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Topological quenching of the tunnel splitting for a particle in a double-well potential on a planar loop

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The motion of a particle along a one-dimensional closed curve in a plane is considered. The only restriction on the shape of the loop is that it must be invariant under a twofold rotation about an axis perpendicular to the plane of motion. Along the curve a symmetric double-well potential is present leading to a twofold degeneracy of the classical ground state. In quantum mechanics, this degeneracy is lifted: the energies of the ground state and the first excited state are separated from each other by a slight difference ΔE, the tunnel splitting. Although a magnetic field perpendicular to the plane of the loop does not influence the classical motion of the charged particle, the quantum-mechanical separation of levels turns out to be a function of its strength B. The dependence of ΔE on the field B is *oscillatory*: for specific discrete values B_n the splitting drops to zero, indicating a twofold degeneracy of the ground state. This result is obtained within the path-integral formulation of quantum mechanics; in particular, the semiclassical instanton method is used. The origin of the quenched splitting is intuitively obvious: it is due to the fact that the configuration space of the system is not simply connected, thus allowing for destructive interference of quantum-mechanical amplitudes. From an abstract point of view this phenomenon can be traced back to the existence of a *topological* term in the Lagrangian and a nonsimply connected configuration space. In principle, it should be possible to observe the splitting in appropriately fabricated mesoscopic rings consisting of normally conducting metal.

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I. INTRODUCTION

The nonlocal character of quantum mechanics manifests itself in the phenomenon of tunneling. A quantum-mechanical particle prepared in one of the minima of a symmetric double-well potential on the real line is influenced by the existence of the second minimum, in spite of the fact that from a classical point of view its energy might not be sufficient to negotiate the intervening potential barrier. The instanton method [1], based on specific *classical* paths in the *inverted* potential, is used successfully to evaluate path integrals [2]: one obtains the quantum-mechanical splitting ΔE of the lowest-lying energy levels in the semiclassical limit.

In this work the instanton method is applied to a system which, although similar to the particle in a double well, is different from a topological point of view. Imagine a particle moving on a one-dimensional loop [3,4] in the xy plane, under the influence of a symmetric double-well potential $V(\varphi)$. The total system is assumed to be invariant under twofold rotations about the z axis. Since two inequivalent classical paths exist which connect the minima, one expects modifications of the tunneling phenomenon compared to the double well on the line (cf. [5]). In addition, a constant magnetic field B pointing along the z axis will be included. This field does not have any influence on the motion of the classical particle since the resulting Lorentz force is always perpendicular to the loop. It will be shown that, nevertheless, the splitting of the ground-state energies of the quantum-mechanical system depends on the strength of the field B; for specific values of B the splitting even drops to zero.

From a general point of view the model studied in this paper is interesting for the following reasons. First of all, the system provides a natural realization of a quantum-mechanical particle moving on a Riemannian manifold which is defined by the loop. Second, it represents an example of a quantum system defined on a configuration space which is multiply connected [6]. Third, such a configuration space allows for "topological terms" in the action being irrelevant in a classical description, but leading to observable effects in the corresponding quantum theory. In the context of topological field theory such effects are known to be caused by Chern-Simons or Wess-Zumino terms [7].

In Sec. II the model is defined and the determination of the splitting ΔE via the instanton method is sketched. Section III contains the semiclassical evaluation of the relevant propagators for the particle on a *circle* without magnetic field in a first step and with nonzero field in a second one. Then, in Sec. IV the modifications necessary for the treatment of *arbitrary* loops are presented. Section V provides a brief summary; the structural similarity of results obtained for quenched tunnel splitting in spin systems is pointed out and related work on transmission properties of mesoscopic rings is discussed. Finally, the topological aspects of the model are briefly restated in general terms.

II. MODEL

The Lagrangian

$$L^A(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}m\dot{\mathbf{q}}^2 - V(\mathbf{q}) + \frac{e}{c}\mathbf{A}(\mathbf{q})\cdot\dot{\mathbf{q}}\,, \qquad \mathbf{q}=(x,y) \tag{1}$$

describes a charged particle under the influence of a potential $V(\mathbf{q})$ moving in the xy plane in the presence of a uniform magnetic field $\mathbf{B}$ pointing along the z axis: $\mathbf{B}(\mathbf{q}) = B\mathbf{e}_z = \nabla\times\mathbf{A}(\mathbf{q})$. The vector potential $\mathbf{A}(\mathbf{q})$ has constant magnitude on circles about the z axis and is tangent to them, $\mathbf{A}(\mathbf{q}) = \alpha(r)\mathbf{e}_\varphi$, with r and φ being polar coordinates in the xy plane. For simplicity, the particle is from now on constrained to move on a circle about the origin, defined by $\partial\Gamma$: $G(x,y) = x^2+y^2-R^2 = 0$; general loops will be studied in Sec. IV. The Lagrangian $L^A(\mathbf{q},\dot{\mathbf{q}})$ becomes

$$L^A(\varphi,\dot\varphi) = \frac{1}{2}mR^2\dot\varphi^2 - V(\varphi) + A(R)\dot\varphi\,, \tag{2}$$

with the constant $A(R) = eR\alpha(R)/c$. The value of the action functional $S^A[\varphi(t)]$ depends on the path connecting $\varphi(t_1)$ and $\varphi(t_2)$

$$S^A[\varphi(t)] = \int_{t_1}^{t_2} L^A(\varphi,\dot\varphi)\,dt\,. \tag{3}$$

It is important to note that the third term of (2), as a total derivative

$$\sigma[\varphi(t)] \equiv \int_{t_1}^{t_2} A(R)\dot\varphi\,dt = A(R)\{\varphi(t_2)-\varphi(t_1)\}\,, \tag{4}$$

does not contribute to the classical equations of motion. This property is easily understood in physical terms by observing that the Lorentz force due to the magnetic field $\mathbf{B}$ is always perpendicular to the ring and therefore does not influence the classical motion. Nevertheless, due to the multiply connected configuration space, the presence of the gauge term $A(R)\dot\varphi$ will lead to observable consequences in a quantum-mechanical setting.

The potential $V(\varphi)$ on the ring is assumed to be invariant under a rotation by π about the z axis, i.e.,

$$V(\varphi+\pi) = V(\varphi),\;\; V(-\varphi) = V(\varphi)\,, \tag{5}$$

as shown in Fig. 1. Qualitatively, the results will be seen to depend only on the symmetry (5) of the potential, not on its actual shape. The classical ground states of the system are given by the particle resting at one of the minima φ_+ or φ_-, as follows immediately from the equations of motion

$$mR^2\frac{d^2\varphi}{dt^2} = -\frac{dV}{d\varphi}\,. \tag{6}$$

The energy $E = mR^2\dot\varphi^2/2 + V(\varphi)$ is non-negative if one requires $V(\varphi_\pm) = 0$.

The goal of this investigation is to calculate the separation ΔE of the two lowest energy levels of the quantum system. To this end appropriate propagators will be evaluated semiclassically within the path-integral formulation of quantum mechanics. In order to establish notation, the basic ingredients of the so-called instanton method are now reviewed briefly. For simplicity, the Lagrangian (2) with vanishing magnetic field is considered first.

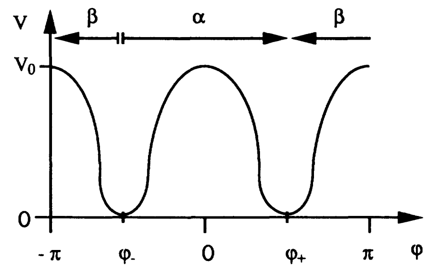

FIG. 1. The symmetric double well on a loop. The instantons α and β, respectively, belong to two inequivalent paths connecting the maxima $\varphi_\pm$ of the inverted potential $-V(\varphi)$.

The quantum-mechanical amplitude for a particle to reach the position eigenstate $|\,\varphi_b\rangle \equiv |\,\varphi(t_b)\rangle$ after the time interval $T \equiv t_b - t_a$ when starting from $|\,\varphi_b\rangle$ at time t_a is governed by the propagator

$$\langle\varphi_b\,|\,e^{-i\widehat{H}T/\hbar}\,|\,\varphi_a\rangle \equiv K(\varphi_b,t_b;\varphi_a,t_a) = \int_{\varphi_a}^{\varphi_b}\mathcal{D}\varphi\,\exp\left(\frac{i}{\hbar}S[\varphi(t)]\right)\,, \tag{7}$$

where $\widehat{H}$ is the Hamiltonian operator of the system and the right-hand side denotes a formal sum over all paths in configuration space connecting the points φ_a and φ_b in time T. The weight of each path $\varphi(t)$ depends on its action $S[\varphi(t)]$. It is convenient to analytically continue the propagator to complex times $\tau = it$. In the resulting *Euclidean* propagator (cf. [8])

$$K_e(\varphi(\tau_b),\tau_b;\varphi(\tau_a),\tau_a) = \sum_{n=0}^{\infty}\phi_n^*(\varphi_b)\phi_n(\varphi_a)\exp(-E_nT/\hbar)\,, \tag{8}$$

all but the contributions from the lowest states will be suppressed exponentially for large T. The functions $\phi_n(\varphi)$ are the quantum-mechanical eigenstates of the Hamiltonian with eigenvalues E_n.

Under the substitution $t\to -i\tau$ the path integral in Eq. (7) takes on its Euclidean form

$$K_e(\varphi_b,\tau_b;\varphi_a,\tau_a) = \int_{\varphi(\tau_a)}^{\varphi(\tau_b)}\mathcal{D}\varphi\,\exp\left(-\frac{1}{\hbar}S_e[\varphi(\tau)]\right)\,, \tag{9}$$

$S_e[\varphi(\tau)]$ standing for the Euclidean action

$$S_e[\varphi(\tau)] = \int_{\tau_a}^{\tau_b} d\tau\left\{\frac{mR^2}{2}\left(\frac{d\varphi}{d\tau}\right)^2 - W(\varphi)\right\}\,; \tag{10}$$

therefore, the action S_e is naturally associated with a particle moving in the *inverted* potential $W(\varphi) \equiv$

$-V(\varphi)$. The associated Euclidean energy $E_e = -mR^2(d\varphi/d\tau)^2/2 - W(\varphi)$ no longer has a definite sign, but it is still a conserved quantity as follows from the Euclidean equation of motion

$$mR^2\frac{d^2\varphi}{d\tau^2} = -\frac{dW}{d\varphi} \equiv \frac{dV}{d\varphi}\,. \tag{11}$$

An exact calculation of the Euclidean propagator K_e usually is not possible; its semiclassical approximation, however, is obtained by taking into account only those contributions to the path integral (9) which come from the stationary points of the action functional $S_e[\varphi(t)]$ and their neighborhoods, i.e., paths which solve Eq. (11) and paths fluctuating about these solutions. Choosing the minima $\varphi_\pm$ of the potential $V(\varphi)$ as initial and final points φ_a and φ_b, respectively, one finds that K_e is dominated by zero-energy solutions of (11) given by

$$\frac{d\varphi}{d\tau} = \pm\sqrt{2V(\varphi)/mR^2}\,. \tag{12}$$

In the limit of $T \to \infty$ such a path (connecting φ_a with φ_b) is known as an *instanton*; if the ficticious particle travels in the opposite sense the solution is called an *anti-instanton*. As mentioned earlier one is interested in the propagator K_e for large T, cf. Eq. (8). The temporal width of one single instanton is finite [8]; in other words, the particle is located most of the time in the neighborhoods near the maxima $\varphi_\pm$. It turns out that in order to obtain an asymptotically correct expression for the propagator, not only single instantons but *strings* of arbitrarily many instantons and anti-instantons have to be taken into account: in classical terms this situation corresponds to the particle going back and forth any number of times between the maxima of the potential $W(\varphi)$. Apart from an error which is exponentially small in time T [9], such strings are approximate solutions of the equations of motion. Furthermore, it is assumed that the centers of the instantons on the τ axis, τ_n, are widely separated: contributions of overlapping instantons ($\tau_n \sim \tau_{n+1}$) can be neglected consistently (cf. [8]); this assumption is also known as the dilute-gas approximation.

Due to the difference in topology of the double-well potential on the ring and on the real line, the sets of paths connecting φ_+ and φ_- are different; the tunnel splitting ΔE is expected to be sensitive to this difference. In the following section, the four propagators

$$K_e(\varphi_\pm, T/2; \varphi_\pm, -T/2)\,, \quad K_e(\varphi_\pm, T/2; \varphi_\mp, -T/2) \tag{13}$$

with external magnetic field $\mathbf{B}$ will be calculated in the limit of large T, the knowledge of which is sufficient for a determination of the separation ΔE by comparison with Eq. (8). The calculations actually will closely parallel work done by Felsager [8], and for various technical details the reader is urged to consult this reference.

III. CALCULATION OF THE TUNNEL SPLITTING

The calculation of the tunnel splitting ΔE as a function of the magnetic field B is divided into two parts. First, the field B is assumed to be zero, the focus being on the enumeration of all possible paths connecting the minima at $\varphi_\pm$. In contrast to the double-well potential on the line, the number of paths with prescribed length increases *exponentially*, not linearly. In a second step, the field term (4) is taken into account, leading to a dependence of the splitting on the field: $\Delta E = \Delta E(B)$.

A. Vanishing magnetic field

Imagine the quantum-mechanical particle to be located at the maximum φ_- of the potential $W(\varphi)$ at time $-T/2$. The (Euclidean) amplitude to find the particle at position φ_+ after time T is given by $K_e(\varphi_+, T/2; \varphi_-, -T/2)$. Approximate evaluation of the path integral in Eq. (9) proceeds as follows. The main contribution comes from the two single instantons denoted by α and β (cf. Fig. 1): the first one visits the minimum at $\varphi = 0$ of the potential $W(\varphi)$ before reaching φ_+ and the second one travels in the opposite direction passing through the point $\varphi = -\pi$ before reaching the maximum at φ_+. Since an instanton effectively needs some finite time only to travel from a point near φ_- to a point near φ_+, for long times T other contributions arising from more complicated paths have to be included, as mentioned before: the fictitious particle may "oscillate" any number of times between the maxima, the only proviso being that it starts at φ_- and finally comes to rest at φ_+. Anti-instantons traveling from φ_+ to φ_- are denoted by $\overline{\alpha}$ and $\overline{\beta}$, respectively. Therefore each path consists of an alternating sequence of instantons and anti-instantons with the positions of their centers on the τ axis given by

$$-T/2 < \tau_1 < \tau_2 < \cdots < \tau_N < T/2\,, \tag{14}$$

with an odd integer N, which will be referred to as the length of the string.

In Fig. 2 a graphic scheme is given to enumerate all possible paths of a given length N. There are two paths $(\alpha;\beta)$ with $N = 1$ connecting φ_- with φ_+, corresponding to instantons taking the upper or the lower branch of the ring when traveling to the other minimum. For $N = 3$ there are eight possible paths: symbolically all strings are given by

$$(\alpha\overline{\alpha}\alpha; \alpha\overline{\alpha}\beta; \alpha\overline{\beta}\alpha; \alpha\overline{\beta}\beta; \beta\overline{\alpha}\alpha; \beta\overline{\alpha}\beta; \beta\overline{\beta}\alpha; \beta\overline{\beta}\beta)\,, \tag{15}$$

and similarly for larger values of N. For a given N, there

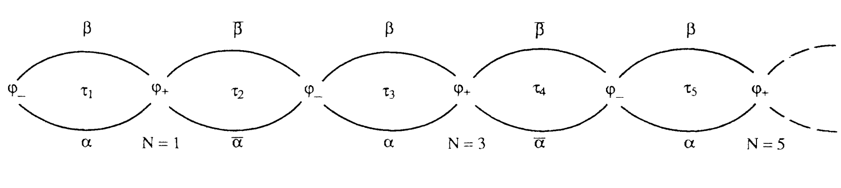

FIG. 2. Graphical enumeration of instanton strings of length N. Each sequence of symbols α (or β) and $\overline{\alpha}$ (or $\overline{\beta}$) corresponds to a possible string.

are 2^N paths: whenever the particle is at one of the maxima $\varphi_\pm$ of the potential $W(\varphi)$ there are *two* ways to reach the other maximum. A multi-instanton path of length N with centers at $\vec{\tau} \equiv (\tau_1, \tau_2, \dots, \tau_N)$ will be denoted by $\widetilde{\varphi}^N_{\vec{\tau}}(\tau)$. For the approximate evaluation of $K_e(\varphi_+, T/2; \varphi_-, -T/2)$ one has to know the behavior of the paths $\varphi(\tau)$ in the neighborhood of the quasistationary paths $\widetilde{\varphi}^N_{\vec{\tau}}(\tau)$ following from the expansion

$$\varphi(\tau) \simeq \widetilde{\varphi}^N_{\vec{\tau}}(\tau) + \eta(\tau), \tag{16}$$

with $\eta(\tau)$ vanishing at the end points $\tau = \pm T/2$. Expanding the potential about the quasistationary paths one obtains

$$W[\varphi(\tau)] \simeq W(\widetilde{\varphi}^N_{\vec{\tau}}) + \frac{1}{2}\frac{d^2W}{d\varphi^2}(\widetilde{\varphi}^N_{\vec{\tau}})\eta^2(\tau), \tag{17}$$

where the dependence of $\widetilde{\varphi}^N_{\vec{\tau}}$ on τ has been suppressed. The contribution of a path $\widetilde{\varphi}^N_{\vec{\tau}}$ and its neighborhood is given by

$$\exp\left\{-S_e(\widetilde{\varphi}^N_{\vec{\tau}})/\hbar\right\} \int_{\eta(-T/2)=0}^{\eta(T/2)=0} \mathcal{D}\eta \, \exp\left\{-\widetilde{S}_e[\eta(\tau)]/\hbar\right\}, \tag{18}$$

with

$$\widetilde{S}_e[\eta(\tau)] = \int_{-T/2}^{T/2} d\tau \left\{\frac{mR^2}{2}\left(\frac{d\eta}{d\tau}\right)^2 - \frac{1}{2}W''(\widetilde{\varphi}^N_{\vec{\tau}})\,\eta^2\right\} \tag{19}$$

representing the action of the deviations from the multi-instanton path.

An exponentially small error only is made if one approximates the action S_E of a multi-instanton path in (18) by

$$S_e(\widetilde{\varphi}^N_{\vec{\tau}}) \sim N S_e^0, \tag{20}$$

S_e^0 being the action of a single instanton or anti-instanton; since the Lagrangian $L(\varphi, \dot{\varphi})$ is invariant under the transformation $\varphi \to -\varphi$, instantons and anti-instantons have the same action. Summing all contributions of the quasistationary paths and integrating over all possible locations $\vec{\tau}$ of instanton centers one finds

$$K_e(\varphi_+, T/2; \varphi_-, -T/2)$$
$$\sim \sum_{\text{odd } N} e^{-NS_e^0/\hbar} \int_{-T/2}^{T/2} \cdots \int_{-T/2}^{\tau_3}\int_{-T/2}^{\tau_2} d\tau_N \cdots d\tau_2 d\tau_1 \times \widetilde{K}_N(0, T/2; 0, -T/2), \tag{21}$$

with $\widetilde{K}_N(0, T/2; 0, -T/2)$ denoting the path integral over the fluctuations $\eta(\tau)$ in Eq. (18). Introduce the quantity

$$\Delta = \frac{\widetilde{K}_N(0, T/2; 0, -T/2)}{K_\omega(0, T/2; 0, -T/2)} \tag{22}$$

as the ratio of the N-instanton propagator $\widetilde{K}_N$ to the Euclidean propagator K_ω of a harmonic oscillator with frequency ω starting and ending at zero:

$$K_\omega(0, T/2; 0, -T/2) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh \omega T}} \sim \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \quad (T \to \infty), \tag{23}$$

the frequency ω being determined by the quadratic approximation of the potential $V(\varphi)$ at the minima $\varphi_\pm$. It is known that in the limit of large T the quantity Δ depends neither on T nor on the position of the center of the instanton; also, the quantity Δ can be evaluated explicitly, giving the relevant contribution to the so-called prefactor of the final expression for the splitting ΔE. Up to this point the calculation is essentially equivalent to that of a double-well potential on the line [8]. The expression for $\widetilde{K}_N(0, T/2; 0, -T/2)$, however, depends on the topological properties of the ring-shaped configuration space. Using $(N-1)$ times the general property of composition for propagators

$$K(\varphi'', \tau''; \varphi, \tau) = \int d\varphi' K(\varphi'', \tau''; \varphi', \tau') K(\varphi', \tau'; \varphi, \tau), \tag{24}$$

one can write

$$\widetilde{K}_N(0, T/2; 0, -T/2) = \sum_{2^N \text{ paths}} \int \cdots \int d\varphi_{N-1} d\varphi_{N-2} \cdots d\varphi_1 \times \widetilde{K}(0, T/2, \varphi_{N-1}, T_{N-1}) \widetilde{K}(\varphi_{N-1}, T_{N-1}, \varphi_{N-2}, T_{N-2}) \cdots \widetilde{K}(\varphi_1, T_1; 0, -T/2). \tag{25}$$

The propagator $\widetilde{K}_N$ is made up of 2^N contributions from the different paths consisting of N instantons and anti-instantons. In Eq. (25) each of these paths is decomposed into a product of N single (anti-)instanton contributions $\widetilde{K}$. Using Eq. (22) in the form

$$\widetilde{K}(\varphi_n, T_n; \varphi_{n-1}, T_{n-1}) \sim \Delta\, K_\omega(\varphi_n, T_n; \varphi_{n-1}, T_{n-1}) \tag{26}$$

and recombining the oscillator propagators according to Eq. (24), one can write

$$\begin{aligned} \widetilde{K}_N(0, T/2; 0, -T/2) &= \sum_{2^N \text{ paths}} \Delta^N K_\omega(0, T/2; 0, -T/2) \\ &\sim \Delta^N K_\omega(0, T/2; 0, -T/2) \sum_{2^N \text{ paths}} 1 \\ &= (2\Delta)^N \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2}. \end{aligned} \tag{27}$$

Consequently, the existence of 2^N paths for given N implies that the propagator $\widetilde{K}_N$ is proportional to $(2\Delta)^N$, to be compared with a single path (leading to Δ^N) in the case of the double well on the line. Finally, the full propagator is obtained from Eq. (21) as

$$K_e(\varphi_+, T/2; \varphi_-, -T/2) \sim \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \sum_{\text{odd } N} \int d\vec{\tau} (2\Delta)^N \exp\left(-NS_e^0/\hbar\right) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \sinh\left\{2\Delta T \exp\left(-S_e^0/\hbar\right)\right\} , \quad (28)$$

where the intermediate steps are identical to those in Ref. [8] after replacing the quantity Δ by 2Δ. Similarly, one obtains the propagator for the particle starting and ending at the point φ_+ by summing over N even

$$K_e(\varphi_+, T/2; \varphi_+, -T/2) \sim \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \cosh\left\{2\Delta T \exp\left(-S_e^0/\hbar\right)\right\} . \quad (29)$$

The remaining two propagators follow from the invariance under $\varphi_\pm \to \varphi_\mp$. Using the ground-state wave functions of a harmonic oscillator centered at the minima $\varphi_\pm$ for the functions $\phi_0(\varphi)$ in Eq. (8) and comparing with the explixit expressions for the propagators for large values of T, Eqs. (28) and (29), allows one to deduce the following result for the energy splitting:

$$\Delta E \equiv E_1 - E_0 = 2(2\Delta) T \exp\left(-S_e^0/\hbar\right) \quad (30)$$

for a particle in a symmetric double-well potential on a ring.

B. Nonzero magnetic field

In the presence of a uniform magnetic field **B** pointing along the z axis the Euclidean action $S_e[\varphi(\tau)]$ is modified by the Euclidean version of the field term (4)

$$S_e^A[\varphi(\tau)] = S_e[\varphi(\tau)] - i\sigma[\varphi(\tau)] = S_e[\varphi(\tau)] - i \int_{\tau_a}^{\tau_b} A(R)\dot{\varphi} d\tau . \quad (31)$$

The path-integral expression for the Euclidean propagator now reads

$$K_e^A(\varphi_b, \tau_b; \varphi_a, \tau_a) = \int_{\varphi_a}^{\varphi_b} \mathcal{D}\varphi \exp\left(-S_e^A[\varphi(\tau)]/\hbar\right) \quad (32)$$

and the calculation of Sec. III A goes through up to the expansion about the various instanton strings. Now a path $\varphi(\tau)$ of length N contributes

$$\exp\left\{S_e^A(\widetilde{\varphi}_\mp^N(\tau))/\hbar\right\} \int_{\eta(-T/2)=0}^{\eta(T/2)=0} \mathcal{D}\eta \exp\left\{-\widetilde{S}_e[\eta(\tau)]/\hbar\right\} , \quad (33)$$

where the action $\widetilde{S}_e^A[\eta(\tau)]$ in the exponent has been replaced by $\widetilde{S}_e[\eta(\tau)]$ since one has

$$\int_{-T/2}^{T/2} d\tau\, \sigma[\eta(\tau)] = \int_{\eta(-T/2)}^{\eta(T/2)} d\eta\, A(R) = A(R)\left\{\eta(T/2) - \eta(-T/2)\right\} = 0 \quad (34)$$

for all paths with $\eta(\pm T/2) = 0$, which are the only ones considered here. Consequently, the path-integral part of Eq. (33) is identical to the one calculated previously; it can be written as $\Delta^N K_\omega(0, T/2; 0, -T/2) \sim \Delta^N \sqrt{m\omega/2\pi\hbar}\exp(-\omega T/2)$. The first factor in Eq. (33), however, now depends on the topology of the path under consideration.

As before, the real part of the Euclidean action of an N-instanton string is given approximately by N times the corresponding single-instanton action S_e^0,

$$S_e^A(\varphi_\mp^N) \equiv S_e(\varphi_\mp^N) - i\sigma(\varphi_\mp^N) \simeq NS_e^0 - i\sigma_0\delta(\varphi_\mp^N) , \quad (35)$$

where $\sigma_0 = \int_{\varphi_-}^{\varphi_+} A(R) d\varphi$ is the contribution of an individual instanton traveling directly from φ_- to φ_+, arising from the field term. The quantity $\delta(\varphi_\mp^N)$ takes on integer values depending on the topological properties of the individual multi-instanton path. It is important that the field term leads to contributions from α- and β-type instantons with equal magnitude but *opposite* sign

$$\int_\alpha d\varphi\, A(R) = \int_{-\pi/2}^{\pi/2} d\varphi\, A(R) = A(R)\pi = -\int_{-\pi/2}^{-3\pi/2} d\varphi\, A(R) = -\int_\beta d\varphi\, A(R) . \quad (36)$$

In other words, the factor δ is $+1$ for a single α-type instanton and equals -1 for the β type.

More explicitly, the contribution of paths with length N has the following structure. Each path comes with factors $\exp(-NS_e^0/\hbar)$ and $\Delta^N K_\omega$ depending on the length N only, whereas the factor $\exp\{-i\sigma_0\delta(\varphi_\mp^N)\}$ depends on the nature of the path taken. One can write, for the contribution of these paths,

$$\widetilde{K}_N^A(\varphi_+, T/2; \varphi_-, -T/2) \sim \Delta^N \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2 - NS_e^0/\hbar} \sum_{2^N \text{ paths}} e^{-i\sigma_0\delta(\varphi_\mp^N)/\hbar} . \quad (37)$$

Without magnetic field ($\sigma_0 = 0$) one recovers the result for K_e given in Eq. (27). The sum in Eq. (37) can be evaluated in the following way. First, divide the set of all paths from φ_- to φ_+ into two parts: those paths starting with an α instanton and those starting with a β instanton

$$\sum_{2^N \text{ paths}} e^{-i\sigma_0\delta/\hbar} = \sum_{2^{N-1} \text{ paths}}^{\alpha} e^{-i\sigma_0\delta/\hbar} + \sum_{2^{N-1} \text{ paths}}^{\beta} e^{-i\sigma_0\delta/\hbar} = 2\cos(\sigma_0/\hbar) \sum_{2^{N-1} \text{ loops}} e^{-i\sigma_0\delta/\hbar} . \quad (38)$$

In the last step the contribution of the first part of the multi-instanton path has been factored out and the remaining sum in Eq. (38) is over all closed loops of length $(N-1)$, starting and ending at φ_+. Taking property Eq. (36) into account, one can see that $\delta(\varphi_\mp^{N-1})$ equals the winding number m of the loop under consideration. For $N=3$, there are four loops of length 2, two of which have $m=0$ and two having $m=\pm1$, respectively. If $N=5$, one has eight paths with winding numbers varying between ±2. The multiplicities of paths with different m are given by binominal coefficients $\binom{N}{k}$, $k=0,1,2,3,\ldots$. For arbitrary N, one finds that

$$\sum_{2^{N-1} \text{ loops}} e^{-i\sigma_0\delta(\varphi_\mp^N)} = \left(e^{i\sigma_0/\hbar} + e^{-i\sigma_0/\hbar}\right)^{N-1} = [2\cos(\sigma_0/\hbar)]^{N-1} , \quad (39)$$

leading finally to the Euclidean propagator

$$K_e^A(\varphi_+, T/2; \varphi_-, -T/2) \sim \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \sum_{\text{odd } N} \int d\vec{\tau}\, [2\Delta\cos(\sigma_0/\hbar)]^N \exp\left(-NS_e^0/\hbar\right) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\omega T/2} \sinh\{2\Delta T\cos(\sigma_0/\hbar)\exp(-S_e^0/\hbar)\}. \quad (40)$$

This result is again obtained by following the calculation in [8], now replacing Δ by $2\Delta\cos(\sigma_0/\hbar)$. Consequenlty, the expression for the tunnel splitting has to be modified accordingly and one obtains

$$E \equiv E_1 - E_0 = 2(2\Delta)\cos\left(\frac{\pi}{2}\frac{\Phi}{\Phi_0}\right)\exp(-S_e^0/\hbar) , \quad (41)$$

with the magnetic flux $\Phi = B\pi R^2$ through the ring and $\Phi_0 = hc/2e$ being the elementary (two-electron) flux quantum. Therefore, the splitting ΔE between the lowest energy eigenvalues is "quenched" whenever the flux Φ through the ring is an odd integer multiple of the flux quantum

$$\Phi(B,R) = (2k+1)\Phi_0 , \qquad k \in \mathbb{Z} . \quad (42)$$

Since the flux Φ is a function of two independent parameters, the tunnel splitting can be made to vanish in two ways: by a variation of either the magnitude B of the field or the radius R of the circle.

IV. TUNNEL SPLITTING FOR GENERAL LOOPS

The quenching of the tunnel splitting will now be shown to persist for particles moving on a large class of planar loops $\partial\Gamma$ instead of a circle. The shape of the loops is an arbitrary smooth curve (Fig. 3) required to be invariant under rotation through an angle π about an axis perpendicular to the plane. Such loops will be called C_2 loops. As before a double-well potential V is assumed to exist along the line, being compatible with the symmetry of the loop, as expressed in Eq. (5). Altogether, the classical system now is described by the Lagrangian $L^A(\mathbf{q}, \dot{\mathbf{q}})$ of Eq. (1) along with the constraint

$$\partial\Gamma: \quad G(x,y) = 0. \quad (43)$$

The twofold rotational symmetry implies that if the point (x_0, y_0) is located on curve then $-(x_0, y_0)$ also is on the curve: $G(-x_0, -y_0) = 0$.

Replace the coordinates $z = x + iy$ by a new pair $w = u + iv$ by means of an analytic function

$$z = g(w) . \quad (44)$$

Expressed in the new coordinates one finds, for the Lagrangian,

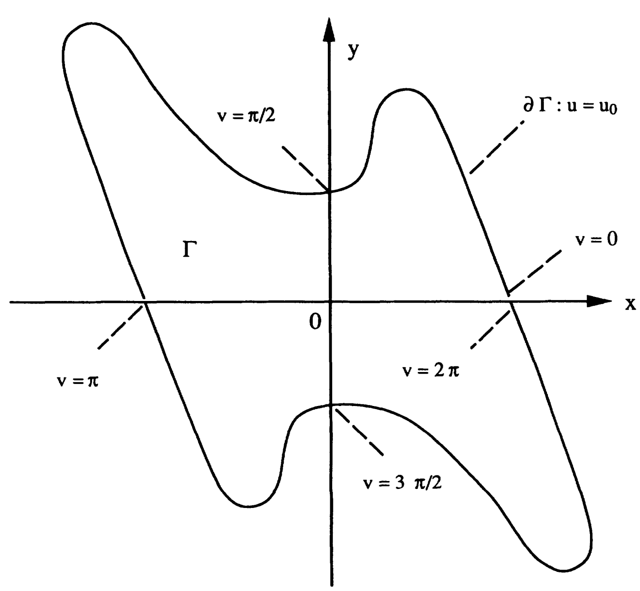

FIG. 3. A general C_2 loop, defined by $\partial\Gamma$: $G(x,y)=0$. The coordinates v and u represent an orthogonal system; $\partial\Gamma$ coincides with the coordinate line $u = u_0$.

$$L^A(w,\dot{w}) = \frac{1}{2}\left|\frac{dg}{dw}\right|^2(\dot{u}^2+\dot{v}^2) + \frac{eB}{2c}\frac{1}{2}\left(\dot{v}\frac{\partial}{\partial u} - \dot{u}\frac{\partial}{\partial v}\right)|\,g(w)\,|^2 - V(u,v)\,. \tag{45}$$

Due to the Riemann mapping theorem [10] there exists a function $g(w)$ such that the curve $\partial\Gamma$ coincides with one of the coordinate lines in the w plane, with $u = u_0$, say. Since $g(w)$ defines a conformal transformation, u and v represent orthogonal coordinates and the motion on $\partial\Gamma$ is described by the Lagrangian

$$L^A(v,\dot{v}) = \frac{1}{2}m(v)\dot{v}^2 + a(v)\dot{v} - V(v)\,, \tag{46}$$

where

$$m(v) = \left|\frac{dg}{dw}\right|^2_{u=u_0} > 0,\quad a(v) = \frac{eB}{4c}\left[\frac{\partial}{\partial u}\,|\,g(w)\,|^2\right]_{u=u_0}. \tag{47}$$

Consequently, for each C_2 loop the Lagrangian is identical to that of a particle moving on a circle with a positive position-dependent mass $m(v)$ and a modified vector potential $a(v)$. The analytic function $g(w) = \exp w$ mediates between Cartesian and polar coordinates: for $u_0 = \ln R$ one recovers the Lagrangian of the particle (with mass $m = 1$) on the circle, Eq. (2). The symmetry of the system implies [cf. Eq. (5)]

$$m(v+\pi) = m(v),\quad a(v+\pi) = a(v)\,. \tag{48}$$

The effect of the magnetic field on the particle is still described by a total derivative $d\mathcal{A}(v)/dv$, where

$$\mathcal{A}(v) = \int_{v_0}^{v} dv'\,a(v')\,, \tag{49}$$

having again no influence on the classical motion. The Lagrangian (46) is a simple example of a particle moving on a Riemannian manifold, $m(v)$ being the metric tensor. A general discussion of this situation can be found in Ref. [6], for example (cf. also the end of this section).

As a result of these modifications, the calculations performed in the previous sections have to be changed in three places; from now on overlined quantities denote quantities referring to general C_2 loops. First of all, the actions of the extremal paths (i.e., the classical solutions) will be different since the instanton now is defined as a solution of

$$\frac{dv}{d\tau} = \pm\sqrt{2V(v)/m(v)} \tag{50}$$

instead of Eq. (11). Due to $m(v) > 0$ and the periodicity of $m(v)$, Eq. (48), one can proceed in analogy to the previous calculation after replacing $2V(v)/mR^2 \to 2V(v)/m(v)$ [11].

Second, the phase shift $\sigma_0/\hbar$ is modified because the effective vector potential $a(v)$ is no longer constant along the loop $\partial\Gamma$. Nevertheless, the symmetry properties of the system still guarantee the shifts for α- and β-type instantons to have equal modulus and opposite sign. Explicitly, the field term now defines $\overline{\sigma}_0$ according to

$$\int_\alpha dv\,a(v) = \int_{-\pi/2}^{\pi/2} dv\,a(v) = \overline{\sigma}_0 = -\int_{-3\pi/2}^{-\pi/2} dv\,a(v) = -\int_\beta dv\,a(v)\,, \tag{51}$$

in analogy to Eq. (36).

The third modification is due to the change in the quantity Δ. It will be shown now that the methods to evaluate it explicitly are still working in spite of the more complicated mass term $m(v)$. The Euclidean path integral reads

$$\overline{K}_e(v_b,\tau_b;v_a,\tau_a) = \int_{v(\tau_a)}^{v(\tau_b)} \mathcal{D}v\,\exp\left\{-\overline{S}_e[v(\tau)]/\hbar\right\}\,, \tag{52}$$

where $\overline{S}_e[v(t)]$ is defined as

$$\overline{S}_e[v(t)] = \int_{\tau_a}^{\tau_b} d\tau\left\{\frac{m(v)}{2}\left(\frac{dv}{d\tau}\right)^2 + V(v)\right\}. \tag{53}$$

The Euclidean equation of motion is given by

$$m(v)\frac{d^2v}{d\tau^2} + \frac{1}{2}m'(v)\left(\frac{dv}{d\tau}\right)^2 - V'(v) = 0 \tag{54}$$

and the prime denotes the derivative with respect to v. Let $\widetilde{v}(t)$ be one of the solutions for $E_e = 0$ of this equation connecting the maxima of the inverted potential $-V(v)$ [i.e., $\widetilde{v}(t)$ satisfies Eq. (50)] and expand the paths entering in $\overline{K}_e$ about it according to $v(\tau) = \widetilde{v}(\tau) + \eta(\tau)$. One obtains

$$\overline{K}_e(v_b,\tau_b;v_a,\tau_a) = \int_{\eta(\tau_a)=0}^{\eta(\tau_b)=0} \mathcal{D}\eta\,\exp\left\{-\overline{S}_e[\widetilde{v}(\tau)+\eta(\tau)]/\hbar\right\} = \exp\left\{-\overline{S}_e^0[\widetilde{v}(\tau)]/\hbar\right\}\int_{\eta(\tau_a)=0}^{\eta(\tau_b)=0} \mathcal{D}\eta\,Z_0[\eta(\tau)]Z_1[\eta(\tau)]\,, \tag{55}$$

where

$$Z_0[\eta(\tau)] = \exp\left(-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau\left\{m(\widetilde{v})\frac{d\widetilde{v}}{d\tau}\frac{d\eta}{d\tau} + V'(\widetilde{v})\eta + \frac{m'(\widetilde{v})}{2}\left(\frac{d\widetilde{v}}{d\tau}\right)^2\widetilde{v}\eta\right\}\right) = 1\,, \tag{56}$$

as follows from partially integrating the first term, using $\eta_a = \eta_b = 0$, and the Euclidean equation of motion (54). Furthermore, since

$$\int_{\tau_a}^{\tau_b} d\tau\,m'(\widetilde{v})\dot{\widetilde{v}}\eta\frac{d\eta}{d\tau} = -\frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau\frac{d^2m(\widetilde{v})}{d\tau^2}\eta^2\,, \tag{57}$$

with $\dot{\widetilde{v}} \equiv d\widetilde{v}/d\tau$, one can write the remaining integrand of (55) in the form

$$Z_1[\eta(\tau)] = \exp\left(-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b} d\tau \left\{\frac{m(\widetilde{v})}{2}\left(\frac{d\eta}{d\tau}\right)^2 + U(\widetilde{v})\eta^2\right\}\right), \tag{58}$$

where

$$U(\widetilde{v}) = \frac{1}{2}\left[V''(\widetilde{v}) - \frac{d^2 m(\widetilde{v})}{d\tau^2} + \frac{1}{2}m''(\widetilde{v})\left(\frac{d\widetilde{v}}{d\tau}\right)^2\right] \tag{59}$$

is a given function of (Euclidean) time τ since it is completely determined by $\widetilde{v}(\tau)$.

Thus the problem of determining the parameter $\overline{\Delta}$ has been reduced to calculating the propagator for an oscillator with time-dependent mass and frequency. This can be done, for example, in analogy to the calculation presented by Felsager [8] after reintroducing real time $t = i\tau$ and partially integrating

$$\overline{K}(v_b,t_b;v_a,t_a) = \exp\left\{-\overline{S}_e^0[\widetilde{v}(t)]/\hbar\right\}\int_{\eta(t_a)=0}^{\eta(t_b)=0}\mathcal{D}\eta \times \exp\left[-\frac{1}{\hbar}\int_{t_a}^{t_b} dt\eta(t)\left\{\frac{d}{dt}\left(m(\widetilde{v})\frac{d}{dt}\right) + U(\widetilde{v})\right\}\eta(t)\right]. \tag{60}$$

The operator in curly brackets is of Sturm-Liouville type [12]; in combination with the boundary conditions $\eta_a = \eta_b = 0$, it is known to have a complete set of eigenfunctions. After expanding an arbitrary path in terms of these functions, the expression (60) turns into an infinite-dimensional Gaussian integral which can be calculated in the usual way. However, as the actual value of the prefactor $\overline{\Delta}$ is not relevant for the quenching of the tunnel splitting, it is not calculated explicitly here.

Finally, some remarks concerning the quantum mechanics of constrained systems are appropriate. It is known that quantization of such systems leads to corrections of the "natural" Hamiltonian in the form of an additional potential term of order $\hbar^2$, related to the spatially variable curvature of the constraining surface [13,14]. In order to obtain well-defined results the constraint should be modeled as the limit of a strong narrow gully as it is discussed thoroughly in [15]. Here the freedom of applying conformal transformations to the interior Γ of the curve $\partial\Gamma$ reflects different choices of the shape of the gully: the function $g(w)$ in Eq. (44) is not defined uniquely. Given the curve $\partial\Gamma$ in the complex plane one can conformally map its interior Γ onto itself by prescribing (i) the image of an arbitrary point and (ii) the angular part of the derivative of $g(w)$ at this point *without* losing the essential property that one of the coordinate lines coincides with the boundary $\partial\Gamma$. The question of ordering the noncommuting operators in the kinetic energy, showing up in the quantized Hamiltonian associated with the Lagrangian $L(v,\dot{v})$, Eq. (45), would lead to ambiguities of order $\hbar^2$. Similarly, making a point transformation in the path integral (52) in order to remove the spatial dependence of the mass would lead to corrections of the order $\hbar^2$ [16]. In the present context a semiclassical evaluation of the path integrals is attempted, being correct to order $\hbar$ only, so that terms of order $\hbar^2$ can be safely neglected.

V. TOPOLOGICAL ASPECTS AND DISCUSSION

If a particle is constrained to move in a symmetric double-well potential on a C_2 loop one finds the following expression for the tunnel splitting:

$$\Delta\overline{E} = 2(2\overline{\Delta})\cos\left(\frac{\pi}{2}\frac{\overline{\Phi}}{\Phi_0}\right)\exp(-\overline{S}_e^0/\hbar). \tag{61}$$

It is interesting to compare this general expression for the splitting $\Delta\overline{E}$ with the corresponding formula obtained for a particle in a double well on the line, where $\Delta E = 2\Delta T\exp(-S_e^0/\hbar)$. Three modifications arise. Consider first a situation without magnetic field. Imagine to have two copies of the part of the potential $V(x)$ on the line between the minima $x_\pm = \pm\pi/2$ and use them to construct the double well on the ring with radius $R = 1$. The parameter relevant for the splitting is seen to acquire a factor of 2 on the ring: $\Delta \to 2\Delta$, which arises since, intuitively speaking, on the ring there are *two* distinct ways for the particle to tunnel from one minimum to the other. This can also be seen immediately if one imagines to cut open the ring at $\varphi = 0$, for example, eliminating in this way the paths containing β- and $\overline{\beta}$-type instantons: the resulting tunnel splitting would be identical to that one on the line. If the external magnetic field is turned on, no interference can arise for the potential on the line: for a given length N there is one and only one path connecting the minima. On the circle, however, the splitting acquires an oscillatory factor depending on the enclosed flux representing the second modification of Eq. (61). The third change is due to deforming the circle to an arbitrary loop with appropriate symmetry, which require quantitative modifications only: $\Delta \to \overline{\Delta}$, $S_e^0 \to \overline{S_e^0}$, and $\Phi \to \overline{\Phi}$. The topological character of the quenching is clearly illustrated by the qualitative insensitivity of the structure of Eq. (61) under smooth deformations.

The quenching of the tunnel splitting due to an external magnetic field investigated here is analogous to the quenching observed in spin systems. Suppose that a crystal field provides two equivalent minima for a magnetic ion with spin $\hat{J}$ and that the location of the minima depends on the external field [5,17]. In a Hamiltonian formulation the classical spin system is equivalent to that one of a fictitious charged particle moving on a sphere representing the phase space of the spin. It is coupled to a fictitious magnetic monopole located at the center of the spherical phase space. Again, the field term is a total derivative thus being irrelevant on the classical level. Quantum mechanically, however, it contributes differently to the actions of the two types of instantons present in the system. As a consequence, the tunnel splitting ΔE also acquires a trigonometric factor $\cos\phi$. The quantity ϕ can be expressed in terms of the magnetic flux of the monopole through the phase-space surface spanned by the two instanton paths, thus becoming a function of the external magnetic field. It follows that specific values of the field strength exist for which the quantum-mechanical ground state of the spin system is twofold degenerate.

In the study of transport properties of mesoscopic systems [18–20] results have been obtained which are closely related to those for the model presented here; the basic questions and the methods, however, are somewhat different. An important example of a mesoscopic system consists of a small ring with attached conducting legs, the transmission properties of which are of particular interest. In a simple model, impurities in the ring are modeled by two (typically different) scatterers, one in each branch. The question of tunneling, however, is not addressed directly. Nevertheless, it turns out that due to the variation of a magnetic field enclosed by the ring, oscillations of the transmission amplitude are observed. Presumably, they represent remnants of the quenching of the tunnel splitting addressed here, requiring the exact symmetry of the potential well. Without this symmetry, the actions associated with the paths α and β, respectively, would be different, no longer allowing for complete destructive interference.

In this context also some general remarks [19] have been made pertaining to the energy spectrum of a perfectly conducting ring with an arbitrary 2π-periodic potential $V(\varphi)$. It has been observed that one can set up an interesting and useful analogy between a quantum-mechanical particle moving on a circle and in a one-dimensional periodic lattice $V(q)$, respectively. At first, there seems to be an important difference in the periodic properties of the wave functions in these systems. On a circle, the wave function $\psi_C(\varphi)$ is required to be transformed into itself under translation about 2π: $\psi_C(\varphi + 2\pi) = \psi_C(\varphi)$, whereas a Bloch wave $\psi_L(x)$ may acquire a phase when shifted over one period x_0 of the lattice: $\psi_L(x + x_0) = \exp(ikx_0)\psi_L(x)$, with k being the wave vector of the eigenfunction under consideration. If, however, a magnetic field is enclosed by the circle, under a rotation about 2π the phase of the state $\psi_C(\varphi)$ is shifted by an amount which is proportional to the strength of the field. Consequently, the familiar dispersion relation $E = E(k)$ for the particle in a one-dimensional crystal can be interpreted to give the energy spectrum $\mathcal{E} = \mathcal{E}(B)$ of the circle enclosing magnetic flux. Choosing a unit cell which contains two identical potential wells in order to appropriately represent the double-well potential on the ring, one finds that there are no energy gaps at the boundaries of the Brillouin zone [21]. As a result there are particular values of k, i.e., values of the field B, for which two orthogonal states with the same energy do exist. Tunneling becomes impossible, thus confirming the results obtained here from the path-integral approach. Also, analogies could be drawn to the phenomenon of flux quantization in superconductivity [22,23].

As mentioned before, the coupling of the particle to the magnetic field gives rise to a term in the Lagrangian which is a total derivative. Therefore it represents a gauge transformation of the Lagrangian and does not influence the classical equations of motion. In quantum mechanics, however, nontrivial topological properties of the configuration space D can lead to observable effects stemming from such a term, because quantum mechanics is sensitive to global features. This is seen particularly well in the path-integral approach to quantum mechanics. The propagator connecting two points q and q' consists of a sum of contributions, each stemming from a specific homotopy sector. In other words, there exist paths from q to q' (characterized by different generalized winding numbers) which cannot be transformed into each other by continuous deformations. In general, one can write [4,7]

$$K(q, q'; T) = \sum_{\gamma \in \pi_1(D)} a[\gamma] K_a(q, q'; T) , \qquad (62)$$

$\pi_1(D)$ being the first homotopy group (or fundamental group) of configuration space D. The factors $a[\gamma]$ have modulus one, but the superposition of paths γ from different sectors (i.e., paths with different winding numbers) leads to interference of the various partial propagators K_a. In the present calculation it turned out to be more convenient to arrange the contributions to the propagator according to the *length* N of the instanton string, not according to their winding numbers. Consequently, the sum over paths in Eq. (37) still contains contributions from various sectors.

The addition of topological terms which do not affect the classical motion but lead to consequences in the corresponding quantum-mechanical systems has attracted interest in various fields. Anyons being candidates for the explanation of the fractional quantum Hall effect are conveniently defined by adding a total derivative to the free particle Hamiltonian in two dimensions, in this way effectively attaching a tube with magnetic flux to the particles [24,7]. In a field theoretical context topological terms in the Lagrangian are also known as Chern-Simons or Wess-Zumino terms. In general, topological field theories are interesting since qualitative properties of the solutions follow from topological arguments alone.

Experimentally, observing the properties of mesoscopic systems is within reach [25]; even experiments with *single* Au loops have been reported [26]. The presence of a persistent current in a ring enclosing magnetic flux [18] has been confirmed, although the quantitative agreement between experimental and theoretical data is still discussed [27,28]. Due to the possibility to fabricate and to handle single mesoscopic loops, an experimental realization of the double-well potential might be possible by means of the presently available technology.

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[1] S. Coleman, in *The Whys of Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1979), p. 805.
[2] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
[3] L. S. Schulman, Phys. Rev. **188**, 1139 (1969).
[4] L. S. Schulman, J. Math. Phys. **112**, 304 (1971).
[5] A. Garg, Europhys. Lett. **22**, 205 (1993).
[6] M. S. Marinov, Phys. Rep. **60**, 1 (1980).
[7] G. Morandi, *The Role of Topology in Classical and Quantum Physics*, Lecture Notes in Physics m7 (Springer, Berlin, 1992).
[8] B. Felsager, *Geometry, Particles and Fields* (Odense University Press, Odense, 1981).
[9] H. Kleinert, *Pfadintegrale* (Wissenschaftsverlag, Mannheim, 1993).
[10] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1974).
[11] This shows that, in the case of the ring, one also could have started with a nonuniform, C_2-periodic magnetic field $B(\varphi)$, stemming from a vector potential with $A(\varphi+\pi) = A(\varphi)$.
[12] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, (Springer, Berlin, 1968), Vol. I.
[13] B. S. De Witt, Rev. Mod. Phys. **29**, 377 (1957).
[14] R. C. T. da Costa, Phys. Rev. A **23**, 1982 (1981).
[15] N. G. van Kampen, Am. J. Phys. **52**, 419 (1984).
[16] B. Sakita, *Quantum Theory of Many-Variable Systems and Fields* (World Scientific, Singapore, 1985).
[17] St. Weigert, Europhys. Lett. **26**, 561 (1994).
[18] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).
[19] M. Büttiker, Y. Imry, and M. Ya. Azbel, Phys. Rev. A **30**, 1982 (1984).
[20] Yuval Gefen, Yoseph Imry, and M. Ya. Azbel, Phys. Rev. Lett. **52**, 129 (1984).
[21] This situation corresponds to a one-dimensional model of a harmonic crystal, consisting of atoms with alternating masses m_a and m_b. Letting the masses approach each other $m_a \to m_b$, one obtains in the limit a unit cell being too large by a factor of 2, implying zero energy gaps at the boundary of the Brillouin zone.
[22] N. Byers and C. N. Yang, Phys. Rev. Lett **7**, 46 (1961).
[23] F. Bloch, Phys. Rev. Lett. **21**, 1241 (1968).
[24] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
[25] L. P. Levy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. **64**, 2074 (1990).
[26] V. Chandrasekhar, R. A. Webb, M. J. Brady, M. B. Ketchen, W. J. Gallagher, and A. Kleinsasser, Phys. Rev. Lett. **67**, 3578 (1991).
[27] A. Altland, S. Iida, A. Müller-Groeling, and H. A. Weidenmüller, Europhys. Lett. **20**, 155 (1992).
[28] R. A. Smith and V. Ambegaokar, Europhys. Lett. **20**, 161 (1992).