusion process in momentum and diagonalizes the cantilever state in the position basis. This latter result is "state reduction." Explicit expressions for the rate of the measurement back action in terms of the device parameters are given. We also calculate the signal-to-noise ratio in the limit of a bad cavity. A comparison to recent experiments suggests that current AFMs are not quantum limited.

PACS number(s): 42.50.Lc, 06.30.Bp, 61.16.Ch, 87.64.Dz

I. INTRODUCTION

If there is a problem of measurement in quantum mechanics, it must surely be this: why is quantum measurement theory apparently so irrelevant to experimental physics? More than 60 years of unresolved debate on the interpretation of quantum mechanics and measurement has not in the least slowed the progress of experimental physics. Perhaps the answer to this question is, until very recently, no measurement has been limited by quantum noise. Thus it has simply not been necessary to consider in detail the action of a measuring device at the quantum level.

This is not to say that there are no experiments that require a quantum explanation. Quite the contrary is the case. However, in all cases the action of the measurement device itself can be described, just as Bohr claimed, as a classical device, limited perhaps by thermal or other classical sources of noise. Furthermore it is not the case that no progress has been made on the theoretical front. Indeed it is only possible to undertake an analysis of realistic quantum limited measurements because of the rapid progress in the field of quantum stochastic process over the past few years [1–4].

In the past decade measurements limited by quantum noise have become important. Perhaps the best example comes, rather surprisingly, from the effort to detect gravitational radiation. Gravitational radiation is very classical; however, for most sources gravitational radiation interacts so weakly with terrestrial detectors as to excite the system to only the first few levels above the ground state. If any detector is to work at all, this very small excitation must be measured in the background of many noise sources. It was realized by Braginsky [5] that, after all classical noise sources had been minimized or eliminated, ultimately the sensitivity of the device would still be insufficient, due solely to the irreducible quantum fluctuations in the ground state of the detector. Consideration of gravitational radiation detectors has forced us to confront the real problems of measurement in quantum mechanics. It was in this context that the concept of a quantum nondemolition (QND) measurement arose [6]. In recent years QND measurements have been carried out on the electromagnetic field [7–9]. In quantum optics measurements are necessarily limited by quantum noise as at optical frequencies thermal noise may be neglected compared to zero-point fluctuations in the electromagnetic field. Furthermore, optical nonlinearities are sufficiently large to generate significant quantum coherence effects, which do not appear at all in a classical treatment. Outside of quantum optics and gravitational radiation detection, it is very hard to find examples of truly quantum-limited measurements.

In this paper we analyze the measurement of displacement with the atomic force microscope [10,11]. The purpose is to show, in the context of a real measurement, how quantum limits appear and furthermore to show how quantum state reduction appears from the viewpoint of the experimentalist. Current atomic force microscopes operate nowhere near the quantum limit. It is not even clear that, for the measurement of atomic position, one would want to operate at the quantum limit. However, the analysis shows clearly how to modify current atomic force microscopes should it become necessary to make measurements so accurate that they are limited only by the uncertainty principle. A recent proposal [12] has suggested that the atomic force microscope (AFM) can be used to image systems of precessing spins. In addition there are many schemes that use the AFM to determine the magnetic properties of surfaces. A quantum-limited atomic force microscope, in this context, might be a realistic alternative to superconducting quantum interference device detection of small magnetic moments [13].

II. AFM MODEL

We will first consider an AFM with interferometric transduction of the cantilever position [14,15]. In this configuration the cantilever forms a perfectly reflecting mirror for one end of a low-Q cavity. The other cavity mirror could be formed by the cleaved end of an optical fiber placed close to the cantilever as in the experiment of Rugar *et al.* [14] or it could be formed by a beam splitter at which input light and back-reflected light are combined as in the experiment of Schoenenberger and Alvarado [15]. In the experiment of Schoenenberger *et al.* there is also a reference cavity which is formed by reflections of a second beam, derived from the same source as the detection beam, from a different position on the cantilever. This configuration permits homodyne detection to be done in the presence of phase fluctuations in the source laser. We will assume that the motion of the cantilever is simple harmonic. At the end of this section we consider the case in which there is no explicit reference to a cavity field.

The Hamiltonian describing the interaction between the cavity field and the cantilever position is identical to that used to describe the interferometric detection of gravitational radiation [16]. The free Hamiltonian is

$$H_0 = \hbar\omega_c a^\dagger a + \frac{\hat{p}^2}{2m} + \frac{m\omega_m^2}{2}\hat{q}^2 \ , \tag{1}$$

where m is the mass of the cantilever, ω_m is the resonant frequency of the cantilever, ω_c is the cavity field, resonance frequency, a is the annihilation operator for the cavity field, and $\hat{q}$ and $\hat{p}$ are the canonical position and momentum operators for the cantilever. The interaction between the cavity field and the cantilever is described by the interaction Hamiltonian [16]

$$H_I = \hbar\frac{\omega_c}{L}a^\dagger a\hat{q} \ , \tag{2}$$

where L is the cavity length. Thus the mirror moves in an effective linear potential with acceleration proportional to the intracavity photon number. It is convenient to define dimensionless canonical variables for the cantilever

$$Q = \left(\frac{2\hbar}{m\omega_m}\right)^{-\frac{1}{2}}\hat{q} \ , \tag{3}$$

$$P = \left(2\hbar m\omega_m\right)^{-\frac{1}{2}}\hat{p} \ . \tag{4}$$

The commutation relation is $[Q,P] = i/2$. This means that we are measuring position and momentum in units of the rms position and momentum fluctuations in the ground state of a harmonic oscillator. The interaction Hamiltonian may then be written as

$$H_I = \kappa a^\dagger a Q, \tag{5}$$

where

$$\kappa = \left(\frac{2\hbar\omega_c^2}{m\omega_m L^2}\right)^{\frac{1}{2}} \ . \tag{6}$$

Estimates of this coupling constant, for realistic AFMs, is given in Sec. VI.

Both the cavity field and the cantilever oscillator are damped and consequently subject to noise. The nature of this noise, however, is quite different for the two subsystems. For the field at optical frequencies, the noise is due entirely to quantum zero-point noise and the correct description of the damping is by the quantum-optics master equation. The cantilever, however, is dominated by thermal fluctuations and the appropriate description of the damping is provided by the Brownian motion master equation, in the high temperature limit [4]. The total dynamics of the system is then described by the master equation

$$\begin{aligned}\frac{dW}{dt} &= -i\omega_m[Q^2+P^2,W] - iE[a+a^\dagger,W] \\ &\quad -i[a^\dagger a(\Delta+\kappa Q),W] \\ &\quad +\frac{\gamma}{2}(2aWa^\dagger - a^\dagger aW - Wa^\dagger a) \\ &\quad -i\frac{\Gamma}{m}[Q,[P,W]_+] - 2\mathcal{N}_T[Q,[Q,W]],\end{aligned}$$

where W is the total system density operator, $[\,,\,]_+$ is an anticommutator, E is the amplitude (assumed real) of the coherent laser field driving the cavity, Δ is the detuning of the laser, γ is the cavity decay rate, Γ is the cantilever decay rate, ω_m is the cantilever frequency, and $\mathcal{N}_T$ is the momentum diffusion constant due to thermal noise

$$\mathcal{N}_T = \frac{\Gamma k_B T}{\hbar m\omega_m} \ . \tag{7}$$

Corresponding to this master equation are the quantum stochastic differential equations [4]

$$\frac{da}{dt} = E - i(\Delta+\kappa Q)a - \frac{\gamma}{2} + \sqrt{\gamma}a^{\text{in}} \ , \tag{8}$$

$$\frac{dQ}{dt} = \omega_m P \ , \tag{9}$$

$$\frac{dP}{dt} = -\omega_m Q - \frac{\Gamma}{m}P - \frac{\kappa}{2}a^\dagger a + \xi(t) \ , \tag{10}$$

where the noise operators for the cantilever are defined by

$$\langle\xi(t)\xi(t')\rangle = \frac{\Gamma kT}{m\hbar\omega_m}\delta(t-t'). \tag{11}$$

These equations may be linearized around the stationary steady state with

$$\Delta = \frac{2E^2\kappa^2}{\gamma^2\omega_m} \ , \tag{12}$$

$$\alpha_0 = \frac{2E}{\gamma} \ . \tag{13}$$

That is to say, we choose Δ so that the cavity is on resonance in the steady state. The linearized quantum stochastic differential equations are

$$\frac{d\delta x}{dt} = -\frac{\gamma}{2}\delta x + \sqrt{\gamma}x^{\rm in}, \tag{14}$$

$$\frac{d\delta y}{dt} = -\frac{\gamma}{2}\delta y - \mu\delta Q + \sqrt{\gamma}y^{\rm in}, \tag{15}$$

$$\frac{d\delta Q}{dt} = \omega_m \delta P \ , \tag{16}$$

$$\frac{d\delta P}{dt} = -\frac{\Gamma}{m}\delta P - \omega_m\delta Q - \mu\delta x + \xi(t) \ , \tag{17}$$

where we have defined the quadrature phase fluctuation operators

$$\delta x = \frac{1}{2}(a - \alpha_0 + \text{H.c.}), \tag{18}$$

$$\delta y = -\frac{i}{2}(a - \alpha_0 - \text{H.c.}) \tag{19}$$

and the coupling constant μ is defined by $\mu = \kappa\alpha_0$.

We now define the Fourier components of the fluctuation variables by

$$\delta y(t) = \int_0^\infty [\delta y(\omega)e^{-i\omega t} + \delta y(\omega)^\dagger e^{i\omega t}] \tag{20}$$

and similar expressions for the other quantities in the linearized equations. Note that as $\delta y(t)$ is Hermitian, then $\delta y(\omega) = \delta y(-\omega)^\dagger$. In frequency space the noise correlations are

$$\langle x^{\rm in}(\omega)x^{\rm in}(\omega')^\dagger\rangle = \delta(\omega - \omega') \ , \tag{21}$$

$$\langle y^{\rm in}(\omega)y^{\rm in}(\omega')^\dagger\rangle = \delta(\omega - \omega') \ , \tag{22}$$

$$\langle x^{\rm in}(\omega)y^{\rm in}(\omega')^\dagger\rangle = i\delta(\omega - \omega') \ . \tag{23}$$

In addition to these equations we have the boundary condition which relates the output fluctuation field from the cavity to the input field and intracavity field,

$$\delta y^{\rm out} = \sqrt{\gamma}\delta y - y^{\rm in}. \tag{24}$$

Measurements are made on the output field from the cavity in order to determine the change in position of the cantilever. It is clear from Eq. (15) that information on the changing position of the cantilever is contained on the out-of-phase quadrature of the field $\delta y(t)$. We thus expect that the corresponding output quadrature component will be correlated with the variable $\delta Q(t)$. To see this we compute the correlation function

$$C = \frac{\langle \delta q(\omega), \ \delta y^{\rm out}(\omega)\rangle_{\rm sym}}{\sqrt{\langle \delta q(\omega)^2\rangle\langle \delta y^{\rm out}(\omega)^2\rangle}}, \tag{25}$$

where

$$\langle AB\rangle_{\rm sym} = \frac{1}{2}\langle AB + BA\rangle - \langle A\rangle\langle B\rangle. \tag{26}$$

Solving the linearized equations in the frequency domain we find that

$$|C|^2 = \frac{D + 2\mathcal{N}_T\left(1 + \frac{4\omega^2}{\gamma^2}\right)}{D + 2\mathcal{N}_T\left(1 + \frac{4\omega^2}{\gamma^2}\right) + \frac{\omega_m^2}{4D}|\delta(\omega_m)|^2} \ , \tag{27}$$

where $\mathcal{N}_T$ is the thermal noise rate constant defined in Eq. (7).

$$D = \frac{2\mu^2}{\gamma} \tag{28}$$

$$= \frac{8\kappa^2E^2}{\gamma^3} \tag{29}$$

and

$$\delta(\omega) = \left(1 + \frac{2i\omega}{\gamma}\right)^2\left(1 - \frac{\omega^2}{\omega_m^2} + i\frac{\Gamma\omega}{m\omega^2}\right) \ . \tag{30}$$

As we show below, the rate constant D determines a diffusion in the momentum of the cantilever and thus represents the quantum back action of position measurement. The correlation function will become small for increasing ω due to the last term in the denominator of Eq. (27), so we consider the resonance case of $\omega = \omega_m$. In Fig. 1 we plot $|C|^2$ versus $\frac{\gamma}{\omega_m}$ for various values of D with $\mathcal{N}_T/D$ fixed. It is clear that the best correlation occurs for γ much larger than the resonant frequency of the cantilever. This is the "bad cavity limit." The field is so rapidly damped that the output field from the cavity responds almost instantly to the moving cantilever. In the bad cavity limit we find

$$C = \frac{1 + 2\frac{\mathcal{N}_T}{D}}{1 + 2\frac{\mathcal{N}_T}{D} + \frac{(\Gamma/m)^2}{4D^2}}. \tag{31}$$

Thus for a good measurement we need, in addition to the bad cavity limit, the condition $D \gg \Gamma/m$. This result is illustrated in Fig. 1. Realistic values (see Sec. VI) are $\gamma/\omega_m \approx 10^5$, which is clearly in the bad cavity limit, and $D = 0.5 \text{ s}^{-1}$ and $\Gamma/m = 0.4 \text{ s}^{-1}$, which just satisfy the good measurement criterion. It should be noted, however, that a measurement for which the correlation function is close to unity is not necessarily quantum limited. Indeed for realistic values $\mathcal{N}_T \approx 10^9 \text{ s}^{-1}$, so that we expect the signal to be limited by thermal noise, not back-action noise. This is discussed further in Sec. V.

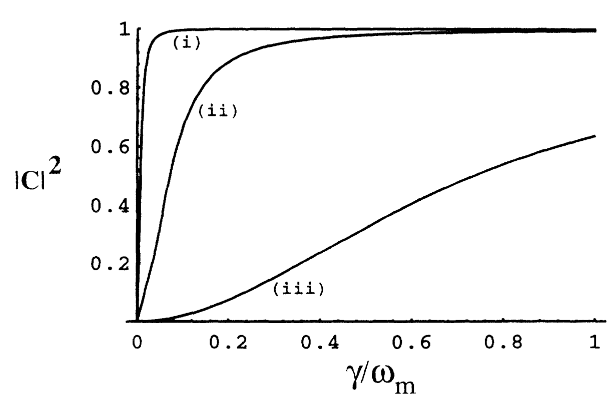

FIG. 1. The correlation coefficient between the output out-of-phase field quadrature amplitude and the cantilever position at the cantilever frequency ω_m versus the field damping rate γ in units of ω_m, $\Gamma/m = 0.001$, and $\mathcal{N}_T/D = 1$. (i) $D = 0.1$, (ii) $D = 0.01$, and (iii) $D = 0.001$.

III. BAD CAVITY LIMIT

In the preceding section we concluded that for a good position measurement we need the cavity to respond rapidly to the changing cantilever position, that is, we need $\gamma \gg \omega_m$. Indeed, as we show in Sec. VI, this is the limit in which current devices are operated. In this section we derive a master equation for the cantilever alone by adiabatically eliminating the cavity field. This is done as follows [17]. In the absence of the coupling to the cantilever, the field reaches a steady state with coherent amplitude α_0 given by

$$\alpha_0 = -\frac{2iE}{\gamma} . \tag{32}$$

We then transform the total state of the system in the steady state by

$$\tilde{W} = D^\dagger(\alpha_0) W D(\alpha_0) . \tag{33}$$

In this "displacement" picture the steady state of the field is close to the vacuum state and we can try an approximate solution of the form

$$\begin{aligned}\tilde{W} = &\rho_0|0\rangle_a\langle 0| + (\rho_1|1\rangle_a\langle 0| + \text{H.c}) \\ &+ \rho_2|1\rangle_a\langle 1| + (\rho_2{}'|2\rangle_a\langle 0| + \text{H.c.}).\end{aligned} \tag{34}$$

The cantilever density operator is then given by

$$\begin{aligned}\rho_M &= \text{tr}W \\ &= \rho_0 + \rho_2 .\end{aligned} \tag{35}$$

The resulting master equation for the cantilever alone is

$$\begin{aligned}\frac{d\rho}{dt} = &-i\omega_m[Q^2+P^2,\rho] - i\kappa|\alpha_0|^2[Q,\rho] + 2if(t)[Q,\rho] \\ &- i\frac{\Gamma}{m}[Q,[P,\rho]_+] - 2\left(\mathcal{N}_T + \frac{D}{2}\right)[Q,[Q,\rho]] ,\end{aligned} \tag{36}$$

where the quantum back-action noise D is given by Eq. (29) and where we have included a possible classical linear driving force $f(t)$. The quantum Langevin equations, corresponding to this master equation, are linear and will be analyzed in Sec. IV.

In the above we have included an explicit cavity field, albeit only to adiabatically eliminate it from the problem. However, we can consider the case in which a multimode traveling wave field is directly coupled to the cantilever. In this case the interaction Hamiltonian is [18]

$$H_I = 2\hbar k b_i^\dagger(t) b_i(t) q , \tag{37}$$

where $b_i(t)$ is the field amplitude operator defined so that $[b_i^\dagger(t'), b_i(t)] = \delta(t'-t)$ and k is the wave number at carrier frequency of the input field. In dimensionless position the interaction becomes

$$H_I = \hbar g b_i^\dagger(t) b_i(t) Q , \tag{38}$$

where

$$g = 2k\left(\frac{2\hbar}{m\omega_m}\right)^{1/2} . \tag{39}$$

To obtain a master equation for the cantilever alone we proceed as follows. Let the field operator be written as a sum of a coherent amplitude and a vacuum fluctuation term

$$b_i(t) = \Delta b_i(t) + \mathcal{E}_i(t) , \tag{40}$$

where $\mathcal{E}_i(t) = \mathcal{E}_i e^{-i\omega L t}$ and $|\mathcal{E}_i|^2$ is the photon flux of the input field. With these assumptions the contribution of this interaction to the master equation is as given in Eq. (36) with the replacements $\kappa|\alpha_0|^2 = g|\mathcal{E}_i|^2$ and

$$D = g^2|\mathcal{E}_i|^2. \tag{41}$$

IV. STATE REDUCTION

Before turning to the most important aspect of the problem, that of determining the frequency response of the signal-to-noise ratio, we first consider the perennial question of state reduction. If the AFM does indeed realize a measurement of position, a naive consideration would indicate that the cantilever state should be "reduced" in the position basis. At the density matrix level, where we do not specify a measurement result but only that the measurement is taking place, state reduction appears as an effective diagonalization of the density operator in some basis, here the position basis [19].

Such is indeed a consequence of Eq. (36). To see this we note that in the position representation the quantum back-action noise term contributes in the form

$$\frac{d\langle q|\rho|q'\rangle}{dt} = (\cdots) - D(q-q')^2\langle q|\rho|q'\rangle , \tag{42}$$

where $(\cdots)$ indicates the dynamics arising from all the other terms in Eq. (36). It is clear from Eq. (42) that the off-diagonal elements of the cantilever density operator, in the position basis, decay at a rate proportional to D and the square of the separation from the diagonal. This is the standard way in which state reduction becomes manifest for continuous measurement in a Markov regime. The parameter D, which determines the rate of diagonalization, also determines the diffusion constant for momentum. This is just the back action required by the uncertainty principle. This is seen by evaluating the contribution of the double commutator term to $\langle P^2\rangle$,

$$\frac{d\langle P^2\rangle}{dt} = (\cdots) + D . \tag{43}$$

Thus for very accurate measurements, D is large, diagonalization is rapid, and, in a complementary fashion, the diffusion in momentum is rapid. Thus D is responsible for diagonalization in position and diffusion in momentum.

The above discussion shows how the density matrix for the cantilever becomes diagonal on the long time scale associated with the adiabatic approximation. In the Appendix we consider the opposite time scale, the time scale in which the cantilever dynamics can be neglected. In this approximation one can obtain the total dynamics of the cantilever and cavity field.

The result in Eq. (A8) indicates that on the time scale $\gamma t << 1$, off-diagonal matrix elements of the cantilever, in the position basis, decay as t^4. Specifically we find

$$\frac{|\rho(q,q',t)|}{|\rho(q,q',0)|} = \exp\left(-\frac{E^2\kappa^2(q-q')^2t^4}{8}\right) . \tag{44}$$

However, on the adiabatic time scale $\gamma t \gg 1$, we find the expected Markov decay rate in Eq. (42).

V. OUTPUT SIGNAL

Of course what is actually measured is some property of the output field from the cavity, not the cantilever position directly. One expects that variations in the cantilever position will appear as phase shifts on the output field, which can then be determined by phase-dependent detection, for example, by homodyne detection. If one does direct photon counting on the output field, the resulting photon current, for unit quantum efficiency, is proportional to the process rate $i(t)$ given by

$$i(t) = \gamma \mathrm{tr}(a^\dagger a W) . \tag{45}$$

The conditional state of the intracavity field and cantilever, given a single count in an interval dt at time t, is given by [17] the count superoperator $\mathcal{J}$, where

$$\tilde{W} = \mathcal{J}W \tag{46}$$
$$= \gamma a W a^\dagger . \tag{47}$$

However, in keeping with the adiabatic approximation we do not wish to make explicit reference to the intracavity field. We are thus led to define the effective count superoperator $\mathcal{J}_M$ by

$$\mathcal{J}_M\rho = \mathrm{tr}(\mathcal{J}W), \tag{48}$$

where tr is a trace over field variables alone. To determine this superoperator explicitly, we need the approximate form of the steady state of the cantilever and field $\tilde{W}$ in the adiabatic approximation and the displacement picture. This is given by

$$\begin{aligned}\tilde{W} = &\left(\rho - \frac{\chi^2}{\gamma^2}Q\rho Q\right)|0\rangle_a\langle 0| \\ &-\left(i\frac{\chi}{\gamma}e^{i\theta}Q\rho|1\rangle_a\langle 0| + \mathrm{H.c.}\right) \\ &+\frac{\chi^2}{\gamma^2}Q\rho Q|1\rangle_a\langle 1| \\ &-\left(\frac{\chi^2}{\sqrt{2}\gamma^2}e^{2i\theta}Q^2\rho|2\rangle_a\langle 0| + \mathrm{H.c.}\right) ,\end{aligned} \tag{49}$$

where $\chi = 2\mu$. In the Schrödinger picture, the steady state is $D(\alpha_0)\tilde{W}D^\dagger(\alpha)$. Substituting Eq. (49) into Eq. (48) we find

$$\mathcal{J}_M\rho = c\rho c^\dagger , \tag{50}$$

where

$$c = \sqrt{\gamma}\left(|\alpha_0| - 2i\frac{\mu}{\gamma}Q\right) \tag{51}$$

with α_0 given by Eq. (32). In the adiabatic approximation terms of higher order in $\frac{\chi}{\gamma}$ have been neglected so Eq. (51) indicates that the position of the cantilever simply causes a phase change in the field reflected from the cavity.

In order to see such a phase change it is necessary to consider a phase-dependent detection scheme. Here we will consider homodyne detection. The output field from the cavity is mixed with a coherent local oscillator field at a beam splitter. The combined field is then subjected to direct photodetection. A full quantum theory of homodyne detection is given in Wiseman and Milburn [17]. The detected field operator may be written as

$$B = \sqrt{\gamma}(a+\beta) , \tag{52}$$

where $\sqrt{\gamma}\beta$ is the local oscillator amplitude in units such that the corresponding amplitude squared is in units of photon flux. In practice the local oscillator is derived from the same laser which is driving the cavity. In order to ensure good phase matching between the local oscillator and output field, it would be desirable to first pass the local oscillator through a cavity identical to that seen by the signal, but with the end mirror fixed in place. This is essentially the scheme of Schoenenberger and Alvarado [15].

The effective count superoperator for the cantilever is then given by

$$\mathcal{J}_M\rho = \gamma \mathrm{tr}[(a+\beta)\rho(a^\dagger + \beta^*)] . \tag{53}$$

If we choose $\beta = -i\alpha_0$, then

$$\mathcal{J}_M = 2\gamma|\alpha_0|^2\rho + \chi|\alpha_0|(Q\rho + \rho Q) , \tag{54}$$

where we have dropped terms of order $\frac{\chi}{\sqrt{\gamma}}$ and higher. The mean photocurrent is then

$$E(i(t)) = 2\gamma|\alpha_0|^2 + 2\chi|\alpha_0|\langle Q\rangle . \tag{55}$$

The mean signal $s(t)$ can then be defined as the photocurrent minus the dc component $2\gamma|\alpha_0|^2$, that is,

$$s(t) = 2\chi|\alpha_0|\langle Q\rangle . \tag{56}$$

The fluctuations in the homodyne current are given by [17]

$$E(i(t+\tau),i(t)) = 2\gamma|\alpha_0|^2[4D\langle Q(t+\tau),Q(t)\rangle_s + \delta(t)] , \tag{57}$$

where D is the same back-action noise parameter defined in Eq. (29) and the symmetrized correlation function $\langle\ ,\ \rangle_s$ is defined by Eq. (26). The last δ-function term is the shot-noise term. The signal spectrum is defined by

$$s(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} s(t) \ . \tag{58}$$

The noise spectrum is defined by

$$\begin{aligned} N(\omega) &= \int_{-\infty}^{\infty} d\tau E(i(t+\tau), i(t)) e^{i\omega\tau} \\ &= 2\gamma|\alpha_0|^2 \left(1 + 4D \int_{-\infty}^{\infty} d\tau e^{i\omega t} \langle Q(t+\tau), Q(t)\rangle_s \right) . \end{aligned} \tag{59}$$

If we define the positive and negative frequency components of $Q(t)$ by

$$Q(t) = \int_0^{\infty} d\omega [Q(\omega)e^{-i\omega t} + Q^{\dagger}(\omega)e^{i\omega t}] \ , \tag{60}$$

then the signal-to-noise ratio $R(\omega)$, defined by

$$R(\omega) = \frac{|s(\omega)|}{\sqrt{N(\omega)2T}}, \tag{61}$$

is given by

$$[R(\omega)]^2 = \frac{4D|\langle Q(\omega)\rangle|^2}{[1 + 4D\langle Q(\omega), Q(\omega)^{\dagger}\rangle_s]2T} \ , \tag{62}$$

where T is the measurement integration time defining the Fourier components and D is given by Eq. (29). It was shown in Secs. II and III that a good measurement should correspond to D large. This is clearly seen to be a good measurement from the point of view of the signal-to-noise ratio; if D is large the signal-to-noise ratio is determined only by the intrinsic signal-to-noise ratio in the cantilever position $Q(t)$.

To proceed we need the quantum Langevin equations corresponding to the master equation for the cantilever Eq. (36). These are

$$\frac{dQ}{dt} = \omega_m P \ , \tag{63}$$

$$\frac{dP}{dt} = -\omega_m Q - \frac{\Gamma}{m}P - \frac{\kappa}{2}|\alpha_0|^2 + f(t) + \xi(t) \ , \tag{64}$$

where the operator-valued white noise source $\xi(t)$ is defined by

$$\langle \xi(t)\xi(t')\rangle = \left(\mathcal{N}_T + \frac{D}{2}\right) \delta(t-t') \ . \tag{65}$$

The constant force proportional to $|\alpha_0|^2$ is just the radiation pressure force. We will assume that this may be removed by applying a constant force to the cantilever and hence neglect it in what follows. Taking the Fourier transform of these equations and ignoring initial transients (as we are only interested in the stationary fluctuations) we find

$$\langle Q(\omega)\rangle = \frac{\omega_m \tilde{f}(\omega)}{\omega_m^2 - \omega^2 - i\frac{\omega\Gamma}{m}} \ , \tag{66}$$

where $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$. The noise is given by

$$\langle Q(\omega)Q(\omega)^{\dagger}\rangle = \frac{\omega_m^2(\mathcal{N}_T + \frac{D}{2})}{|\omega_m^2 - \omega^2 - i\frac{\omega\Gamma}{m}|^2} \ . \tag{67}$$

For a dc force, $f(t) = f_0$ and $\tilde{f}(\omega) = f_0 T$, with T the measurement integration time. Then at $\omega = 0$,

$$[R(0)]^2 = \frac{8Df_0^2 T}{\omega_m^2 + 4D\left(\mathcal{N}_T + \frac{D}{2}\right)} \ . \tag{68}$$

Thus the minimum detectable force (signal-to-noise ratio is one) is determined by

$$f_{\min}^2 = \frac{1}{4T}\left(\frac{\omega_m^2}{2D} + 2\mathcal{N}_T + D\right) . \tag{69}$$

We can express this in terms of a minimum detectable displacement using

$$f_0 = (2\hbar m\omega_m)^{-1/2} K d_0 \ , \tag{70}$$

where K is the spring constant in newtons per meter and d_0 is the displacement in meters. Thus

$$d_{\min} = \frac{\hbar m\omega_m}{2K^2 T}\left(\frac{\omega_m^2}{2D} + 2\mathcal{N}_T + D\right) \ . \tag{71}$$

The term proportional to D^{-1} is the error due to shot noise on the output intensity. This may be made small by increasing the laser power, i.e., increasing D. The second term is the error due to the thermal fluctuations in the cantilever. The final term is the error due to the quantum back-action noise, which in physical terms arises from radiation pressure fluctuations. Clearly there is an optimal value of D, which is easily found to be

$$D_{\text{opt}} = \frac{\omega_m}{\sqrt{2}} \ . \tag{72}$$

At the optimal value for D, and in the quantum limit of $D >> \mathcal{N}_T$, the minimum detectable displacement squared is

$$d_{\min}^2 = \frac{\hbar}{\sqrt{2}m\omega_m^2 T} \ . \tag{73}$$

This is the "standard quantum limit" [19] for this problem.

VI. DISCUSSION AND CONCLUSION

Current AFMs do not operate anywhere near the quantum limit. However, it is instructive to estimate the quantities $\mathcal{N}_T$ and D for operational systems. We will consider two examples: the optical system of Schoenenberger and Alvarado [15] and the optical trap AFM of Ghislain and Webb [20].

Schoenenberger and Alvarado use a cantilever with an optical interferometric transducer. The cantilever forms a mirror and has a resonant frequency of around 4.5×10^3 s^{-1}. The Q of the cantilever is 200 and the spring constant is 20 Nm^{-1}. At room temperature this leads to a value for the thermal noise constant $\mathcal{N} = 4 \times 10^9$ s^{-1}. The wavelength of the light used in the detection scheme is 6.3×10^{-7} m^{-1} at a power of around 0.1 mW. This gives an intensity in photon flux $I = 3 \times 10^{16}$ s^{-1}. In this case D [Eq. (41)] is 0.5 s^{-1} , rather smaller than $\mathcal{N}_T$. Clearly this device is nowhere near the quantum limit, and it would seem unlikely that any realistic device could ever operate near the quantum limit due to the smallness of D compared to the thermal noise.

However, the optical trap scheme of Ghislain and Webb [20] seems more promising. In this scheme instead of using a cantilever as the force transducer, an optically trapped microparticle is used. This device has a very small spring constant (10^{-5} Nm^{-1}) and consequently can be used to detect much weaker forces. In the Ghislain and Webb proposal the microparticle is also suspended in water and thus the performance is limited by thermal fluctuations in the liquid. However, with this scheme there is nothing in principle to prevent operation in ultra high vacuum, provided sufficiently small particles and high power trapping lasers can be used. In that case the effective thermal noise is due only to laser intensity fluctuations and incoherent scattering of light by the microparticle. An analysis of these noise sources will be given in a subsequent paper. Using the figures quoted by Ghislain and Webb, we estimate that $\mathcal{N}_T = 2.5 \times 10^{12}$ s^{-1} and $D = 5 \times 10^5$ s^{-1}, which is clearly heading in the right direction.

In this paper we have analyzed in detail the noise limitations of atomic force microscope transducers including both classical and quantum back-action noise. We have shown how state reduction occurs when the device is fully quantum-limited and that the parameter that determines the rate of state reduction appears from an experimental point of view as the noise constant for quantum back-action noise. The analysis indicates how difficult it will be to make quantum limited AFMs unless a way can be found to overcome thermal noise. Perhaps feedback schemes might be useful in this context [21]. However, replacing a cantilever transducer with an optical trap transducer indicates that a quantum-limited AFM may be possible.

ACKNOWLEDGMENTS

G.J.M. would like to thank Howard Wiseman for useful discussions. We thank the Australian Research Council, the New Zealand Foundation of Research, Science and Technology, and the University of Auckland Research Committee for financial support.

APPENDIX: STATE REDUCTION

In this appendix we show how the monitoring of the cantilever position leads to a diagonalization of the cantilever density operator in the position basis. Two time scales are considered, a short non-Markov time scale and a longer, exponential decay, time scale.

Assume that the time scale for diagonalization is such that we can ignore the motion of the cantilever due to free evolution or damping. Thus the master equation that governs the field-cantilever dynamics is

$$\frac{d\rho}{dt} = -i\kappa[a^\dagger a Q, \rho] - iE[a + a^\dagger, \rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \; . \tag{A1}$$

Now define a field operator

$$\rho_F(q, q', t) = \langle q|\rho(t)|q'\rangle \; , \tag{A2}$$

which is the matrix element of the density operator in the cantilever position basis. Then

$$\frac{\partial \rho_f(q, q', t)}{\partial t} = -i\kappa x[a^\dagger a, \rho_F] - i\kappa y\{a^\dagger a, \rho_F\} -iE[a + a^\dagger, \rho_F] +\frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \; , \tag{A3}$$

where $x = (q + q')/2$ and $y = (q - q')/2$. We now use the positive-P representation [4] for the operator ρ_F,

$$\rho_F = \int P(\alpha, \beta)\Lambda(\alpha, \beta)d^2\alpha d^2\beta \; . \tag{A4}$$

Then

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(iE + \frac{\gamma}{2}\alpha + i\kappa x\alpha + i\kappa y\alpha\right)P + \frac{\partial}{\partial t}\left(-iE + \frac{\gamma}{2}\beta - i\kappa x\beta + i\kappa y\beta\right)P - 2i\kappa y\alpha\beta P \; . \tag{A5}$$

Note that when $y = 0$, β can equal α^* and we have a Fokker-Planck equation for the (unnormalized) density operator $\rho_F(q, q, t)$. If we integrate $P(\alpha, \beta)$ over α and β, we obtain the off-diagonal matrix element $\rho(q, q', t)$ for the cantilever at time t. Equation (A6) can be solved by the method of characteristics. If we assume that the field is initially in a vacuum state and the cantilever is in an arbitrary pure state $|\phi\rangle$, then we find

$$\rho(q, q', t) = \exp\left[-2i\kappa y\left(\frac{E^2 t}{[\frac{\gamma}{2} + i\kappa(x + y)][\frac{\gamma}{2} - i\kappa(x - y)]} - \frac{E^2(1 - e^{-[\gamma/2 + i\kappa(x+y)]t})}{[\frac{\gamma}{2} + i\kappa(x + y)]^2[\frac{\gamma}{2} - i\kappa(x - y)]} - \frac{E^2(1 - e^{-[\gamma/2 - i\kappa(x-y)]t})}{[\frac{\gamma}{2} + i\kappa(x + y)][\frac{\gamma}{2} - i\kappa(x - y)]^2} + \frac{E^2(1 - e^{-(\gamma + 2i\kappa y)t})}{[\frac{\gamma}{2} + i\kappa(x + y)][\frac{\gamma}{2} - i\kappa(x - y)](\gamma + 2i\kappa y)}\right)\right]\rho(q, q', 0) \; . \tag{A6}$$

We now consider two limits, the long time limit and the short time limit. The long time limit is $\gamma t \gg 1$ and

$$\frac{|\rho(q,q',t)|}{|\rho(q,q',0)|} = \exp\left(-\frac{2\kappa^2 E^2 y^2 \gamma t}{[\gamma^2/4+\kappa^2(x^2-y^2)]^2+\gamma^2\kappa^2 y^2}\right). \tag{A7}$$

In this case we have a simple exponential decay and thus identify this as the Markov limit for decay of coherence. Note that the rate of decay is proportional to $(q-q')^2$, as noted in Sec. III. This leads to a rapid diagonalization of the cantilever density operator in the position basis. It is easy to see that in the adiabatic limit of large γ the decay rate is $D(q-q')^2$, as expected, with D given by Eq. (29).

The short-time limit is $\gamma t << 1$. In this case we find

$$\frac{|\rho(q,q',t)|}{|\rho(q,q',0)|} = \exp\left(-\frac{E^2\kappa^2 y^2 t^4}{2}\right). \tag{A8}$$

Clearly this is a non-Markov decay on short time scales. However, it is still the case that the coherence decay rate is proportional to $(q-q')^2$. The reason why coherences initially decay at a rather slower rate is due to the need for free evolution to develop correlations between the field and the cantilever.

[1] A. Barchielli, Phys. Rev. A **34**, 1642 (1986).
[2] L. Diosi, Phys. Lett. A **129**, 31 (1988).
[3] V.P. Belavkin, J. Phys. A **22**, L1109 (1989).
[4] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[5] V.B. Braginsky, Zh. Eksp. Teor. Fiz. **53**, 1434 (1967) [Sov. Phys. JETP **26**, 831 (1968)].
[6] C.M. Caves, K.S. Thorne, R.W.P. Drever, V.D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[7] H.-A. Bachor, M.D. Levenson, D.F. Walls, S.H. Perlmutter, and R.M. Shelby, Phys. Rev. A **38**, 180 (1988).
[8] P. Grangier, J.F. Roch, and G. Roger, Phys. Rev. Lett. **66**, 1418 (1991).
[9] J.Ph. Poizat and P. Grangier, Phys. Rev. Lett. **70**, 271 (1993).
[10] G. Binnig, C.F. Quate, and C. Gerber, Phys. Rev. Lett. **56**, 930 (1986).
[11] D. Rugar and P. Hansma, Phys. Today **43** (10), 23 (1990).
[12] D. Rugar, C.S. Yannoni, and J.A. Sidles, Nature **360**, 563 (1992).
[13] D.D. Awschalom, J.R. Rozen, M.B. Ketchen, W.J. Gallagher, A.W. Kleinsasser, R.L. Sandstone, and B. Bumble, Appl. Phys. Lett. **53**, 2108 (1988).
[14] D. Rugar, H.J. Mamin, and P. Guethner, Appl. Phys. Lett. **55**, 2588 (1989).
[15] C. Schoenenberger and S.F. Alvarado, Rev. Sci. Instrum. **60**, 3131 (1989).
[16] A.F. Pace, M.J. Collett, and D.F. Walls, Phys. Rev. A **47**, 3173 (1993).
[17] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47**, 642 (1993).
[18] H.M. Wiseman, Phys. Rev. A **49**, 2133 (1994).
[19] D.F. Walls and G.J. Milburn, *Quantum Optics* (Springer, Heidelberg, 1994).
[20] L.P. Ghislain and W.W. Webb, Opt. Lett. **18**, 1678 (1993).
[21] J. Mertz, O. Marti, and J. Mlynek, Appl. Phys. Lett. **62**, 2344 (1993).