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Statistical and phase properties of the binomial states of the electromagnetic field

A. Vidiella-Barranco and J.A. Roversi
Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, 13083-970 Campinas, São Paulo, Brazil

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We investigate the nonclassical properties of the single-mode binomial states of the quantized electromagnetic field. We concentrate our analysis on the fact that the binomial states interpolate between the coherent states and the number states, depending on the values of the parameters involved. We discuss their statistical properties, such as squeezing (second and fourth order) and sub-Poissonian character. We show how the transition between those two fundamentally different states occurs, employing quasiprobability distributions in phase space, and we provide, at the same time, an interesting picture for the origin of second-order quadrature squeezing. We also discuss the phase properties of the binomial states using the Hermitian-phase-operator formalism.

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I. INTRODUCTION

The binomial states of the quantized electromagnetic field (denoted here as $|p,M\rangle$), consist simply of a linear combination of number states $|n\rangle$ weighed by a binomial distribution [1]. Some of their properties [1,2], methods of generation [1–3], as well as their interaction with atoms [4] have already been discussed in the literature. However, as far as we know, not enough attention has been paid to the fact that they actually interpolate between the "most classical" pure state allowed in quantum theory (the coherent state $|\alpha\rangle$) and the "less classical" one (the number state $|n\rangle$). In our opinion, this interesting feature places the binomial states in a unique situation, justifying by itself a more careful examination of their interpolation properties. This is going to be the main purpose of this paper.

The binomial states are such that there is a maximum of M photons present in the field, as well as a characteristic probability p of having each photon occurring. By continuously varying p (and also M), we are able to interpolate between the coherent state $|\alpha\rangle$ (α real), which is formed in the limit $p \to 0$ and $M \to \infty$ (keeping $pM = \alpha^2$ constant), and the number state $|M\rangle$, formed when $p = 1$. Therefore, we would like to present a detailed study of how some nonclassical properties of a field prepared in a binomial state change as a function of the probability p. The nonclassical properties we are going to focus on are their squeezing properties (second- and fourth-order squeezing), sub-Poissonian character, behavior of the quasiprobability distributions, as well as their phase properties. We also would like to emphasize mainly the pictorial usefulness of the quasiprobabilities distributions in phase space in the elucidation of nonclassical features of light, as it is generally possible to establish a link between the various nonclassical effects present in a quantized field and its corresponding phase-space representations.

This paper is organized as follows. In Sec. II we present a short review of the binomial states, discussing some of their nonclassical features. In Sec. III we discuss their nonclassical (and interpolation) properties in the light of the quasiprobability distributions in phase space (Q and Wigner functions) [5]. In Sec. IV we investigate the phase properties of binomial states, using the Pegg-Barnett Hermitian-phase-operator formalism [6,7]. We summarize our conclusions in Sec. V.

II. STATISTICAL PROPERTIES OF BINOMIAL STATES

A. Definition

The single-mode binomial states of the quantized electromagnetic field can be defined [1] as the following expansion in the number state basis $|n\rangle$:

$$|p,M\rangle = \sum_{n=0}^{M} B_n^M \, |n\rangle, \tag{1}$$

where

$$B_n^M = \left[\frac{M!}{n!\,(M-n)!}\, p^n (1-p)^{M-n}\right]^{1/2}. \tag{2}$$

This means that their photon number distribution $P_m^M = |\langle m|p,M\rangle|^2 = |B_m^M|^2$ is simply a binomial distribution, i.e., there is a probability P_m^M of occurrence of m photons (each one with probability p), having M independent ways of doing it. It is interesting to note that there is a maximum permissible number of photons, which is just M, in a field prepared in a binomial state, i.e., $P_m^M = 0$ for $m > M$.

From Eqs. (1) and (2) we can see that given any (finite) M, if $p = 0$, $|p,M\rangle$ is reduced to the vacuum state $|0\rangle$. On the other hand, if $p = 1$, we obtain the number state

$|n = M\rangle$. Moreover, in the limit $p \to 0$ and $M \to \infty$, but with $pM = \alpha^2$ constant (as we have already mentioned), the binomial distribution turns into a Poisson distribution and $|p, M\rangle$ becomes a coherent state $|\alpha\rangle$. However, the coherent states obtained are not the most general ones, but only those having a real amplitude $\alpha = |\alpha|$. It is interesting to see that by changing two parameters (p and M) in a single (binomial) state, we obtain fundamentally different states of the electromagnetic field. We would like to note that some of the points addressed in this section have already been discussed in Ref. [1], but because of their importance, it would be interesting to emphasize them in our context. Our strategy in most of this paper will be to keep M fixed and then study the modifications in the various properties of the binomial state as we change p. As a first example, we examine the overlap between the binomial states with the same maximum photon number M, but different probabilities p and q, i.e., $|p, M\rangle$ and $|q, M\rangle$. Their scalar product is given by

$$\langle p, M|q, M\rangle = \sum_{n=0}^{M} \frac{M!}{n!(M-n)!}\left[(pq)^{1/2}\right]^n \times \left[(1-p)^{1/2}(1-q)^{1/2}\right]^{M-n}. \quad (3)$$

If $p = q$, because of the properties of the binomial coefficients, we have that $\langle p, M|q, M\rangle = 1$, i.e., the states are normalized. For other values of p and q, the states will have a nonzero overlap (nonorthogonal), but if $p = 0$ and $q = 1$, for instance, $\langle 0, M|1, M\rangle = 0$, as we would expect.

B. Electric field and squeezing

In order to discuss the electric field and the squeezing properties of the single-mode binomial states, we need to know the action of powers of the annihilation operator on the states themselves:

$$\hat{a}^n|p, M\rangle = \begin{cases} \left[\frac{p^n M!}{(M-n)!}\right]^{1/2} |p, M-n\rangle, & n \le M \\ 0, & n > M \end{cases} \quad (4)$$

as well as the matrix elements

$$\langle p, M|p, M'\rangle = \sum_{n=0}^{M_<} B_n^M B_n^{M'}, \quad (5)$$

where $M_<$ is the smaller of M and M'. Both results can be found in Ref. [1].

For instance, we can immediately see that the mean photon number $\overline{n} = \langle \hat{a}^\dagger \hat{a}\rangle$ in a binomial state is simply given by

$$\overline{n} = pM. \quad (6)$$

This means that by varying the probability of emission of "individual" photons p from 0 to 1 and keeping M constant, the mean energy of the field is increased, as one would expect. We also have that the fluctuations in photon number are

$$\langle \Delta\hat{n}^2\rangle = p(1-p)M, \quad (7)$$

where there is a quadratic dependence of the fluctuations on p, with a maximum in M, and zeros in $p = 0$ (vacuum) and $p = 1$ (number state $|M\rangle$).

We are considering a single-mode electromagnetic field (frequency ω), inside an optical cavity of volume V. The quantized electric field can then be written as

$$\hat{E}(z) = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin(kz)\,(\hat{a} + \hat{a}^\dagger), \quad (8)$$

at $t = 0$, for instance.

First we would like to discuss the mean electric field in a binomial state. Using the expressions in (4), we obtain

$$\langle \hat{E}\rangle = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin(kz)\mathrm{E}(M, p), \quad (9)$$

where

$$\mathrm{E}(M, p) = 2(pM)^{1/2} \sum_{n=0}^{M-1} B_n^M B_n^{M-1}. \quad (10)$$

We notice that in the extreme values of $p = 0$ (vacuum state) and $p = 1$ (number state), the mean electric field is zero. However, it is nonzero for intermediate values of p having a maximum that depends on M. The larger M is, the closer the maximum of $p = 1$ will be. This feature can be better appreciated in Fig. 1, where we have plots of the "dimensionless" mean electric field $\mathrm{E}(M, p)$, in Eq. (10), as a function of p for different values of M. We can also see in (9) that in the (simultaneous) limit $p \to 0$, $M \to \infty$, with $pM = \alpha$ const, we obtain, for the mean electric field,

$$\langle \hat{E}\rangle = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin(kz)\, 2\alpha, \quad (11)$$

that is, a nonzero electric field characteristic of a coherent state with (real) amplitude α.

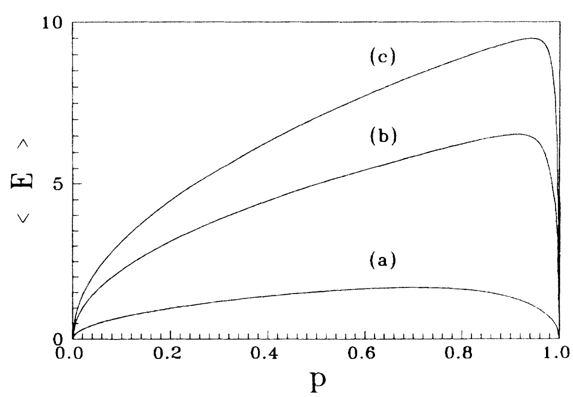

FIG. 1. Dimensionless mean electric field of a single-mode binomial state as a function of p for three different values of M: (a) $M = 5$, (b) $M = 50$, and (c) $M = 100$.

1. Second-order squeezing

We will now turn our attention to the quadrature operators of the field, related to the electric and magnetic field operators, but with the purpose of discussing the phenomenon of squeezing [8]. The quadrature operators, here denoted as $\hat{X}$ and $\hat{Y}$, are defined as

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2}, \qquad \hat{Y} = \frac{\hat{a} - \hat{a}^\dagger}{2i}. \tag{12}$$

The quadrature operators do not commute, i.e., $[\hat{X}, \hat{Y}] = i/2$, and as a consequence their variances obey the uncertainty relation

$$\langle \Delta \hat{X}^2 \rangle \langle \Delta \hat{Y}^2 \rangle \geq 1/16. \tag{13}$$

Second-order (or ordinary) squeezing occurs if any of the quadratures present a reduction in their second-order moments below the vacuum level, i.e., either $\langle \Delta \hat{X}^2 \rangle < 1/4$ or $\langle \Delta \hat{Y}^2 \rangle < 1/4$. It is convenient to define a second-order squeezing index as

$$S_x^{(2)} = \frac{\langle \Delta \hat{X}^2 \rangle - 1/4}{1/4}. \tag{14}$$

If $S_x < 0$ ($S_y < 0$), there will be (ordinary or second-order) squeezing in the $\hat{X}$ ($\hat{Y}$) quadrature.

For the binomial state, the squeezing indices are given by

$$S_x^{(2)} = 2pM + 2p[M(M-1)]^{1/2} \sum_{n=0}^{M-2} B_n^M B_n^{M-2} - 4pM \left(\sum_{n=0}^{M-1} B_n^M B_n^{M-1} \right)^2 \tag{15}$$

and

$$S_y^{(2)} = 2pM - 2p[M(M-1)]^{1/2} \sum_{n=0}^{M-2} B_n^M B_n^{M-2}. \tag{16}$$

In this case squeezing exists for the $\hat{X}$ quadrature and thus we produced a plot of $S_x^{(2)}$ as a function of p for different values of M, which can be appreciated in Fig. 2. We note that the binomial state is squeezed within a considerable range of values of p, with a maximum of squeezing (minimum of S_x) that depends on M. There is obviously no squeezing when $p = 0$ (vacuum) and when $p \to 1$ (number state). We recall that the binomial states are not minimum uncertainty states, except in the coherent state limit ($M \to \infty$), which also includes the vacuum state as a special case. The second-order squeezing properties will be discussed in more detail in Sec. III, using the quasiprobability distributions in phase space.

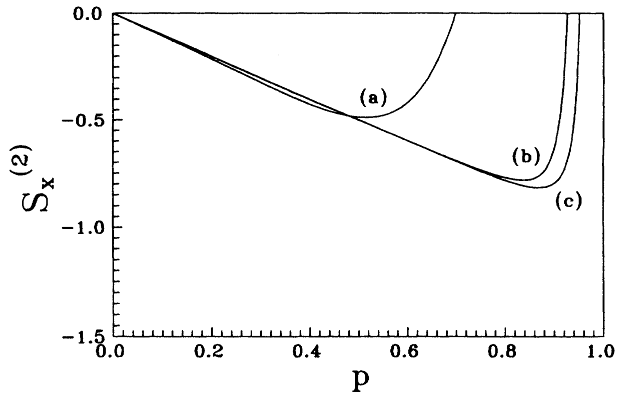

FIG. 2. Index of second-order squeezing $S_x^{(2)}$ of a binomial state as a function of p for three different values of M (as in Fig. 1).

2. Fourth-order squeezing

In order to study the noise properties of higher-order moments of the quadratures in nonclassical states of light the concept of higher-order squeezing has been introduced [9]. This corresponds to a generalization of the concept of squeezing as discussed above. A field is considered to be Nth-order squeezed if the Nth-order moment of the quadrature operator $\langle \Delta \hat{X}^N \rangle$ is smaller than its value in a completely coherent state of the field. The Nth moment of $\Delta \hat{X}$ can be written as [9]

$$\langle \Delta \hat{X}^N \rangle = \sum_{l=0}^{N/2-1} \frac{N^{(2l)}}{l!} \left(\frac{C}{2} \right)^l \langle : \Delta \hat{X}^N : \rangle + C^{N/2}(N-1)!!, \tag{17}$$

where for even N we use the notation $N^{(r)} = N(N-1) \cdots (N-r+1)$. Due to the fact that all the normally ordered moments $\langle : \Delta \hat{X}^N : \rangle$ vanish for a coherent state, the field is squeezed to order N if

$$\langle \Delta \hat{X}^N \rangle < C^{N/2}(N-1)!!, \tag{18}$$

which means that the squeezing condition can be written as

$$\sum_{l=0}^{N/2-1} \frac{N^{(2l)}}{l!} \left(\frac{C}{2} \right)^l \langle : \Delta \hat{X}^N : \rangle < 0. \tag{19}$$

Here we are going to consider only the fourth-order squeezing, i.e., the fourth-order moments of the quadrature operator $\hat{X}$, $\langle \Delta \hat{X}^4 \rangle$. In this case, the condition for the verification of fourth-order squeezing is just $\langle \Delta \hat{X}^4 \rangle < 3/16$ and the fourth-order moment of $\hat{X}$ is given by

$$\langle \Delta \hat{X}^4 \rangle = \langle \hat{X}^4 \rangle - 4 \langle \hat{X}^3 \rangle \langle \hat{X} \rangle + 6 \langle \hat{X}^2 \rangle \langle \hat{X} \rangle^2 - 3 \langle \hat{X} \rangle^4. \tag{20}$$

If we now use Eqs. (4), we obtain, for the various expectation values,

$$\langle \hat{X} \rangle = (pM)^{1/2} \langle p, M | p, M-1 \rangle, \tag{21}$$

$$\langle \hat{X}^2 \rangle = \frac{1}{4} \{ 1 + 2pM + 2p[M(M-1)]^{1/2} \langle p, M | p, M-2 \rangle \}, \tag{22}$$

$$\langle \hat{X}^3 \rangle = \frac{1}{8} \{ 6p^{3/2} M(M-1)^{1/2} \langle p, M-1 | p, M-2 \rangle + 6(pM)^{1/2} \langle p, M | p, M-1 \rangle + 2p^{3/2} [M(M-1)(M-2)]^{1/2} \langle p, M | p, M-3 \rangle \}, \tag{23}$$

$$\langle \hat{X}^4 \rangle = \frac{1}{16} \{ 3 + 12pM + 6p^2 M(M-1) + 12p[M(M-1)]^{1/2} \langle p, M | p, M-2 \rangle + 8p^2 M[(M-1)(M-2)]^{1/2} \langle p, M-1 | p, M-3 \rangle + 2p^2 [M(M-1)(M-2)(M-3)]^{1/2} \langle p, M | p, M-4 \rangle \}. \tag{24}$$

These results, used in conjunction with (5), yield a rather long expression for (20), and it would be again suitable to perform a numerical evaluation. We can also define, as we did for the second-order (ordinary) squeezing, an index of fourth-order squeezing $S_x^{(4)}$:

$$S_x^{(4)} = \frac{\langle \Delta \hat{X}^4 \rangle - 3/16}{3/16}. \tag{25}$$

In Fig. 3 we have a plot of $S_X^{(4)}$ as a function p for different values of M. We note that the fourth-order squeezing increases with p up to a maximum degree, which is considerably larger than the maximum of second-order squeezing (see Fig. 2), and then is progressively lost as $p \to 1$.

3. Mandel's Q parameter

Another important quantity to characterize nonclassical behavior is Mandel's Q parameter [10], which in its normalized form is defined as

$$Q = \frac{\langle \Delta \hat{n}^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}. \tag{26}$$

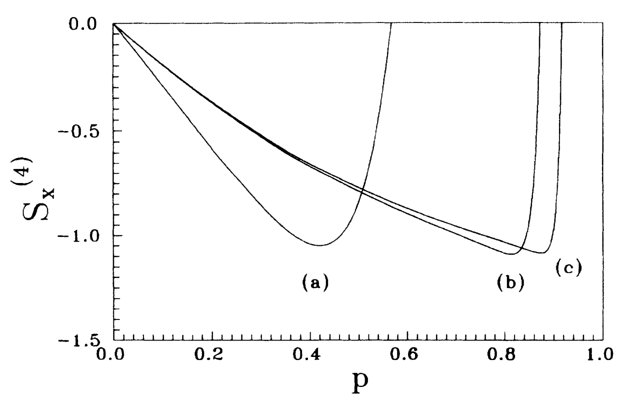

FIG. 3. Index of fourth-order squeezing $S_x^{(4)}$ of a binomial state as a function of p for three different values of M (as in Fig. 1).

If $Q = 0$, as in the case of a coherent state, the field is called Poissonian. If $Q > 0$ ($Q < 0$), it is called super(sub)-Poissonian, respectively. A signature of nonclassical behavior, for instance, would be a distribution narrower than a Poissonian, i.e., when $Q < 0$. In the case of the binomial states, Mandel's Q parameter is simply given by

$$Q = \frac{p(1-p)M - pM}{pM} = -p. \tag{27}$$

This means that binomial states are intrinsically sub-Poissonian, for any value of p, except in the coherent state limit (also including the vacuum state), and this fact does not depend on M. The extreme case is the number state, i.e., when $p = 1$.

III. QUASIPROBABILITIY DISTRIBUTIONS

A. Definitions

The formulation of quantum mechanics in phase space [5] has been particularly useful in quantum optics. The so-called quasiprobability distributions in the coherent state basis have been providing not only an alternative way of calculating quantum mechanical expectation values (e.g., correlation functions), but also useful pictures and explanations of nonclassical features of light fields [11–13].

If a field is prepared in a quantum state described by $\hat{\rho}$, we can define an infinite number of (s-parametrized) quasiprobability distributions in phase space as [14]

$$P(\beta; s) = \frac{1}{\pi^2} \int d^2\xi \, C(\xi; s) \, \exp(\beta \xi^* - \beta^* \xi), \tag{28}$$

where the quantum characteristic function is

$$C(\xi; s) = \mathrm{Tr}[\hat{D}(\xi)\hat{\rho}] \exp(s|\xi|^2/2). \tag{29}$$

Here $\beta = x + iy$, with (x, y) being the c numbers corresponding to the quadratures $(\hat{X}, \hat{Y})$, respectively, and

$\hat{D}(\xi) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is Glauber's displacement operator. For particular values of s we obtain the well-known distributions, e.g., for $s = 0$ we have the Wigner function, and for $s = -1$ the (Husimi) Q function. The latter is positive definite at any point of the phase space for any quantum state. However, not all Q functions can be associated with a density operator, among other problems. The former, on the other hand, may assume negative values for some states and this is generally regarded as a signature of nonclassical effects. Therefore, both of them cannot be considered true probability distributions and hence the name quasiprobabilities. These two functions are the most convenient for our purposes. However, instead of the phase-space integration method [see Eq. (28)], we would like to use the series representation of quasiprobabilities [15]. It is possible to write any s-parametrized quasiprobability distribution as an infinite series [15]

$$P(\beta; s) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{(1+s)^k}{(1-s)^{k+1}}\langle\beta,k|\hat{\rho}|\beta,k\rangle, \qquad (30)$$

where $|\beta, k\rangle = \hat{D}(\beta)|k\rangle$ are the so-called displaced number states. The expression above is suitable for straightforward numerical evaluation, if one wants to avoid a phase-space integration, which sometimes can be quite tedious. If we take $s = -1$, for instance, Eq. (30) is reduced to the familiar expression for the Q function

$$Q(\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}|\beta\rangle. \qquad (31)$$

B. The Q and Wigner functions of a binomial state

If we now insert $\hat{\rho} = |p, M\rangle\langle p, M|$ [see Eq. (1)] into Eq. (31), we obtain the Q function of the binomial state, which can be written as

$$Q(\beta) = \frac{\exp(-|\beta|^2])}{\pi} \times \left|\sum_{n=0}^{M}\left[\frac{M!}{(M-n)!}p^n(1-p)^{M-n}\right]^{1/2}\frac{\beta^n}{n!}\right|^2. \qquad (32)$$

We also want an expression for the Wigner function of a binomial state. By taking $s = 0$ in Eq. (30), we obtain a series representation for the Wigner function

$$W(\beta) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\langle\beta,k|\hat{\rho}|\beta,k\rangle. \qquad (33)$$

Now we simply insert Eq. (1) into (33), which yields

$$W(\beta) = \frac{2}{\pi}\sum_{k=0}^{\infty}\left|\sum_{n=0}^{M}\left[\frac{M!}{(M-n)!}p^n(1-p)^{M-n}\right]^{1/2}\chi_{nk}(\beta)\right|^2. \qquad (34)$$

In the expression above, the matrix elements $\chi_{nk}(\beta) = \langle n|\hat{D}|k\rangle$ are given by [13]:

$$\chi_{nk}(\beta) = \begin{cases} \sqrt{\frac{k!}{n!}}\exp(-|\beta|^2/2)\beta^{n-k}\mathcal{L}_k^{n-k}(|\beta|^2) & \text{if } n \geq k \\ \sqrt{\frac{n!}{k!}}\exp(-|\beta|^2/2)(\beta^*)^{k-n}\mathcal{L}_n^{k-n}(|\beta|^2) & \text{if } n \leq k, \end{cases} \qquad (35)$$

where $\mathcal{L}_n^\alpha(|\beta|^2)$ are the generalized Laguerre polynomials. Both the Q function in Eq. (32) and the Wigner function in Eq. (34) are in an appropriate form for numerical evaluation.

Now we would like to discuss changes occurring in the quasiprobabilities as we vary the "parameters" p and M. We are going to use the Q function as a first way of illustrating the transition from the vacuum state $|0\rangle$ ($p = 0$) to the number state $|M\rangle$ ($p = 1$), for instance. In Fig. 4 there is a plot of the Q function of a binomial state for different values of p, with $M = 5$. When $p = 0$, we note the well-known Gaussian characteristic of the vacuum state [Fig. 4(a)]. By increasing the probability of having individual photons to $p = 0.5$, the state acquires a positive "coherent" amplitude, which corresponds to a gain of energy in the field, as can be seen in Fig. 4(b). By further increasing the probability to $p = 0.9$, we clearly see a deformation in the Q function toward $\beta = 0$ [Fig. 4(c)], and finally, when $p = 1$, the Q function representing the number state $|M = 5\rangle$ is formed [Fig. 4(d)]. We can now understand the "shaping" of a coherent state, under the phase-space point of view, that occurs in the limit $p \to 0$ and $M \to \infty$, with pM constant. For instance, by taking, just as an approximation, the (finite) values of $M = 1000$ and $p = 0.01$, we have a Q function, shown in Fig. 5, that resembles the Q function of a coherent state with coherent amplitude $\alpha = \sqrt{pM} = \sqrt{10}$, as expected.

Another interesting aspect that also deserves examination is the quadrature squeezing. In phase space, squeezing manifests itself as an actual deformation of the quasiprobabilties. We verified that the contours are progressively "compressed" as the distribution is shifted (p increased), until a maximum of squeezing is reached. This can be considered a turning point from which a number state starts being formed and squeezing is increasingly lost. In Fig. 6 we have a plot of the contours of the Q function of two particular binomial states, with $p = 0.2$ and $p = 0.82$ (both having $M = 50$). For $p = 0.2$, the contours are not very different from those of the vacuum state, although they show a small deformation. However, when $p = 0.82$, we clearly notice a "compression" of them along the x direction, which corresponds to squeezing in the $\hat{X}$ quadrature.

The Wigner function also can be used to trace the nonclassical behavior of light fields, in a similar way as we have already done using the Q function. In Fig. 7 we have plots of the Wigner function of a binomial state [from numerical evaluation of Eq. (34)] for different values of p and $M = 1$. The case $p = 0$ has been excluded because the corresponding Wigner function is just a Gaussian centered in the origin, as in Fig. 4(a). In Fig. 7(a) we have the Wigner function for $p = 0.1$. A small displacement

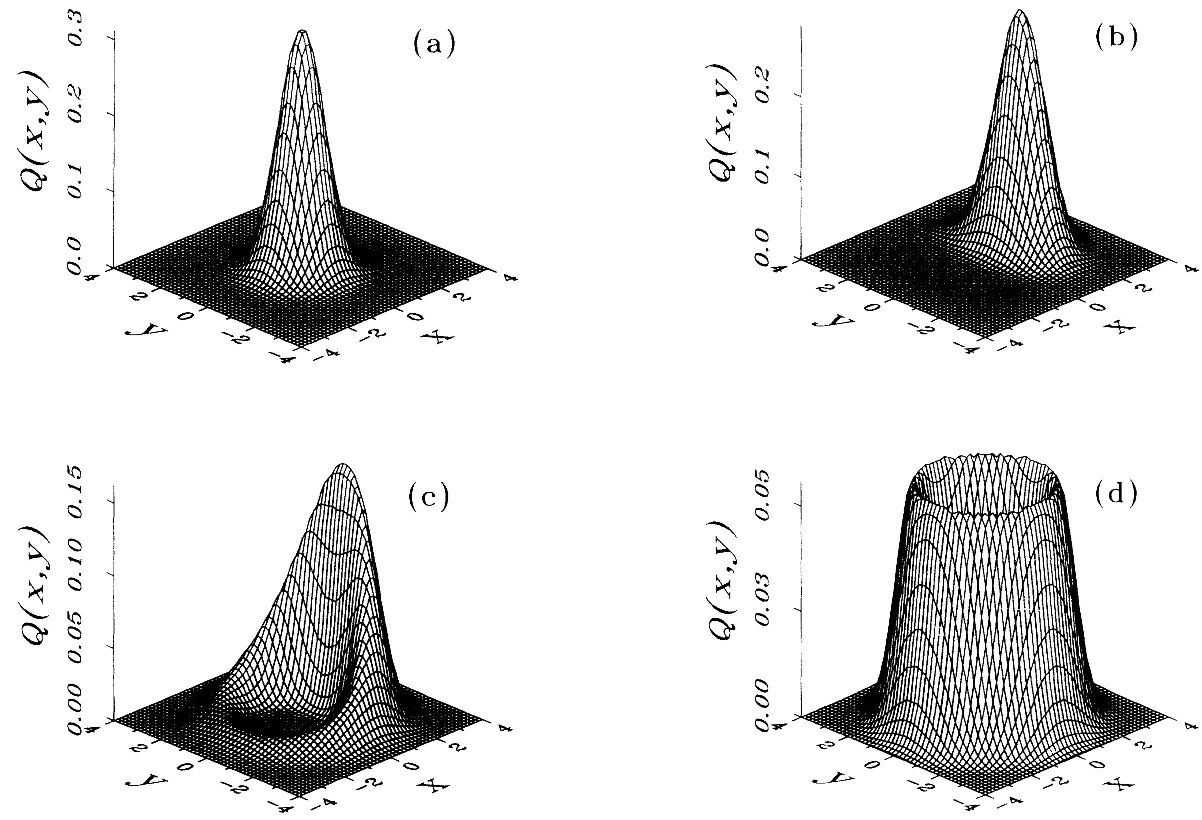

FIG. 4. Q function of a binomial state for four different values of p: (a) $p = 0$, (b) $p = 0.5$, (c) $p = 0.9$, and (d) $p = 1$. In all cases $M = 5$.

of the distribution is already noticeable, but it does not show any negativity yet. In Fig. 7(b), for $p = 0.5$, we note that the negative part of the distribution is already pronounced. For $p = 0.9$ [Fig. 7(c)], the negative part is even larger, and finally, in Fig. 7(d), we have the full Wigner function of a number state $|M = 1\rangle$ ($p = 1$). The negativity of the Wigner function is a sufficient but not necessary condition for having nonclassical effects. Just as a matter of speculation, we could draw a parallel between the negative growth of the Wigner function and the linear increase of the sub-Poissonian character of the binomial states [see Eq. (27)]. The Wigner function becomes more and more negative almost linearly as p is increased.

We would like to point out that because the coefficients in the expansion (1) are real, there is a natural definition of a privileged direction in phase space, along the real (x) axis. Also because we chose the positive square root of pM in the calculations, a positive amplitude is gained by the field if we increase p.

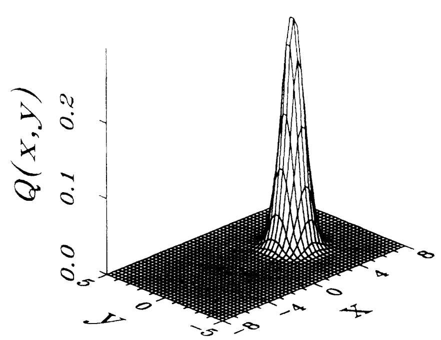

FIG. 5. Q function of a binomial state with $M = 1000$ and $p = 0.01$.

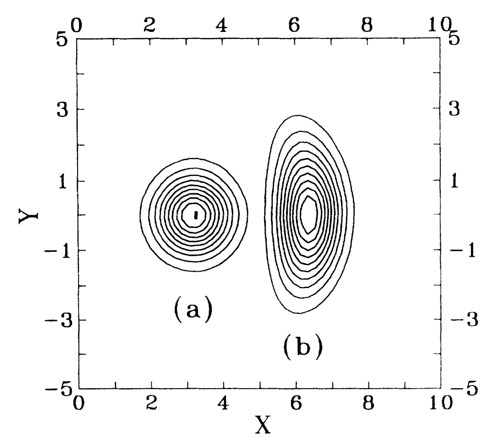

FIG. 6. Contours of the Q function of two binomial states with (a) $p = 0.2$ and (b) $p = 0.82$. In both cases $M = 50$.

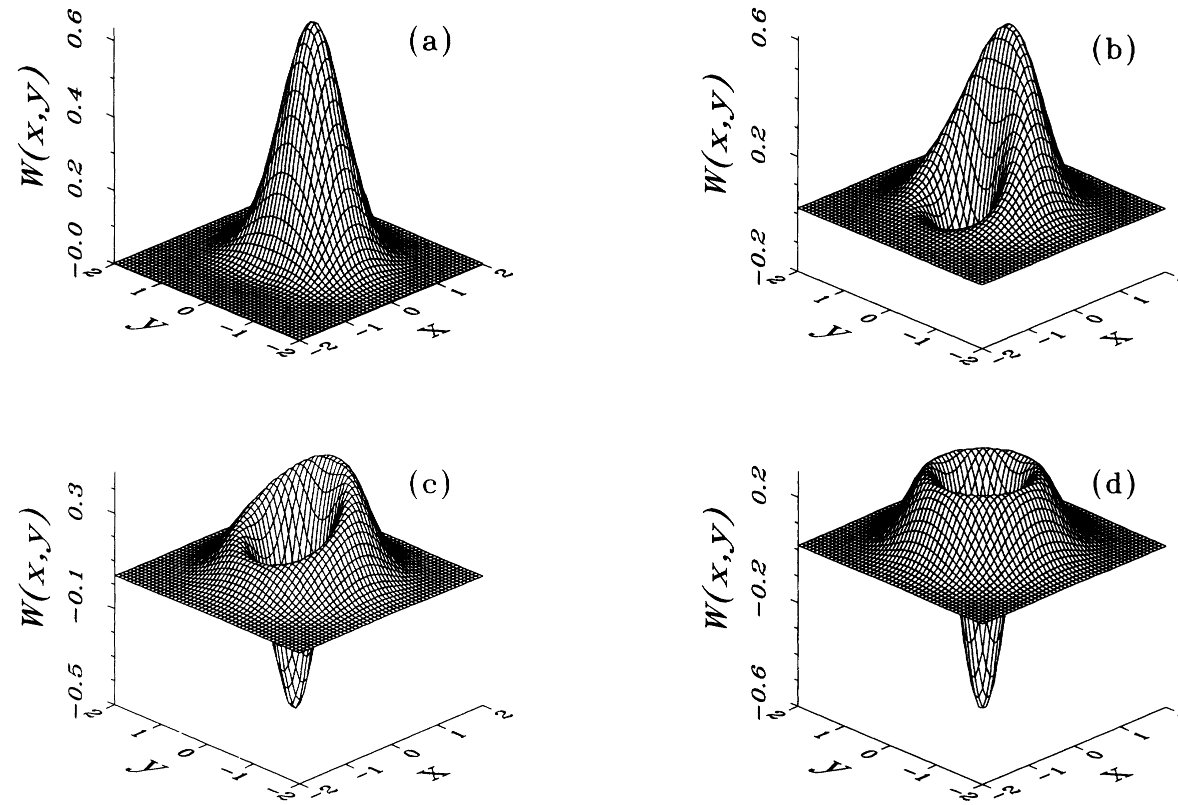

FIG. 7. Wigner function of a binomial state for different values of p: (a) $p = 0.1$, (b) $p = 0.5$, (c) $p = 0.9$, and (d) $p = 1$. In all cases $M = 1$.

IV. PHASE PROPERTIES

A. Hermitian-phase-operator formalism

In order to complete the characterization of the binomial states, now we would like to discuss their phase properties based on the Hermitian-phase-operator formalism [6,7]. The formalism is based on the construction of states of well-defined phase, or phase states $|\theta\rangle$, which can be defined as

$$|\theta_m\rangle = \frac{1}{(s+1)^{1/2}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle. \tag{36}$$

The phase states in Eq. (36) form an orthonormal set provided that we have

$$\theta_m = \theta_0 + 2\pi m/(s+1), \qquad m = 0, 1, ..., s, \tag{37}$$

where θ_0 is an arbitrary reference phase. It is important to note that after the calculations are performed, the limit $s \to \infty$ has to be taken in order to be consistent with the quantum mechanical formalism, in which an infinite state space is used. A Hermitian phase operator can be constructed using the phase states themselves:

$$\hat{\Phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{38}$$

It is also possible to define a continuous phase probability distribution for a general field state described by $\hat{\rho}$ as

$$dP(\theta) = \langle\theta_m|\hat{\rho}|\theta_m\rangle d\theta. \tag{39}$$

It is not difficult then to obtain expressions [6] for the moments of the phase operator, as well as for the phase distribution, when the state is in the form

$$|b\rangle = \sum_{n=0}^{s} b_n \exp(in\mu)\,|n\rangle, \tag{40}$$

where b_n is real and positive. These states are called partial phase states; their mean phase is just

$$\langle b|\hat{\Phi}_\theta|b\rangle = \mu. \tag{41}$$

The phase variance is given by

$$\langle\Delta\hat{\Phi}_\theta^2\rangle = \frac{\pi^2}{3} + \sum_{n>n'} b_n b_{n'} (-1)^{(n-n')}(n-n')^{-2} \tag{42}$$

and the phase probability distribution can be written as

$$P(\theta) = \frac{1}{2\pi}\left\{1 + 2\sum_{n>n'} b_n b_{n'} \cos[(n-n')\theta]\right\}, \quad (43)$$

after taking the limit $s \to \infty$.

B. Phase properties of binomial states

The binomial states in Eq. (1) are in fact partial phase states, provided that we have $b_n = B_n^M$ if $n \leq M$, $b_n = 0$ if $n > M$, and $\mu = 0$. So we immediately have, from (41), that the mean phase in a binomial state is always zero, independently of p and M. If we insert the coefficients from Eq. (2) into (42), we have the variance of the phase operator in a binomial state. The upper limit in the sum is M, because the B_n's are zero for $n > M$. In Fig. 8 we have the variance as a function of p, in the case of having $M = 1$. We see that for both $p = 0$ and $p = 1$ the variance assumes the value of $\pi^2/3$, which is the characteristic value of the variance for a state with randomly distributed phase [7]. The variance decreases, having a minimum at $p = 0.5$, and then there is a (symmetrical) increase up to $p = 1$. This can be compared to the behavior of the mean photon number variance in Eq. (7). As we would expect from the uncertainty relation phase photon number, an increase in the dispersion of one implies in a decrease in the dispersion of the other. For larger values of M, however, a sharper decrease of the variance is verified. This means, as we would expect, that by increasing the maximum number of photons in the field (M), its phase becomes better defined for almost any p, excluding, naturally, the cases $p = 0$ and $p = 1$.

We also would like to show what happens with the phase probability distribution as we vary p. Therefore a three-dimensional plot is required, where we can simultaneously appreciate the modifications in the distribution as a function of both p and θ. This is shown in Fig. 9, after a numerical evaluation of Eq. (43), and using the value $M = 1$. We notice that for both the vacuum case $p = 0$ and the number state case $p = 1$, the phase is uniformly distributed, with probability equal to $1/2\pi$. For intermediate values of p, however, a peak, centered at $\theta = 0$ starts being shaped, meaning that the phase becomes better defined around a mean value. This is in complete agreement with the phase-space pictures al-

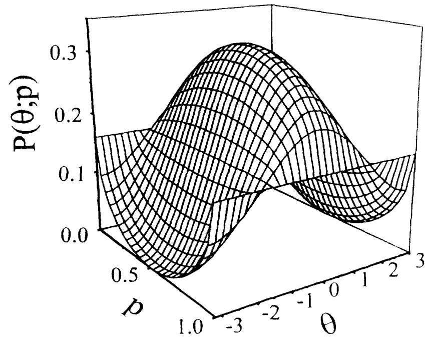

FIG. 9. Phase probability distribution of a binomial state as a function of p and θ for $M = 1$.

ready discussed in Sec. II (see, for instance, Fig. 4). As the Q function is shifted from the origin (as p goes from 0 to 1), there is a better definition of phase, up to a minimum value for its dispersion (variance). Then, as the number state starts being "formed," there is again a loss in that sharpness, bringing up, eventually, a situation of randomly distributed phase.

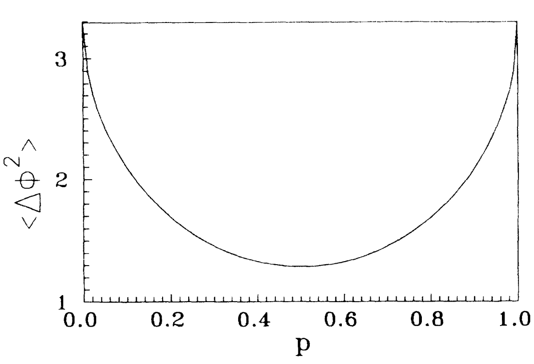

FIG. 8. Variance of the phase operator of a binomial state as a function of p for $M = 1$.

V. CONCLUSIONS

We have studied the nonclassical statistical properties of the binomial states of the electromagnetic field. One of their more striking features is that they can be actually continuously "tuned" from the vacuum state $|0\rangle$ to the number state $|M\rangle$ by changing the probability of "emission" of individual photons (p) from $p = 0$ to $p = 1$, respectively, in a maximum total number of independent photon emissions M. We also have that in the limit $p \to 0$ and $p \to \infty$, but keeping pM constant, the binomial state becomes a coherent state $|\alpha = \sqrt{pM}\rangle$. These interpolation properties have represented the core of our analysis. We have shown that different nonclassical properties are sensitive in a different way to variations in p. For instance, quadrature squeezing (second and fourth order) is relatively intense in the binomial states, there existing a maximum of squeezing that depends on M (see Figs. 2 and 3). On the other hand, the sub-Poissonian degree, characterized by Mandel's Q parameter, decreases monotonically with p, being, at the same time, independent of M (the parameter is simply given by $Q = -p$).

A very clear way of verifying the changes in the field as we vary p and M, as we have shown in Sec. III, is through the quasiprobability distributions in phase space. The increase in the mean energy of the field, as p increases, corresponds first to a shift in the distribution (see, for instance, the changes in the Q function in Fig. 4). At the same time we have a growth in the quadrature squeezing, which can be interpreted as an actual compression of the distribution. This effect is actually connected to the progressive elongation of the distribution necessary to construct a ringlike structure corresponding to a number state. The transition towards a coherent state in suitable limits is also transparent under the phase-space point of view (see Fig. 5). We also established a link between the quasiprobability representations and the actual phase properties of the field. In other words, it is possible to obtain a qualitative picture of the phase properties of the field using the quasiprobabilities and this is in agreement with a more precise treatment using the Hermitian-phase-operator formalism. These examples are additional evidence that the quasiprobability distributions are valuable tools for understanding subtle aspects of nonclassical behavior in light fields.

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