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Simultaneous sharp measurability of position and momentum in infinite quantum systems

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Joint position-momentum observables are defined to be covariant POV measures on the group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ of space translations and boosts. For elementary or finite quantum systems, it is shown that there is no covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ into the von Neumann algebra generated by average position and momentum. So these cannot be measured sharply at the same time. For an infinite quantum system, we construct a covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ into the von Neumann algebra generated by average position and momentum. Therefore average position and momentum of an infinite quantum system can be measured sharply at the same time.

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I. INTRODUCTION

Position and momentum of an elementary quantum system obey (on a dense domain of common self-adjointness) the commutation relations

$$[P,Q] = i\hbar \mathbf{1}. \tag{1}$$

Therefore the accuracy of joint position and momentum measurements is bound from below by $\Delta p \Delta q \geq \hbar/2$. Average position and momentum Q^n, P^n of an n-particle quantum system obey the commutation relations

$$[P^n, Q^n] = \frac{i\hbar}{n}\mathbf{1}. \tag{2}$$

In the limit of an infinite quantum system it is clear that the right hand side of (2) tends to zero. But it is not immediately clear how to interpret the left hand side because the norm limits of the operators P^n, Q^n do not exist.

A solution to this problem is provided in algebraic quantum mechanics. The limits of the averages P^n, Q^n exist in the strong operator topology and are global observables. As such they commute with all other observables and are classical. (It was proven in [1] that under certain circumstances infinite quantum systems admit a classical momentum operator which is not necessarily an averaged observable.)

In this paper we treat the problem of joint measurability of position and momentum in the framework of covariant positive-operator-valued measures [2–4], POV measures for short. POV measures are more general observables than projection-valued measures (for short, PV measures) or, equivalently, self-adjoint operators (definitions of POV measures and PV measures can be found in the Appendix). It is necessary to use this more flexible framework because in the traditional (i.e., von Neumann's) formalism noncommuting observables do not have joint probability distributions. So in von Neumann's formalism it is not even possible to talk about joint measurability of position and momentum. In the framework of covariant POV measures, noncommuting observables can have *unsharp* joint probability measures [5,6]. In this framework it is possible to describe joint measurements of noncommuting observables, as long as the measurement is sufficiently unsharp.

In Sec. II a joint position-momentum observable is defined to be a covariant POV measure on the additive group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$. Then it is shown that there is no covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ into the von Neumann algebra generated by average position and momentum of elementary (Sec. III) and finite (Sec. IV) quantum systems. So these cannot be measured sharply at the same time. In Sec. V we construct the strong operator limits of the average position and momentum observables. Also, we construct a covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ into the von Neumann algebra generated by the average position and momentum of infinite quantum systems. This means that the average position and momentum of an infinite quantum system can be measured sharply at the same time and are classical observables.

For the reader unfamiliar with the formalism of covariant POV measures the basic definitions and concepts are gathered in the Appendix.

II. JOINT POSITION-MOMENTUM OBSERVABLES

Let us start with an example.

Example 1. Let $\{U_a, V_b : a,b \in \mathbb{R}^3\}$ be an irreducible σ-weakly measurable unitary representation of the Weyl relations $U_a V_b = \exp(iab) V_b U_a$ on the Hilbert space $L^2(\mathbb{R}^3)$ (ab denotes the scalar product of the vectors $a, b \in \mathbb{R}^3$).

On $\mathcal{B}(L^2(\mathbb{R}^3))$ the additive group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ is represented by an automorphic action α,

$$\alpha_{(a,\hat{b})}(x) := U_a V_b x V_b^* U_a^* =: \alpha_{(a,b)}(x), \quad x \in \mathcal{B}(L^2(\mathbb{R}^3)). \tag{3}$$

[The characters $\hat{b}$ on $\mathbb{R}^3$ and the elements $b \in \mathbb{R}^3$ are in a one-to-one relation by $\hat{b}(s) = \exp(ibs)$ for $s \in \mathbb{R}^3$. So one can say that $\alpha_{(a,b)}$ is an automorphic representation of $\mathbb{R}^3 \times \mathbb{R}^3$ and that $\alpha_{(a,\hat{b})}$ is an automorphic representation of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$.]

Since the representation $\{U_a, V_b\}$ is σ-weakly measurable, the action α is pointwise σ-weakly continuous. Therefore $(\mathcal{B}(L^2(\mathbb{R}^3)), \mathbb{R}^3 \times \widehat{\mathbb{R}^3}, \alpha)$ fulfills the requirements of a W^* system (see Appendix). The action α is ergodic (see Appendix) on $\mathcal{B}(L^2(\mathbb{R}^3))$ since, due to the irreducibility of the representation $\{U_a, V_b\}$, the only $x \in \mathcal{B}(L^2(\mathbb{R}^3))$ which fulfill $\alpha_{(a,\hat{b})}(x) = x$ for all $(a, \hat{b})$ are multiples of the identity.

Denote by P the self-adjoint generator of the unitary group U_a and by Q the self-adjoint generator of the unitary group V_b. Each character χ of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ can be written as $\chi(a, \hat{b}) = \exp(iat + isb)$ for some $s, t \in \mathbb{R}^3$. Now observe that for each character χ there is a unitary operator $u_\chi \in \mathcal{B}(L^2(\mathbb{R}^3))$, namely $u_\chi = U_t V_s$, such that u_χ transforms under α according to χ:

$$\alpha_{(a,\hat{b})}(u_\chi) = \chi(a, \hat{b}) u_\chi. \tag{4}$$

According to [7], since $(\mathcal{B}(L^2(\mathbb{R}^3)), \mathbb{R}^3 \times \widehat{\mathbb{R}^3}, \alpha)$ is ergodic, it follows from (4) that the automorphic representation α is integrable in the sense of Connes and Takesaki (see Appendix). By ([8], Theorem II.7) this in turn implies that there is a POV measure a on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with values in $\mathcal{B}(L^2(\mathbb{R}^3))$ which fulfills the covariance condition

$$\alpha_{(a,\hat{b})}(a(\Delta)) = a(\Delta + (a, \hat{b})), \tag{5}$$

for any Borel subset Δ of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ and for any $(a, \hat{b}) \in \mathbb{R}^3 \times \widehat{\mathbb{R}^3}$. The POV measure a was derived from the unitary groups U_a, V_b whose generators are the position and momentum operators Q, P. This motivates the following central definition.

Definition. Let $(\mathcal{M}, \mathbb{R}^3 \times \widehat{\mathbb{R}^3}, \alpha)$ be a W^* system of the additive group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$. A POV measure a on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with values in $\mathcal{M}$ fulfilling the covariance condition (5) is called a *joint position-momentum observable.* A covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ is called a *sharp* joint position-momentum observable.

Remarks. (1) We use the additive group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ instead of the additive group $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ because we do not want to consider a six-dimensional position observable as a joint position-momentum observable. $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ describes the group of space translations and boosts.

(2) As illustrated by Example 1, the justification of calling a covariant POV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ a joint position-momentum observable stems from the fact that the operators in its image have the transformation properties of position and momentum. This idea of defining observables through their transformation properties is due to Weyl.

III. ELEMENTARY SYSTEMS

We say that a physical system is *elementary* if it is described by an *ergodic* W^* system.

This definition (see also [8]) generalizes the quantum mechanical notion of elementarity: In traditional quantum mechanics an elementary system is described by an irreducible ray representation U of the symmetry group G. Its algebra of observables $\mathcal{M} = \{U(g) : g \in G\}''$ is a type I factor. In this case the two notions of elementarity coincide: Every automorphism α_g is inner; it can be written as $\alpha_g = U_g \cdot U_g^*$ for some $U_g \in \mathcal{M}$. According to ([9], theorem 67.2) a type I factor system $(\mathcal{M}, G, \alpha)$ is ergodic if and only if the corresponding ray representation $U(G)$ is irreducible.

Theorem 1. Let $(\mathcal{M}, G, \alpha)$ be a W^* system with respect to a locally compact separable group G. If $(\mathcal{M}, G, \alpha)$ is *ergodic* and $\mathcal{M}$ is not Abelian, then there exists *no* covariant PV measure on G with values in $\mathcal{M}$.

Proof (see [8], Theorem IV.1). A covariant PV measure on G with values in $\mathcal{M}$ can be extended to a covariant * morphism $\pi: L^\infty(G) \to \mathcal{M}$. According to ([10], Sec. II.2.2) the existence of such a covariant * morphism is equivalent to the existence of a W^* algebra $\mathcal{N}$ and of a coaction δ on $\mathcal{N}$ such that $\{\mathcal{M}, \alpha\} \cong \{\mathcal{N} \otimes_\delta G, \hat{\delta}\}$. Therefore $\mathcal{M}^\alpha \cong \mathcal{N}$. But since $(\mathcal{M}, G, \alpha)$ is ergodic, $\mathcal{M}^\alpha = \{\lambda 1\}$. Therefore $\mathcal{M} \cong \{\lambda 1\} \otimes_\delta G \cong L^\infty(G)$, which contradicts the assumption that $\mathcal{M}$ is not Abelian. Q.E.D.

Corollary 1. Let $\{U_a, V_b : a, b \in \mathbb{R}^3\}$ be an irreducible σ-weakly measurable unitary ray representation of the Weyl relations on the Hilbert space $L^2(\mathbb{R}^3)$. There exists no covariant PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with values in $\mathcal{B}(L^2(\mathbb{R}^3))$. There exists a covariant POV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with values in $\mathcal{B}(L^2(\mathbb{R}^3))$.

Proof. The nonexistence of a covariant PV measure follows from Theorem 1 and the fact that $(\mathcal{B}(L^2(\mathbb{R}^3)), \mathbb{R}^3 \times \widehat{\mathbb{R}^3}, U_a V_b \cdot V_b^* U_a^*)$ is an ergodic non-Abelian W^* system. The existence of a covariant POV measure was proved in Example 1.

Remarks. (1) For elementary systems there exists a sharp joint position-momentum observable if and only if $\mathcal{M}$ is Abelian, i.e., if and only if the system is classical.

(2) One should be cautious not to infer prematurely from the existence of an unsharp joint position-momentum observable that position and momentum can be measured at the same time. Caution is necessary because for two observables the notion of being measurable *at the same time* carries the connotation of being transitive, whereas the existence of a joint unsharp observable is *not* transitive.

(3) Taking as G the phase space, Theorem 1 generalizes to arbitrary von Neumann algebra results [11] that for (elementary) quantum systems there exist no covariant *projection* valued measures on the *phase* space. There exist, however, covariant PV measures on the *configuration* space of an elementary particle. These are the systems of imprimitivity as introduced by Mackey. Also, there exist covariant POV measures on the phase space $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ if and only if $(\mathcal{M}, \mathbb{R}^3 \times \widehat{\mathbb{R}^3}, \alpha)$ is integrable ([12] or [8], Theorem II.7). These are unsharp joint position-momentum observables.

(4) Note that the problem is not to find covariant PV measure on G with values in *some* W^* system of the group G. It is always possible to take the W^* system $(\mathcal{B}(L^2(G)), G, U \cdot U^*)$ where U is the representation of G on $L^2(G)$ defined by $U_{g'}\psi(g) := \psi(g - g')$, and to take as PV measure E on G simply $E(\Delta)\psi(g) := \chi_\Delta(g)\psi(g)$, where χ_Δ is the characteristic function of the subset Δ of G. (Note that this W^* system is not ergodic.) The problem is to find a covariant POV measure with values

in a *given* W^* system.

Theorem 1 suggests that there are two scenarios in which it is possible to have sharp joint position-momentum observables: Either the von Neumann algebra $\mathcal{M}$ should be Abelian, or the action α should not be ergodic.

IV. FINITE SYSTEMS

The assumption of ergodicity is essential in Theorem 1. This is illustrated by the following example.

Example 2. As in Example 1, let $\{U_a, V_b : a, b \in \mathbb{R}^3\}$ be an irreducible representation of the Weyl relations $U_a V_b = \exp(iab) V_b U_a$ on $L^2(\mathbb{R}^3)$. It describes the one-particle Weyl system. Consider the two-particle Weyl system whose algebra of observables is $\mathcal{B}(L^2(\mathbb{R}^6)) = \mathcal{B}(L^2(\mathbb{R}^3)) \otimes \mathcal{B}(L^2(\mathbb{R}^3))$. Define a representation $\overline{\alpha}$ of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ by extension to linear combinations of

$$\overline{\alpha}_{(a,\hat{b})}(x_1 \otimes x_2) := U_a V_b x_1 V_b^* U_a^* \otimes U_a V_b x_2 V_b^* U_a^*.$$

This W^* system is not ergodic since $1/2(1 + W)$, where $W(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1$ is the unitary operator implementing a permutation of the two particles, is a projection invariant under all $\overline{\alpha}_{(a,\hat{b})}$.

The condition of ergodicity being violated, we can construct a PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ which is covariant with respect to $\overline{\alpha}$. Let Q be the self-adjoint generator of the unitary group V_b and let E^Q be its spectral measure: $Q = \int_{\mathbb{R}^3} \lambda E^Q(d\lambda)$. Since $a \mapsto U_a$ is a strongly continuous unitary representation of the additive group $\mathbb{R}^3$, a corollary of Stone's theorem (see, e.g., [13], Theorem VIII.12) implies that there exists a PV measure $\hat{E}^P$ on $\widehat{\mathbb{R}^3}$ with values in $\mathcal{B}(L^2(\mathbb{R}^3))$ such that $\langle \phi, U_a \psi \rangle = \int_{\widehat{\mathbb{R}^3}} \hat{\lambda}(a) \langle \phi, \hat{E}^P(d\hat{\lambda}) \psi \rangle$ for all $\phi, \psi \in L^2(\mathbb{R}^3)$. Then

$$(\Delta_1, \Delta_2) \mapsto E^Q(\Delta_1) \otimes \hat{E}^P(\Delta_2)$$

can be extended to a PV measure on $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with values in $\mathcal{B}(L^2(\mathbb{R}^3)) \otimes \mathcal{B}(L^2(\mathbb{R}^3))$. It is covariant since

$$U_a V_b E^Q(\Delta_1) V_b^* U_a^* \otimes U_a V_b \hat{E}^P(\Delta_2) V_b^* U_a^* = E^Q(\Delta_1 + a) \otimes \hat{E}^P(\Delta_2 + \hat{b}).$$

This sharp joint position-momentum observable describes the position of one particle and the momentum of the other particle.

We should exclude such trivial joint sharp measurements of position and momentum for nonelementary systems. But it seems difficult to find a noncommutative algebra of observables such that translations and boost act ergodically for finite systems but not for infinite systems. So we will abandon the first scenario and focus on the second one: We try to find an algebra of observables which is non-Abelian for finite systems but Abelian for infinite systems. This will be done by considering average position and momentum. To this purpose we extend the framework of [14] from spin systems to systems on infinite-dimensional Hilbert spaces.

Let $\mathbf{\Lambda}$ be the set of finite subsets of an infinte index set Π, and for $\Lambda \in \mathbf{\Lambda}$ let $|\Lambda|$ be the cardinality of Λ. Let $\{U_a, V_b : a, b \in \mathbb{R}^3\}$ be an irreducible representation of the Weyl relations $U_a V_b = \exp(iab) V_b U_a$ on $L^2(\mathbb{R}^3)$. Denote by $u_n : L^2(\mathbb{R}^3) \to \mathcal{H}_n$ unitary mappings from $L^2(\mathbb{R}^3)$ onto copies $\mathcal{H}_n$ of $L^2(\mathbb{R}^3)$. Define $\pi_n(x) := u_n x u_n^{-1}$ for $x \in \mathcal{B}(L^2(\mathbb{R}^3))$. Now let

$$\mathcal{A}^\Lambda := \bigotimes_{n \in \Lambda} \pi_n(\mathcal{B}(L^2(\mathbb{R}^3)))$$

be the W^* tensor product of $|\Lambda|$ copies of $\mathcal{B}(L^2(\mathbb{R}^3))$. $\mathcal{A}^\Lambda$ is the algebra of observables of a system consisting of $|\Lambda|$ nonrelativistic particles.

As before, a pointwise σ-weakly continuous representation of the additive group $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ on $\mathcal{B}(L^2(\mathbb{R}^3))$ is given by $\alpha_{(a,\hat{b})}(x) := U_a V_b x V_b^* U_a^*$. On $\mathcal{A}^\Lambda$, define an action $\overline{\alpha}$ of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ by extension to linear combinations of

$$\overline{\alpha}_{(a,\hat{b})}\Big(\bigotimes_{n \in \Lambda} \pi_n(x_n)\Big) := \bigotimes_{n \in \Lambda} \pi_n(\alpha_{(a,\hat{b})}(x_n)). \tag{6}$$

$\overline{\alpha}$ describes translations and boosts of the system as a whole.

We will now define the position and momentum operators of the one-particle system. Usually these are taken to be the self-adjoint generators P of U_a and Q of V_b. Since P, Q are unbounded, they do not belong to $\mathcal{B}(L^2(\mathbb{R}^3))$. Rather they are affiliated to it. To make this mathematically precise, write $P = P_+ - P_-$ and $Q = Q_+ - Q_-$ as differences of positive unbounded self-adjoint operators $P_\pm, Q_\pm$, also affiliated with $\mathcal{B}(L^2(\mathbb{R}^3))$. So (see, e.g., [15]) there exist mappings (which I will also denote by $P_\pm, Q_\pm$) from $\mathcal{B}(L^2(\mathbb{R}^3))^1_*$ into the positive reals which are linear and semicontinuous from below. For every $\epsilon > 0$, $(1 + \epsilon P_\pm)^{-1} P_\pm$, $(1 + \epsilon Q_\pm)^{-1} Q_\pm$ are in $\mathcal{B}(L^2(\mathbb{R}^3))$ (see, e.g., [16], paragraph 5.3.10). For every normal state ρ in $\mathcal{B}(L^2(\mathbb{R}^3))^*_1$ we have

$$P_\pm(\rho) = \lim_{\epsilon \to 0} \rho[(1 + \epsilon P_\pm)^{-1} P_\pm],$$

$$Q_\pm(\rho) = \lim_{\epsilon \to 0} \rho[(1 + \epsilon Q_\pm)^{-1} Q_\pm].$$

P and Q transform under α covariantly according to

$$\alpha_{(a,\hat{b})}(P) = P - b\mathbf{1}, \quad \alpha_{(a,\hat{b})}(Q) = Q - a\mathbf{1}. \tag{7}$$

They describe position and momentum of the one-particle system.

Next construct the average position and momentum operators for the system of $|\Lambda|$ particles. For $x \in \mathcal{B}(L^2(\mathbb{R}^3))$ define the *averaged* element $x^\Lambda \in \mathcal{A}^\Lambda$ by

$$x^\Lambda := \frac{1}{|\Lambda|} \sum_{n \in \Lambda} \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \pi_n(x) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}. \tag{8}$$

$[(1+\epsilon P_\pm)^{-1} P_\pm]^\Lambda$ and $[(1+\epsilon Q_\pm)^{-1} Q_\pm]^\Lambda$ are in $\mathcal{A}^\Lambda$ for all $\epsilon > 0$. Therefore we can define positive unbounded self-adjoint operators $P^\Lambda_\pm, Q^\Lambda_\pm$ affiliated with $\mathcal{A}^\Lambda$ by taking

$$P^\Lambda_\pm(\rho) := \lim_{\epsilon \to 0} \rho\{[(1 + \epsilon P_\pm)^{-1} P_\pm]^\Lambda\}$$

$$Q^\Lambda_\pm(\rho) := \lim_{\epsilon \to 0} \rho\{[(1 + \epsilon Q_\pm)^{-1} Q_\pm]^\Lambda\}$$

for any normal state $\rho \in (\mathcal{A}^\Lambda)^1_*$. Define $P^\Lambda := P^\Lambda_+ - P^\Lambda_-$

and $Q^\Lambda := Q^\Lambda_+ - Q^\Lambda_-$. Formally (i.e., ignoring the fact that π_n is only defined on bounded operators) one can write

$$Q^\Lambda = \frac{1}{|\Lambda|}\sum_{n\in\Lambda} \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\pi_n(Q)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}, \quad (9)$$

$$P^\Lambda = \frac{1}{|\Lambda|}\sum_{n\in\Lambda} \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\pi_n(P)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}. \quad (10)$$

Readers less worried about technical details can take these last equations to define Q^Λ, P^Λ. Q^Λ, P^Λ describe the average position and momentum of the Weyl system with $|\Lambda|$ particles.

Lemma 1. The P^Λ, Q^Λ defined above transform under the action $\bar{\alpha}$ defined in (6) covariantly according to

$$\bar{\alpha}_{(a,\hat{b})}(P^\Lambda) = P^\Lambda - b\mathbf{1}^\Lambda, \quad \bar{\alpha}_{(a,\hat{b})}(Q^\Lambda) = Q^\Lambda - a\mathbf{1}^\Lambda. \quad (11)$$

To check this use the proper definition of P^Λ to verify the following formal calculation:

$$\begin{aligned}\bar{\alpha}_{(a,\hat{b})}(P^\Lambda) &\overset{(6,10)}{=} \frac{1}{|\Lambda|}\sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\pi_n[\alpha_{(a,\hat{b})}(P)]\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\\ &\overset{(7)}{=} \frac{1}{|\Lambda|}\sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\pi_n(P-b\mathbf{1})\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\\ &\overset{(8,10)}{=} P^\Lambda - b\mathbf{1}^\Lambda.\end{aligned}$$

Similarly for Q^Λ.

Lemma 2. $\exp(iaP^\Lambda), \exp(ibQ^\Lambda)$ fulfill the commutation relations

$$\exp(iaP^\Lambda)\exp(ibQ^\Lambda) = e^{iab/|\Lambda|}\exp(ibQ^\Lambda)\exp(iaP^\Lambda).$$

This follows by straight calculation from the Weyl relations

$$\exp(ia|\Lambda|^{-1}P)\exp(ib|\Lambda|^{-1}Q) = e^{iab|\Lambda|^{-2}}\exp(ib|\Lambda|^{-1}Q)\exp(ia|\Lambda|^{-1}P).$$

Define $\mathcal{M}^\Lambda$ to be the von Neumann subalgebra of $\mathcal{A}^\Lambda$ generated by $\exp(iaP^\Lambda)$ and $\exp(ibQ^\Lambda)$:

$$\mathcal{M}^\Lambda := \{\exp(iaP^\Lambda)\exp(ibQ^\Lambda) : a,b\in\mathbf{R}^3\}'' \subset \mathcal{A}^\Lambda. \quad (12)$$

Loosely speaking (i.e., neglecting the fact that P^Λ, Q^Λ are unbounded and therefore not contained in $\mathcal{M}^\Lambda$) one can say that $\mathcal{M}^\Lambda$ is the von Neumann algebra generated by average position and momentum Q^Λ, P^Λ. $\mathcal{M}^\Lambda$ contains exactly the averaged observables: $\mathcal{M}^\Lambda = \{x^\Lambda : x \in \mathcal{B}(L^2(\mathbf{R}^3))\}$. For $|\Lambda|$ finite, $\mathcal{M}^\Lambda$ is a factor of type I_∞.

Lemma 3. For every finite index set Λ, the action $\bar{\alpha}$ defined in (6) acts ergodically on the von Neumann algebra $\mathcal{M}^\Lambda$ defined by (12).

Proof. Assume that

$$\bar{\alpha}_{(a,\hat{b})}\left(\sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\pi_n(x)\otimes\cdots\otimes\mathbf{1}\right) = \sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\pi_n(x)\otimes\cdots\otimes\mathbf{1} \quad (13)$$

for all $(a,\hat{b}) \in \mathbb{R}^3\times\widehat{\mathbb{R}^3}$. We will prove that $\sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\pi_n(x)\otimes\cdots\otimes\mathbf{1}$ is a multiple of the identity.

First of all note that if $\mathcal{M}_1, \mathcal{M}_2$ are von Neumann algebras, then for any $A\in\mathcal{M}_2, B\in\mathcal{M}_1$, $\mathbf{1}_1\otimes A = B\otimes\mathbf{1}_2$ implies that $A = c\mathbf{1}_2, B = c\mathbf{1}_1$ for some complex number c.

Denoting by n_0 the first element of the index set Λ, (13) implies

$$[\pi_{n_0}(x) - \pi_{n_0}(\alpha_{(a,\hat{b})}(x))]\otimes\mathbf{1}\cdots\otimes\mathbf{1} = \mathbf{1}\otimes\left(\sum_{\substack{n\in\Lambda\\ n\neq n_0}}\mathbf{1}\otimes\cdots\otimes[\pi_n(\alpha_{(a,\hat{b})}(x)) - \pi_n(x)]\otimes\cdots\otimes\mathbf{1}\right).$$

This in turn implies that

$$\pi_{n_0}(x) - \pi_{n_0}(\alpha_{(a,\hat{b})}(x)) = c\mathbf{1}, \quad (14)$$

$$\sum_{\substack{n\in\Lambda\\ n\neq n_0}}\mathbf{1}\otimes\cdots\otimes[\pi_n(\alpha_{(a,\hat{b})}(x)) - \pi_n(x)]\otimes\cdots\otimes\mathbf{1} = c\mathbf{1}\otimes\cdots\otimes\mathbf{1}. \quad (15)$$

From (14) we infer that

$$x - \alpha_{(a,\hat{b})}(x) = c\mathbf{1}, \quad (16)$$

which implies that $\pi_n(x) - \pi_n(\alpha_{(a,\hat{b})}(x)) = c\mathbf{1}$ for all $n\in\Lambda$. So (15) can be written as

$$-\left(\sum_{n\in\Lambda, n\neq n_0} c\mathbf{1}\otimes\cdots\otimes\mathbf{1}\right) = c\mathbf{1}\otimes\cdots\otimes\mathbf{1}.$$

This implies that $-|\Lambda|c\mathbf{1}\otimes\cdots\otimes\mathbf{1}=0$, which can only be the case if $c=0$. Thus (16) reads $x=\alpha_{(a,\hat{b})}(x)$ for all $(a,\hat{b})\in\mathbb{R}^3\times\widehat{\mathbb{R}^3}$, which by ergodicity of α on $\mathcal{B}(L^2(\mathbb{R}^3))$ implies that x is a multiple of the identity. The same is true for $\sum_{n\in\Lambda}\mathbf{1}\otimes\cdots\otimes\pi_n(x)\otimes\cdots\otimes\mathbf{1}$. Q.E.D.

From Lemmata 1 and 3 and Theorem 1 we obtain the following theorem.

Theorem 2. For $|\Lambda|$ finite, there is no covariant PV measure on $\mathbb{R}^3\times\widehat{\mathbb{R}^3}$ with values in $\mathcal{M}^\Lambda$ which is covariant with respect to the action $\overline{\alpha}$ defined by (6). This shows that average position and momentum of a finite quantum system cannot be measured sharply at the same time.

V. INFINITE SYSTEMS

The first idea of how to treat the case of infinitely many particles would be to take the norm limit of the average position and momentum operators Q^Λ, P^Λ. From Lemma 2 one would expect that their commutator tends to zero, and that therefore one can measure them sharply at the same time. But we will see that there is a problem: the norm limits of the average position and momentum operators do not exist.

Let $\mathcal{A}$ be the C^*-inductive limit of the net $\{\mathcal{A}^\Lambda\}_{\Lambda\in\mathbf{\Lambda}}$. $\mathcal{A}$ is simple and has a quasilocal structure. One can embed each $\mathcal{A}^\Lambda$ into $\mathcal{A}$ by $\iota:\ x\mapsto x\otimes\mathbf{1}\otimes\mathbf{1}\otimes\cdots$. The action $\overline{\alpha}$ on $\mathcal{A}^\Lambda$ defined by (6) can be extended continuously to an action on $\mathcal{A}$, and further to one on $\mathcal{A}^{**}$. Let us denote this action also by $\overline{\alpha}$.

The following Lemma is taken from [14].

Lemma 4. For any $y\in\mathcal{A}$, $\lim_\Lambda\|[y,\iota x^\Lambda]\|$, if it exists, is zero. Since $\mathcal{A}$ is simple, Lemma 4 implies that any norm limit of x^Λ would be a multiple of the identity. This can only be the case if x is a multiple of the identity (see [14]). Therefore the norm limits of the average position and momentum observables P^Λ, Q^Λ do not exist.

But it can be shown [17] that the strong operator limits

$$s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_\omega(x^\Lambda)\in\pi_\omega(\mathcal{A})'',$$

where π_ω denotes the GNS representation of $\mathcal{A}$ associated to the state ω, exist for many states ω on $\mathcal{A}$, in particular for the permutation invariant states. The existence of this limit is equivalent to the existence of the limit with ω replaced by a state quasiequivalent to ω.

Lemma 5. For any normal state ω on $\mathcal{A}$ and for any $x\in\mathcal{B}(L^2(\mathbb{R}^3))$, the strong operator limits $s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_\omega(x^\Lambda)$ of the average elements x^Λ, if they exist, are in the center of $\pi_\omega(\mathcal{A})''$.

Lemma 5. It follows directly from Lemma 4 because for all $y\in\mathcal{A}$ we have

$$[\pi_\omega(y), s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_\omega(x^\Lambda)]=s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_\omega([y,\iota x^\Lambda])=0.$$

Now we will define the average position and momentum observables Q^Π, P^Π of the infinite quantum system. Denote by π_u the universal representation of $\mathcal{A}$ and by $\mathcal{H}_u$ its Hilbert space. For any state ω on $\mathcal{A}$ we identify $\pi_\omega(x)$ with $\pi_u(x)c(\pi_\omega)$, where $c(\pi_\omega)$ is the central support of the representation π_ω. Define P_G as the largest central projection C in $\mathcal{A}^{**}$ for which the limits

$$P^\Pi_{\pm,\epsilon}:=s\text{-}\lim_\Lambda\pi_u[(1+\epsilon P^\Lambda_\pm)^{-1}P^\Lambda_\pm]C,$$

$$Q^\Pi_{\pm,\epsilon}:=s\text{-}\lim_\Lambda\pi_u[(1+\epsilon Q^\Lambda_\pm)^{-1}Q^\Lambda_\pm]C$$

exist. $P_G\mathcal{A}^{**}$ is the direct sum of all representations associated to states ω for which the strong operator limits of the averaged observables x^Λ exist. From Lemma 5 we know that $P^\Pi_{\pm,\epsilon}, Q^\Pi_{\pm,\epsilon}$ are in the center of $P_G\mathcal{A}^{**}$. Then we can define positive unbounded self-adjoint operators $P^\Pi_\pm, Q^\Pi_\pm$ affiliated with the center of $P_G\mathcal{A}^{**}$ by

$$\rho(P^\Pi_\pm):=\lim_{\epsilon\to 0}\rho(P^\Pi_{\pm,\epsilon}),\qquad \rho(Q^\Pi_\pm):=\lim_{\epsilon\to 0}\rho(Q^\Pi_{\pm,\epsilon})$$

for every $\rho\in(P_G\mathcal{A}^{**})^1_*$. Now take $P^\Pi:=P^\Pi_+-P^\Pi_-$ and $Q^\Pi:=Q^\Pi_+-Q^\Pi_-$. Q^Π and P^Π can formally be written as

$$Q^\Pi=s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_u(Q^\Lambda)P_G, \tag{17}$$

$$P^\Pi=\text{s-}\lim_{\Lambda\in\mathbf{\Lambda}}\pi_u(P^\Lambda)P_G. \tag{18}$$

Again, the reader less worried about technical details can think of P^Π, Q^Π as being defined by (17), (18). They are the average position and momentum observables of the infinite system. By Lemma 5 they are affiliated to the center of $P_G\mathcal{A}^{**}$ and therefore commute with each other.

Define $\mathcal{M}^\Pi$ to be the von Neumann subalgebra of $P_G\mathcal{A}^{**}$ generated by $\exp(iaP^\Pi)$ and $\exp(ibQ^\Pi)$:

$$\mathcal{M}^\Pi:=\{\exp(iaP^\Pi)\exp(ibQ^\Pi):a,b\in\mathbf{R}^3\}''\subset P_G\mathcal{A}^{**}. \tag{19}$$

Loosely speaking (i.e., neglecting the fact that P^Π, Q^Π are unbounded and therefore not contained in $\mathcal{M}^\Lambda$) one can say that $\mathcal{M}^\Pi$ is the von Neumann algebra generated by average position and momentum Q^Π, P^Π. $\mathcal{M}^\Pi$ contains exactly the strong limits of the averaged observables (8), and is contained in the center of $P_G\mathcal{A}^{**}$.

So we arrive at Theorem 3.

Theorem 3. There exists a covariant PV measure E_G on $\mathbb{R}^3\times\widehat{\mathbb{R}^3}$ with values in the von Neumann algebra $\mathcal{M}^\Pi$ generated by the average position and momentum. E_G can be extended to a (covariant) isomorphism between $L^\infty(\mathbb{R}^3\times\widehat{\mathbb{R}^3})$ and $\mathcal{M}^\Pi$.

Proof. The only thing which we did not prove yet is the existence of a covariant PV measure on $\mathbb{R}^3\times\widehat{\mathbb{R}^3}$ with values in $\mathcal{M}^\Pi$.

Due to the linear dependence of aP^Π on a and of bQ^Π on b,

$$(\hat{b},a)\mapsto\exp(ibQ^\Pi)\exp(iaP^\Pi)$$

is a strongly continuous unitary representation of the additive group $\widehat{\mathbb{R}}^3\times\mathbb{R}^3$ (the dual of $\mathbb{R}^3\times\widehat{\mathbb{R}^3}$) in the Hilbert space $P_G\mathcal{H}_u$. According to a corollary of Stone's theorem ([13], Theorem VIII.12) there is a unique projection valued measure E_G on the dual group of $\widehat{\mathbb{R}}^3\times\mathbb{R}^3$ with values in the central projectors of $P_G\mathcal{A}^{**}$ such that

$$\langle\phi, \exp(ibQ^{\Pi})\exp(iaP^{\Pi})\psi\rangle = \int_{\mathbb{R}^3\times\widehat{\mathbb{R}^3}} (s,\hat{t})(\hat{b},a)\langle\phi, E_G(ds,d\hat{t})\psi\rangle$$

for all $\phi,\psi \in P_G\mathcal{H}_u$. [Here we identified $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$ with the dual of $\widehat{\mathbb{R}}^3 \times \mathbb{R}^3$ and wrote $(s,\hat{t})(\hat{b},a)$ for the value of the character $(s,\hat{t})$ on the group element $(\hat{b},a) \in \widehat{\mathbb{R}}^3 \times \mathbb{R}^3$.] The support of E_G is equal to the spectra of its self-adjoint generators P^{Π}, Q^{Π} [see, e.g., [18], Proposition 3.2.40, (6)], and so is equal to the whole of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$.

From the definition of P_G it is obvious that it is invariant under $\overline{\alpha}$. Also, using the proper definition of P^{Π}, Q^{Π} one verifies the following formal calculation:

$$\begin{aligned}\overline{\alpha}_{(a,\hat{b})}(P^{\Pi}) &\overset{(18)}{=} \overline{\alpha}_{(a,\hat{b})}(s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}} \pi_u(P^{\Lambda})P_G) \\ &\overset{(11,6)}{=} s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}} \pi_u(P^{\Lambda} - b\mathbf{1}^{\Lambda})P_G \\ &\overset{(18)}{=} P^{\Pi} - b\, s\text{-}\lim_{\Lambda\in\mathbf{\Lambda}} \pi_u(\mathbf{1})P_G \\ &= P^{\Pi} - b\mathbf{1}^{\Pi},\end{aligned}$$

and similarly for Q^{Π}:

$$\bar{\alpha}_{(a,\hat{b})}(Q^{\Pi}) = Q^{\Pi} - a\mathbf{1}^{\Pi}.$$

(Here $\mathbf{1}^{\Pi}$ denotes the identity in the von Neumann algebra $P_G\mathcal{A}^{**}$.) This can be written as covariance of the measure E_G:

$$\bar{\alpha}_{(a,\hat{b})}(E_G(\Delta)) = E_G([\Delta] + (a,\hat{b}))$$

for all Borel sets Δ of $\mathbb{R}^3 \times \widehat{\mathbb{R}^3}$. Q.E.D.

Remarks. (1) The action $\overline{\alpha}$ on the von Neumann algebra $\mathcal{M}^{\Pi}$ is ergodic. (This can be seen by taking the strong limit of Lemma 3. The proof of Lemma 3 does not depend on $|\Lambda|$.) Let us compare this with Theorem 1: The existence of a covariant representation in spite of the ergodicity of the action is due to the fact that $\mathcal{M}^{\Pi}$ is Abelian, whereas for $|\Lambda|$ finite, $\mathcal{M}^{\Lambda}$ is a factor.

(2) E_G is a joint position-momentum observable derived from the limit of the average position and momentum observables. It describes the average position and momentum of the infinite quantum system. Since E_G is a projection valued measure, it describes a *sharp* joint position-momentum observable. Furthermore, since E_G takes values in the *center* of $P_G\mathcal{A}^{**}$, average position and momentum are classical observables.

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APPENDIX: COVARIANT POV MEASURES

Davies and Lewis [3], as well as Ludwig [4], proposed a generalized concept of observable which we use in the specific context of von Neumann algebras. Let $\mathcal{M}$ be a W^* algebra with separable predual. So $\mathcal{M}$ is isomorphic to a σ-weakly closed subalgebra of the bounded operators on some separable Hilbert space.

Definition. A *POV measure* on a Borel space $(X, \Sigma(X))$ is a mapping a from $\Sigma(X)$ into $\mathcal{M}$ such that (i) $a(X) = \mathbf{1}$, (ii) $\mathbf{0} \le a(\Delta) \le \mathbf{1}$ for all $\Delta \in \Sigma(X)$, and (iii) for each family $\{\Delta_i\}$ of mutually disjoint subsets Δ_i of X we have $a(\bigcup_{i=1}^{\infty}\Delta_i) = \sum_{i=1}^{\infty} a(\Delta_i)$, where the right hand side converges in the σ-weak topology of $\mathcal{M}$. If we have $a(\Delta) = a(\Delta)^2$ for all $\Delta \in \Sigma(x)$, then a is a projection-valued measure (PV measure).

Definition. A *W^* system* $(\mathcal{M}, G, \alpha)$ consists of a W^* algebra $\mathcal{M}$, a locally compact, separable group G, and a representation $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ of G as a group of automorphisms of $\mathcal{M}$ such that (i) $\alpha_{g_1g_2} = \alpha_{g_1}\alpha_{g_2}$ and (ii) for all operators $x \in \mathcal{M}$ the function $g \mapsto \alpha_g(x)$ is σ-weakly continuous.

Example. The systems of traditional quantum mechanics can be regarded as W^* systems in the following way. A quantum mechanical system is specified by a unitary ray representation U of a kinematical group G, which is, for example, the Poincaré group or the Galilei group. This unitary representation U fulfills

$$U(g_1)U(g_2) = c(g_1,g_2)U(g_1g_2), \quad g \in G$$

where c is a complex number of modulus 1. Associated to U is a representation α of G as a group of automorphisms of $\mathcal{B}(\mathcal{H})$ defined by

$$\alpha_g(x) := U_g x U_g^*, \quad g \in G, x \in \mathcal{B}(\mathcal{H}).$$

Then $(\mathcal{B}(\mathcal{H}), G, \alpha)$ is a type I factor W^* system. The choice of a type I factor for $\mathcal{M}$ is a consequence of von Neumann's irreducibility postulate.

Conversely, every W^* system, where $\mathcal{M}$ is a factor of type I, can be brought into the form $(\mathcal{B}(\mathcal{H}), G, U\cdot U^*)$. This is due to the fact that all automorphisms α of a type I factor are inner: they are induced by a unitary operator $U \in \mathcal{M}$ by $\alpha(x) = UxU^*$.

Example. Systems of classical mechanics can also be regarded as W^* systems. Such a system is described by a symplectic manifold X and a representation $s_g : X \to X$ of G by canonical transformations. To each real-valued function on X there corresponds an observable. The *-algebraic operations on the observables are defined pointwise. The Liouville measurable and essentially bounded functions form a W^* algebra $L^{\infty}(X)$. We define

$$[\alpha_g(f)](x) := f(s_{g^{-1}}(x)), \quad g \in G, x \in X, f \in L^{\infty}(X).$$

If s is measurable and X is separable, α is pointwise σ-weakly continuous and $(L^{\infty}(X), G, \alpha)$ is a W^* system. Statistical states correspond to normalized L_1 functions on X.

Definition. A W^* system $(\mathcal{M}, G, \alpha)$ is *ergodic* if the equation

$$\alpha_g(x) = x \quad \forall g \in G$$

is satisfied only by multiples $x = \lambda\mathbf{1}$ of the identity operator.

Definition. A W^* system $(\mathcal{M}, G, \alpha)$ is called *integrable* if the set

$$\left\{ x \in \mathcal{M} : \int_G \alpha_g(x^*x)dg < \infty \right\}$$

is σ-weakly dense in $\mathcal{M}$. [Here $\int_G \alpha_g(x^*x)dg < \infty$ means that the set $\{\int_K \alpha_g(x^*x)dg : K \subset G, K \text{ compact}\}$ is bounded in $\mathcal{M}$. The integral $\int_K \alpha_g(x^*x)dg$ is taken in the σ-weak topology.]

Remark. Every W^* system $(\mathcal{M}, G, \alpha)$ of a compact group G is integrable. Some results of the theory of integrable ergodic W^* systems can be found in [19].

Definition. A POV measure a on a transitive G space X with values in $\mathcal{M}$, together with a pointwise σ-weakly continuous representation $\alpha : g \mapsto \alpha_g \in \text{Aut}(\mathcal{M})$ of G is called a *system of covariance* (a, α) *based on* X if a acts covariantly with respect to α,

$$\alpha_g(a(\Delta)) = a(g\Delta),$$

where the Borel set $g\Delta$ is obtained from the Borel set Δ by pointwise action of G. If a is projection valued then (a, α) is called a *transitive system of imprimitivity* [20].

Remarks. (1) Since G acts transitively on X, X is homeomorphic to G/H for some closed subgroup $H \subseteq G$.

(2) Traditionally systems of imprimitivity as introduced by Mackey [20] are covariant PV measures based on a generalized configuration space on which a symmetry group acts transitively. The covariance condition leads to canonical commutation relations between the configuration observables and the generators of the representation $U(G)$. By admitting for systems of covariance not only projection-valued measures it becomes possible to study covariant measures on, for example, the phase space, and not only on the configuration space.

(3) Traditionally (see, e.g., the review of Ali [2]) systems of covariance are defined as POV measures which act covariantly with respect to a unitary ray representation of G. But defining covariance with respect to unitary ray representations of G is only possible in the special case that $\mathcal{M}$ has separable predual and that all automorphisms α_g are inner. Then one can replace the automorphic representation of G by a σ-weakly measurable unitary ray representation. In general automorphisms α need not be representable by unitary ray representations as $\alpha = U \cdot U^*$, even if we allow for $U \notin \mathcal{M}$. Our more general definition allows also for the description of classical systems and of systems with classical *and* quantum properties.

(4) Every covariant POV measure a based on G/H with values in a von Neumann algebra $\mathcal{M}$ can be extended σ-weak continuously to a positive, linear, normalized, covariant map ϕ from $L^\infty(G/H)$ into $\mathcal{M}$. ϕ and a are related by $a(\Delta) = \phi(\chi_\Delta)$, where $\Delta \in \Sigma(G/H)$ and where χ_Δ is the characteristic function of the Borel set Δ. ϕ is a covariant *representation* of $L^\infty(G/H)$ in $\mathcal{M}$ if and only if a is a PV measure.

(5) The map ϕ has been called a covariant embedding [8] and is closely related to generalized coherent states [21, 2]. ϕ maps functions on phase space G/H (classical observables) into operators on a Hilbert space (quantum observables). It can be regarded as a quantization map (see [22]).

(6) The map ϕ induces a dual map $\phi^* : \mathcal{M}^* \to L^\infty(G/H)^*$ defined by $\phi^* : \rho \mapsto \rho \circ \phi$. The covariant embedding ϕ is called *normal* if it satisfies $\phi^*(\mathcal{M}_*) \subseteq L_1(G/H)$.

(7) If G/H is a phase space, a normal covariant embedding ϕ induces a *phase space representation* ϕ^* (see, e.g., [2]). ϕ^* associates to every density matrix $\rho \in \mathcal{M}_*$ a positive normalized L_1 function $\phi^*(\rho)$ on G/H such that

$$\text{tr}[\rho\phi(f)] = \int_{G/H} [\phi^*(\rho)](sH) f(sH) d\mu(sH)$$

for all $f \in L^\infty(G/H)$. In particular, if the system is in a state ρ the probability of getting a measurement result in the Borel set $\Delta \subset G/H$ is given by

$$\int_\Delta [\phi^*(\rho)](gH)d(gH) = \text{tr}[\rho\phi(\chi(\Delta))].$$

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