Displaced and squeezed parity operator: Its role in classical mappings of quantum theories

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The displaced parity operators are shown to have properties that bear a deep relationship with those of the Wigner functions. By exploiting these properties we show that these operators play an important role in linking together many of the aspects of the various exact phase-space mappings of quantum mechanics. These include the Wigner and Weyl representations, coherent states and the Bargmann representation, the P and Q representations, the Weyl correspondence, and the Moyal star product formalism. We also introduce corresponding displaced Fourier operators and show that their squares are just the displaced parity operators. The formalism is extended to squeezed and displaced parity operators, and their corresponding central role in the theory of squeezed coherent states and general squeezing is explained. We also elucidate the part played by the displaced parity operators in the Moyal star product and its extensions, as a first step towards a potential application of these operators in such modern developments as deformation theory and quantum groups. Finally, we indicate how the apparatus developed might also find applications in other recent exact classical mappings of many-particle quantum mechanics or quantum field theory, which are *not* special cases of deformation theory. Prime examples here include the powerful so-called independent-cluster method techniques, which incorporate the coupled-cluster method formalism with its inbuilt supercoherent states. Throughout the work we stress the central and unifying role played by the displaced parity operator and its generalizations.

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I. INTRODUCTION

The displayed parity operator has recently been considered by several authors [1–4] within the context of providing an integral form which gives a direct correspondence (namely, a unitary map) between a function defined on the classical phase space and the equivalent operator in the quantum-mechanical Hilbert space. This work is thus clearly part of a long tradition of generalized phase-space techniques for the treatment of problems in quantum mechanics and quantum statistical mechanics, which originated with the pioneering work of Weyl [5] and Wigner [6]. The latter work, in particular, first introduced what is now known as the Wigner distribution function. Important work by Groenewold [7] and Moyal [8] also developed Wigner's techniques further. These authors especially laid the basis for showing that the Wigner correspondence between classical and quantum-mechanical functions implies a definite rule of ordering for the noncommuting operators in the latter case, namely, the so-called Weyl ordering or Weyl symmetrization.

Other rules of association have been developed between classical phase-space functions and their corresponding quantum-mechanical Hilbert space operators, some of which also lead to alternative phase-space distribution functions [9–17]. For example, in quantum optics particular use has been made of both the P representation of Glauber [12] and Sudarshan [13] and the Q representation of Husimi [9] and Kano [15], and their implied rules of association for the noncommuting operators of normal ordering and antinormal ordering, respectively. Each such phase-space distribution function has lent itself to a reformulation of problems in quantum mechanics or quantum statistical mechanics as equivalent quasiclassical problems, where the distribution function plays the role of a quasiprobability distribution in the appropriate phase space. The relationships between the various distribution functions or, equivalently, between the various representations of a given quantum-mechanical operator have been investigated by many authors [18–21].

Although the Wigner distribution function was originally defined in terms of the Schrödinger wave function, both it and the alternative distribution functions find a more natural formulation in the Bargmann Hilbert space [22,23] and the corresponding holomorphic (or Bargmann) representation of wave functions. A very natural role is thus played by the (Glauber) coherent states. Furthermore, these original coherent states have themselves now been generalized to the so-called squeezed (or two-photon or paired) coherent states [24–27]. The use of the Wigner function for such squeezed states has been suggested by a number of different authors [28–30], and a recent rather complete discussion of the linear canonical transformations of coherent and squeezed states in the Wigner phase space has been given in Ref. [31].

Finally, whereas all of the previous phase-space representations of quantum-mechanical operators have been based on pure (coherent) states, the present authors [32] have considered the generalization to the so-called *coherent mixed states.* The particular case of *thermal coherent states,* which describe displaced harmonic oscillators in thermodynamic equilibrium with a heat bath at some nonzero temperature T, was studied in detail. It was shown how an equivalent description of this formalism can be given in terms of a superposition of coherent states and thermal noise and thus how the formulation applies to the common and important practical situation of an admixture of coherent signal and noise. We also showed how the usual P and Q representations can thereby be generalized into their nonzero-temperature counterparts, and we demonstrated their explicit relationships to the Wigner-Weyl representation.

The intention of the present paper is to show how all of the above developments can be drawn together by exploiting the properties of the displaced and squeezed parity operator. Apart from shedding light on each of these topics, especially by exploring deeper connections between them, we also indicate how further extensions and applications may be practicable for other situations. These include theories in which a central role is played by some *star product* analogous to that of Moyal [8] for the Wigner representation of the product of two operators in terms of their individual Wigner representations.

The outline of the remainder of the paper is as follows. After a preliminary discussion in Sec. II of the parity operator and relationships between it and the displacement operator, the displaced parity operator is examined in detail in Sec. III. The Weyl and Wigner representations are then considered in depth in Sec. IV as an illustrative example of the previous formalism. The relationship of the parity operator with the Fourier operator that generates Fourier transforms is explored in Sec. V, and the formalism is extended in Sec. VI to incorporate squeezing as well as displacement. The Moyal star product and related classical mappings of quantum theories are discussed in Sec. VII. Finally, our results are summarized in Sec. VIII, where we also indicate further extensions and applications of them.

II. PRELIMINARIES AND THE PARITY OPERATOR

We consider the linear harmonic oscillator with mass m and angular frequency ω. Its Hamiltonian is given by

$$H_0=\tfrac{1}{2}m\omega^2\hat{x}^2+\frac{\hat{p}^2}{2m}=\hbar\omega(a^\dagger a+\tfrac{1}{2}I)\ , \tag{2.1}$$

in terms of the identity operator I and creation and destruction operators $a^\dagger$ and a, respectively,

$$\begin{aligned} a^\dagger &\equiv (2m\hbar\omega)^{-1/2}(m\omega\hat{x}-i\hat{p})\ ,\\ a &\equiv (2m\hbar\omega)^{-1/2}(m\omega\hat{x}+i\hat{p})\ , \end{aligned} \tag{2.2}$$

which obey the usual canonical commutation relations of the Heisenberg algebra $\mathcal{A}$,

$$[\hat{x},\hat{p}]=iI \Longleftrightarrow [a,a^\dagger]=I\ . \tag{2.3}$$

As always, all of the ensuing formalism also applies to a single mode of the electromagnetic field of angular frequency ω, if we replace m by $\hbar\omega/c^2$. Henceforth, we freely use units where $\hbar=m=\omega=1$. The eigenstates of H_0 are the usual orthonormal set of number eigenstates $\{|n\rangle; n=0,1,2,\dots\}$,

$$\begin{aligned} a^\dagger a|n\rangle &= n|n\rangle\ ,\\ |n\rangle &= (n!)^{-1/2}(a^\dagger)^n|0\rangle\ ,\quad n=0,1,2,\dots\ , \end{aligned} \tag{2.4}$$

where the vacuum state $|0\rangle$ is defined as usual by

$$a|0\rangle=0\ . \tag{2.5}$$

These states provide a resolution of the identity operator

$$\sum_{n=0}^{\infty}|n\rangle\langle n|=I\ . \tag{2.6}$$

The corresponding Hilbert space $\mathcal{H}$ is now decomposed as the direct sum

$$\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\ , \tag{2.7}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the subspaces spanned, respectively, by the odd number eigenstates and the even number eigenstates. We introduce the corresponding projection operators onto the two subspaces as follows:

$$\begin{aligned} \Pi_1 &\equiv \sum_{n=0}^{\infty}|2n+1\rangle\langle 2n+1|\ ,\\ \Pi_2 &\equiv \sum_{n=0}^{\infty}|2n\rangle\langle 2n|\ . \end{aligned} \tag{2.8}$$

They have the following trivial properties on $\mathcal{H}$:

$$\begin{aligned} \Pi_1+\Pi_2 &= I\ ,\\ \Pi_i\Pi_j &= \Pi_i\delta_{ij}\ ,\quad i,j=1,2\ . \end{aligned} \tag{2.9}$$

The *parity operator* U_0 is then readily defined in terms of these projectors as

$$U_0\equiv\Pi_2-\Pi_1=\exp(i\pi a^\dagger a)\ , \tag{2.10}$$

where the equality follows readily by post-multiplication with the identity operator of the form of Eq. (2.6). It has the easily proven properties

$$U_0=U_0^\dagger\ ,\quad U_0^2=I \tag{2.11}$$

and is thus clearly an observable with eigenvalues ±1.

We define position and momentum eigenfunctions $|x\rangle_x$ and $|p\rangle_p$, respectively, as usual,

$$\hat{x}|x_0\rangle_x=x_0|x_0\rangle_x\ ,\quad \hat{p}|p_0\rangle_p=p_0|p_0\rangle_p\ , \tag{2.12}$$

with normalizations as implied by the following resolutions of the identity operator:

$$I=\int_{-\infty}^{\infty}dx'|x'\rangle_{x\,x}\langle x'|\ ,\quad I=\int_{-\infty}^{\infty}dp'|p'\rangle_{p\,p}\langle p'|\ , \tag{2.13}$$

and hence, to be compatible with Eq. (2.3), with the explicit overlap

$$ {}_x\langle x'|p'\rangle_p=(2\pi)^{-1/2}e^{ip'x'} . \quad (2.14)$$

We readily prove from Eqs. (2.2) and (2.5) that the coordinate-space (or Schrödinger) and momentum-space representations of the (normalized) vacuum state are given, respectively, by

$$ {}_x\langle x|0\rangle=\pi^{-1/4}e^{-(1/2)x^2} , \quad {}_p\langle p|0\rangle=\pi^{-1/4}e^{-(1/2)p^2} , \quad \langle 0|0\rangle=1 . \quad (2.15)$$

By making explicit use of the parity relations for the number eigenstates

$$ {}_x\langle x_0|n\rangle=(-1)^n {}_x\langle x_0|n\rangle , \quad {}_p\langle p_0|n\rangle=(-1)^n {}_p\langle p_0|n\rangle , \quad (2.16)$$

we readily prove

$$ U_0|x_0\rangle_x=|-x_0\rangle_x , \quad U_0|p_0\rangle_p=|-p_0\rangle_p , \quad (2.17)$$

as expected.

We similarly introduce the usual set of (normalized) canonical coherent states $\{|z\rangle; z\in\mathbb{C}\}$ as the eigenstates in $\mathcal{H}$ of the destruction operator

$$ a|z\rangle=z|z\rangle . \quad (2.18)$$

They are readily constructed explicitly either in terms of the number eigenstates as

$$ |z\rangle=\exp(-\tfrac{1}{2}|z|^2)\sum_{n=0}^{\infty}(n!)^{-1/2}z^n|n\rangle \quad (2.19)$$

or in terms of the unitary Weyl displacement operator $D(z)$ as

$$ |z\rangle=D(z)|0\rangle , \quad D(z)\equiv\exp(za^\dagger-z^*a) . \quad (2.20)$$

One again easily proves that the action of the parity operator on the coherent states is given by

$$ U_0|z=\zeta\rangle=|z=-\zeta\rangle . \quad (2.21)$$

One may also prove directly from Eq. (2.10)

$$ U_0aU_0^\dagger=-a , \quad U_0a^\dagger U_0=-a^\dagger \quad (2.22)$$

and hence that

$$ U_0f(a,a^\dagger)U_0^\dagger=f(-a,-a^\dagger) \quad (2.23)$$

for an arbitrary regular function $f=f(a,a^\dagger)$. In particular we have

$$ U_0D(z)U_0^\dagger=D(-z)\Longleftrightarrow U_0D(z)=D(-z)U_0 . \quad (2.24)$$

The coordinate-space and momentum-space representations of the coherent states are readily shown to be given by

$$ {}_x\langle x|z\rangle=\pi^{-1/4}\exp(-\tfrac{1}{2}|z|^2-\tfrac{1}{2}z^2+\sqrt{2}xz-\tfrac{1}{2}x^2) , \quad (2.25a)$$

$$ {}_p\langle p|z\rangle=\pi^{-1/4}\exp(-\tfrac{1}{2}|z|^2+\tfrac{1}{2}z^2-\sqrt{2}ipz-\tfrac{1}{2}p^2) \quad (2.25b)$$

and the overlap between two coherent states by

$$ \langle z|z'\rangle=\exp(-\tfrac{1}{2}|z|^2-\tfrac{1}{2}|z'|^2+z^*z') . \quad (2.26)$$

By making use of Eqs. (2.2) and (2.20), together with the simple Baker-Campbell-Hausdorff relation

$$ e^{A+B}=e^Ae^Be^{-(1/2)[A,B]} , \quad (2.27)$$

valid for any two operators A and B, each of which commutes with their mutual commutator, we find that the displacement operator can be written in the form

$$ D(z)=\exp(iz_Rz_I)\exp(-\sqrt{2}iz_R\hat{p})\exp(\sqrt{2}iz_I\hat{x}) , \quad (2.28)$$

where $z_R\equiv\text{Re}z$ and $z_I\equiv\text{Im}z$. Equation (2.28) leads simply to the following relations for the matrix elements of $D(z)$:

$$ {}_p\langle p|D(z)|x\rangle_x=(2\pi)^{-1/2}\exp[i(z_Rz_I-\sqrt{2}iz_Rp+\sqrt{2}iz_Ix-px)] \quad (2.29)$$

and

$$ {}_x\langle x'|D(z)|x\rangle_x=\exp[iz_I(x+x')/\sqrt{2}]\times\delta(x-x'+\sqrt{2}z_R) , \quad (2.30a)$$

$$ {}_p\langle p'|D(z)|p\rangle_p=\exp[-iz_R(p+p')/\sqrt{2}]\times\delta(p-p'+\sqrt{2}z_I) . \quad (2.30b)$$

Equations (2.30a) and (2.30b) readily yield that

$$ \text{Tr}D(z)=\pi\delta(z_I)\delta(z_R)\equiv\pi\delta^{(2)}(z) . \quad (2.31)$$

They also lead to the important relations

$$ \int_{-\infty}^{\infty}dz_RD(z)=\sqrt{2}\pi|z_I/\sqrt{2}\rangle_{p\,p}\langle -z_I/\sqrt{2}| , \quad (2.32a)$$

$$ \int_{-\infty}^{\infty}dz_ID(z)=\sqrt{2}\pi|z_R/\sqrt{2}\rangle_{x\,x}\langle -z_R/\sqrt{2}| . \quad (2.32b)$$

In turn, either Eq. (2.32a) or (2.32b) leads to the fundamental relation between the displacement and parity operators

$$ \int\frac{d^2z}{2\pi}D(z)=U_0 , \quad d^2z\equiv d(\text{Re}z)d(\text{Im}z)=(2i)^{-1}dz\,dz^* , \quad (2.33)$$

where we have made use of Eqs. (2.11), (2.13), and (2.17).

Equations (2.30a) and (2.30b) are particularly useful, for example, in deriving relationships between different representations of state vectors or operators in $\mathcal{H}$, which can otherwise be difficult to derive and/or whose origins often remain obscure. Thus a state vector $|f\rangle\in\mathcal{H}$ may be represented by its Schrödinger (or coordinate-space) representation $f_x(\alpha)$ or its Fourier-inverse momentum-space representation $f_p(\beta)$ defined, respectively, as

$$ f_x(\alpha)\equiv{}_x\langle\alpha|f\rangle , \quad f_p(\beta)\equiv{}_p\langle\beta|f\rangle . \quad (2.34)$$

Alternatively, the Bargmann (or holomorphic) representation [22,23] of a state vector $|f\rangle$, whose decomposition in the number-state basis is

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle , \tag{2.35}$$

is defined as

$$f_B(z) \equiv \exp(\tfrac{1}{2}|z|^2)\langle z^*|f\rangle = \sum_{n=0}^{\infty} (n!)^{-1/2} f_n z^n . \tag{2.36}$$

We thus have

$$\langle 0|D(-z^*)|f\rangle = \exp(-\tfrac{1}{2}|z|^2) f_B(z) . \tag{2.37}$$

For all normalizable states $|f\rangle \in \mathcal{H}$, the function $f_B(z)$ is a holomorphic or entire function of the complex number z of order $\rho \le 2$ (and type $\tau \le \frac{1}{2}$ if $\rho = 2$). By taking matrix elements of Eqs. (2.32a) and (2.32b) between states $\langle 0|$ and $|f\rangle$, and by making use of Eq. (2.15), we find

$$\int_{-\infty}^{\infty} dz_R \exp(-\tfrac{1}{2}z_R^2) f_B(-z^*) = 2^{1/2}\pi^{3/4}\exp(\tfrac{1}{4}z_I^2) f_p(-2^{-1/2}z_I) , \tag{2.38a}$$

$$\int_{-\infty}^{\infty} dz_I \exp(-\tfrac{1}{2}z_I^2) f_B(-z^*) = 2^{1/2}\pi^{3/4}\exp(\tfrac{1}{4}z_R^2) f_x(-2^{-1/2}z_R) . \tag{2.38b}$$

III. THE DISPLACED PARITY OPERATOR

The displacement operator $D(z)$ does not commute with the projection operators Π_1 and Π_2 of Eq. (2.8). Hence the decomposition of Eq. (2.7) of the Hilbert space $\mathcal{H}$ into $\mathcal{H}_1$ and $\mathcal{H}_2$ is not invariant under the action of $D(z)$. In order to define the "displaced version" of Eq. (2.7) we introduce the displaced number eigenstates

$$|n;z\rangle \equiv D(z)|n\rangle \tag{3.1}$$

and generalize the operators Π_1 and Π_2 into their displaced counterparts

$$\Pi_1(z) \equiv \sum_{n=0}^{\infty} |2n+1;z\rangle\langle 2n+1;z| = D(z)\Pi_1 D^\dagger(z) ,$$
$$\Pi_2(z) \equiv \sum_{n=0}^{\infty} |2n;z\rangle\langle 2n;z| = D(z)\Pi_2 D^\dagger(z) . \tag{3.2}$$

They clearly obey relationships analogous to those expressed in Eq. (2.9). We may also write as the analogous extension of Eq. (2.7)

$$\mathcal{H} = \mathcal{H}_1(z) \oplus \mathcal{H}_2(z) , \tag{3.3}$$

where $\mathcal{H}_1(z)$ and $\mathcal{H}_2(z)$ are the Hilbert subspaces spanned, respectively, by the odd and even displaced number eigenstates.

The displaced parity operator $U(z)$ is now defined as

$$U(z) \equiv D(z) U_0 D^\dagger(z) = \exp[i\pi(a^\dagger - z^* I)(a - zI)] , \tag{3.4}$$

where, in the second line, we have used Eq. (2.10). This definition, together with Eq. (2.11), immediately yields the relations

$$U^2(z) = I, \quad U(z) = U^\dagger(z) \tag{3.5}$$

from which we see that $U(z)$ is an observable, with eigenvalues equal to ± 1. It may also be written in the equivalent forms

$$U(z) = \sum_{n=0}^{\infty} (-1)^n |n;z\rangle\langle n;z| = \Pi_2(z) - \Pi_1(z) . \tag{3.6}$$

By making use of either Eq. (2.13) or Eqs. (2.31) and (2.33), we can evaluate the trace of $U(z)$,

$$\mathrm{Tr}\,U(z) = \mathrm{Tr}\,U_0 = \tfrac{1}{2} , \tag{3.7}$$

which is compatible with the sum of the alternating divergent series obtained from Eq. (3.6). It should be pointed out that the trace of the operator $\exp[(i\pi + \epsilon)a^\dagger a]$, where ϵ is a real infinitesimal, exists for $\epsilon < 0$, but not for $\epsilon > 0$. In the case $\epsilon = 0$, which is of interest to us here, the trace thus exists as the one-sided limit.

Furthermore, by making use of Eq. (2.24), we find

$$U(z) = D(2z) U_0 = U_0 D(-2z) , \tag{3.8}$$

where we have also made use of the group relation

$$D(z_1) D(z_2) = \exp[\tfrac{1}{2}(z_1 z_2^* - z_1^* z_2)] D(z_1 + z_2) \tag{3.9a}$$

for the Weyl displacement operators, which is readily proven from Eqs. (2.20) and (2.27). In fact, by making use of Eq. (3.8), we see that the Weyl operators and the displaced parity operators taken together form a larger group [1,2], with the additional group multiplication properties given by

$$U(z_1) U(z_2) = \exp[2(z_1^* z_2 - z_1 z_2^*)] D(2z_1 - 2z_2) , \tag{3.9b}$$

$$D(z_1) U(z_2) = \exp(z_1 z_2^* - z_1^* z_2) U(\tfrac{1}{2}z_1 + z_2) , \tag{3.9c}$$

$$U(z_1) D(z_2) = \exp(z_1^* z_2 - z_1 z_2^*) U(z_1 - \tfrac{1}{2}z_2) . \tag{3.9d}$$

We may straightforwardly show from Eqs. (2.17) and (2.28) that the mode of action of the displaced parity operator on the position and momentum eigenstates is given by

$$U(z)|x\rangle_x = \exp[iz_I(4z_R - 2^{3/2}x)]|-x + 2^{3/2}z_R\rangle_x , \tag{3.10a}$$

$$U(z)|p\rangle_p = \exp[-iz_R(4z_I - 2^{3/2}p)]|-p + 2^{3/2}z_I\rangle_p , \tag{3.10b}$$

in the derivation of which we have made use of the simply proven relationships

$$e^{-i\alpha\hat{p}}|x\rangle_x = |x+\alpha\rangle_x , \quad e^{i\beta\hat{x}}|p\rangle_p = |p+\beta\rangle_p . \tag{3.11}$$

Similarly, the mode of action of the displaced parity operator on the position and momentum operators is given by

$$U(z)\hat{x}U^\dagger(z) = -\hat{x} + 2^{3/2}z_R , \tag{3.12a}$$

$$U(z)\hat{p}U^\dagger(z) = -\hat{p} + 2^{3/2}z_I . \tag{3.12b}$$

By making use of Eqs. (2.32a), (2.32b), and (3.8), we may also prove the important relations

$$\int_{-\infty}^{\infty} dz_R\, U(z) = 2^{-1/2}\pi|\sqrt{2}z_I\rangle_{p\,p}\langle\sqrt{2}z_I| \ , \qquad (3.13a)$$

$$\int_{-\infty}^{\infty} dz_I\, U(z) = 2^{-1/2}\pi|\sqrt{2}z_R\rangle_{x\,x}\langle\sqrt{2}z_R| \ , \qquad (3.13b)$$

from either one of which we may also prove

$$\frac{2}{\pi}\int d^2z\, U(z) = I \ , \qquad (3.14)$$

with the aid of Eq. (2.13).

We next consider the Q representations of the operators $D(z)$ and $U(z)$, where the Q representation of an arbitrary operator Θ is defined in the usual way as

$$Q(\Theta;\zeta) \equiv \langle\zeta|\Theta|\zeta\rangle \ . \qquad (3.15)$$

By making use of Eqs. (2.20), (2.26), and (3.9a), we readily find

$$Q(D(z);\zeta) = \exp(-\tfrac{1}{2}|z|^2 + z\zeta^* - z^*\zeta) \qquad (3.16)$$

and

$$Q(U(z);\zeta) = \exp(-2|z-\zeta|^2) \ , \qquad (3.17)$$

from which we may also prove the relation

$$\int d^2z\, e^{iz\cdot w} Q(D(-\tfrac{1}{2}iz);\zeta) = 8\pi Q(U(w);\zeta) \ , \qquad (3.18)$$

where we have introduced the convenient scalar product notation

$$z\cdot w \equiv \tfrac{1}{2}(zw^* + z^*w) = z_R w_R + z_I w_I \qquad (3.19)$$

between two arbitrary complex numbers z and w.

We note that Klauder [33] has shown that an operator Θ is uniquely determined by its Q representation in $\mathcal{H}$. Thus, if we write Θ in normal-ordered form, we have, from Eq. (2.18),

$$\Theta = f_n(a^\dagger, a) \Longleftrightarrow Q(\Theta;z) = f_n(z^*, z) \ , \qquad (3.20)$$

where $f_n(a^\dagger,a) = :f_n(a^\dagger,a):$ and the normal-ordering operator :( ): indicates that in every term all of the destruction operators stand to the right of all the creation operators

$$f_n(a^\dagger, a) = \sum_{k,l} \phi_{kl}(a^\dagger)^k a^l \ . \qquad (3.21)$$

We next introduce the notation $\tilde{f}(w) \equiv \tilde{f}(w,w^*)$ for the two-dimensional Fourier transform of the general function $f(z) \equiv f(z,z^*)$,

$$\tilde{f}(w) \equiv \int d^2z\, e^{iz\cdot w} f(z) \Longleftrightarrow f(z) = \int \frac{d^2w}{(2\pi)^2} e^{-iz\cdot w}\tilde{f}(w) \ . \qquad (3.22)$$

Thus, if we define the operator $d(z)$ as

$$d(z) \equiv D(-\tfrac{1}{2}iz) \ , \qquad (3.23)$$

we see from Eq. (3.18) that the Q representations of the operators $d(z)$ and $8\pi U(w)$ are Fourier transforms of each other, as defined by Eq. (3.22). By making use of Eq. (3.20), we now define the *Q-parametric* Fourier transform of an operator $\Theta \equiv \Theta(z)$ defined in terms of a complex parameter z to be that (unique) operator $\tilde{\Theta} \equiv \tilde{\Theta}(w)$ whose Q representative is the Fourier transform with respect to the parameter z of $Q(\Theta;z)$. Hence

$$\tilde{d}(w) = 8\pi U(w) \ . \qquad (3.24)$$

The reader should note very carefully that the Q-parametric Fourier transform of an operator defined here is quite distinct from the functional Fourier transform introduced later in Sec. V, with which it bears no direct relationship and with which it should not be confused.

IV. WEYL AND WIGNER REPRESENTATIONS

Moyal [8] first proved the very important relation for displacement operators

$$\frac{1}{\pi}\int d^2z\langle\alpha|D^\dagger(z)|\gamma\rangle\langle\beta|D(z)|\delta\rangle = \langle\alpha|\delta\rangle\langle\beta|\gamma\rangle \ , \qquad (4.1)$$

which is valid for arbitrary states $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, and $|\delta\rangle$. Equation (4.1) is perhaps most simply proved by employing Eq. (2.13) to put the arbitrary states into either the Schrödinger or momentum-space representations and by making subsequent use of Eqs. (2.29) or (2.30a) and (2.30b). In exactly the same way we also make use of Eq. (3.8) to prove an analogous relation for the displaced parity operators

$$\frac{4}{\pi}\int d^2z\langle\alpha|U(z)|\gamma\rangle\langle\beta|U(z)|\delta\rangle = \langle\alpha|\delta\rangle\langle\beta|\gamma\rangle \ . \qquad (4.2)$$

By assuming that the above four arbitrary states are members of a complete set of states in $\mathcal{H}$, and by expanding the general operator Θ in the usual (nondiagonal) decomposition in terms of them, Eqs. (4.1) and (4.2) can be written in the equivalent forms

$$\Theta = \frac{1}{\pi}\int d^2z\, \mathrm{Tr}[D^\dagger(z)\Theta]D(z) \ , \qquad (4.3a)$$

$$\Theta = \frac{4}{\pi}\int d^2z\, \mathrm{Tr}[U(z)\Theta]U(z) \ . \qquad (4.3b)$$

It is convenient and conventional to make a trivial change of variables in order to rewrite Eq. (4.3a) in the form

$$\Theta = \int \frac{d^2w}{(2\pi)^2}\tilde{W}(\Theta;w)d(w) \ , \qquad (4.4)$$

where the function $\tilde{W}(\Theta;w)$ is defined as

$$\tilde{W}(\Theta;w) \equiv \pi\,\mathrm{Tr}[D(\tfrac{1}{2}iw)\Theta] \ . \qquad (4.5)$$

The representation of Eq. (4.4) for the arbitrary operator Θ as an expansion in terms of displacement operators is just the usual so-called *Weyl representation* and $\tilde{W}(\Theta;w)$ is the corresponding Weyl or W representation of Θ. By making use of the familiar Parseval formula of Fourier analysis, Eq. (4.4) may be rewritten in the form

$$\Theta = \int \frac{d^2z}{(2\pi)^2} W(\Theta;z)\tilde{d}(z) \ , \qquad (4.6)$$

using the notation introduced in Eq. (3.22) for Fourier transforms.

Further use of Eq. (3.24), and a comparison with Eq. (4.3b), then shows that Eq. (4.6) may be rewritten as

$$\Theta=\frac{2}{\pi}\int d^2z\, W(\Theta;z)U(z)\ , \tag{4.7a}$$

where $W(\Theta;z)$, the Fourier transform of the Weyl representation of Θ, is explicitly given by

$$W(\Theta;z)=2\,\mathrm{Tr}[U(z)\Theta]\ . \tag{4.7b}$$

Henceforth, we propose to refer to the representation of Eqs. (4.7a) and (4.7b) for an arbitrary operator Θ as an expansion in terms of the displaced parity operators, as the *Wigner representation,* for reasons that will become clearer below. The Wigner and Weyl representations of Θ, $W(\Theta;z)$ and $\tilde{W}(\Theta;w)$, respectively, are thus simply related by being the Fourier transforms of each other. We note that the Wigner representations of the identity operator I and the parity operator U_0 are given from Eqs. (3.7), (3.8), and (2.31) as

$$W(I;z)=1,\quad W(U_0;z)=\tfrac{1}{2}\pi\delta^{(2)}(z)\ . \tag{4.8}$$

We note that Eqs. (4.4) and (4.8) lead to the Weyl representation for the parity operator

$$U_0=\frac{1}{2\pi}\int d^2\zeta\, D(\zeta)=\frac{1}{2\pi}\int d^2\zeta \exp(\zeta a^\dagger-\zeta^* a) \tag{4.9a}$$

and hence to the following alternative form for the displaced parity operator:

$$U(z)=\frac{1}{2\pi}\int d^2\zeta \exp[\zeta(a^\dagger-z^* I)-\zeta^*(a-zI)]\ . \tag{4.9b}$$

By making use of Eqs. (3.22), (3.23), and (4.4), we may readily show that the operators whose Wigner representations are the simple monomials $(\sqrt{2}z_R)^m(\sqrt{2}z_I)^n$ and $z^m z^{*n}$ are given, respectively, by

$$\begin{aligned} &W(\Theta;z)=(\sqrt{2}z_R)^m(\sqrt{2}z_I)^n \Longleftrightarrow \Theta=\{\hat{x}^m\hat{p}^n\}_W\ ,\\ &W(\Theta;z)=z^m z^{*n}\Longleftrightarrow \Theta=\{a^m(a^\dagger)^n\}_W\ . \end{aligned} \tag{4.10}$$

Here the symbol $\{\hat{A}^m\hat{B}^n\}_W$ represents the so-called Weyl-ordered or Weyl-symmetrized form of the product $\hat{A}^m\hat{B}^n$, namely, the equally weighted linear combination of all possible distinct products involving the operator $\hat{A}$ taken m times and the operator $\hat{B}$ taken n times. It may be expressed formally as

$$\{\hat{A}^m\hat{B}^n\}_W=\left[\frac{\partial}{\partial\sigma}\right]^m\left[\frac{\partial}{\partial\tau}\right]^n \exp(\sigma\hat{A}+\tau\hat{B})|_{\sigma=\tau=0}\ . \tag{4.11}$$

By making use of Eq. (2.30a), we may rewrite Eq. (4.5) in the following more explicit form in terms of position eigenstates:

$$\tilde{W}(\Theta;z)=\pi\int_{-\infty}^{\infty}dx\ {}_x\langle x+2^{-3/2}z_I|\Theta|x-2^{-3/2}z_I\rangle_x \times\exp(i2^{-1/2}z_R x)\ . \tag{4.12}$$

Further, either by directly taking the Fourier transform of Eq. (4.12) or by an explicit evaluation from Eq. (4.8) in terms of position eigenstates we find

$$W(\Theta;z)=\int_{-\infty}^{\infty}dq\ {}_x\langle\sqrt{2}z_R+\tfrac{1}{2}q|\Theta|\sqrt{2}z_R-\tfrac{1}{2}q\rangle_x \times\exp(-i\sqrt{2}z_I q)\ . \tag{4.13a}$$

The latter form is particularly well known as the *Wigner function* in the case that Θ is a density operator ρ; it is for just this reason that, more generally, we now refer to Eqs. (4.7a) and (4.7b) as the Wigner representation of an arbitrary operator Θ.

It is interesting to note that Eq. (4.13a) may also be written in the alternative and particularly suggestive form of a unitary operator acting on the product of the mixed coordinate-space and momentum-space representations of the operator Θ and the identity operator

$$W(\Theta;z)=\exp\left[\frac{i}{4}\frac{\partial^2}{\partial z_R\partial z_I}\right]2\pi\ {}_x\langle\sqrt{2}z_R|\Theta|\sqrt{2}z_I\rangle_p \times{}_p\langle\sqrt{2}z_I|\sqrt{2}z_R\rangle_x\ . \tag{4.13b}$$

The equivalence of Eqs. (4.13a) and (4.13b) is perhaps most easily established by the insertion of a complete set of coordinate-space eigenstates from Eq. (2.13) into Eq. (4.13b), so that the operator Θ is given a wholly Schrödinger representation, and by making use of the Taylor theorem.

Furthermore, by making use of Eqs. (3.13a), (3.13b), and (3.14), together with Eq. (4.1), we may rather simply prove the following relations for the Wigner representation of an operator Θ:

$$\int_{-\infty}^{\infty}dz_R\, W(\Theta;z)=\sqrt{2}\pi\ {}_p\langle\sqrt{2}z_I|\Theta|\sqrt{2}z_I\rangle_p\ , \tag{4.14a}$$

$$\int_{-\infty}^{\infty}dz_I\, W(\Theta;z)=\sqrt{2}\pi\ {}_x\langle\sqrt{2}z_R|\Theta|\sqrt{2}z_R\rangle_x\ , \tag{4.14b}$$

$$\frac{1}{\pi}\int d^2z\, W(\theta;z)=\mathrm{Tr}\Theta\ . \tag{4.14c}$$

It is precisely these relations that provide the usual interpretation of the Wigner function $W(\rho;z)$ for a density operator ρ (with $\mathrm{Tr}\rho=1$) as a quasiprobability distribution in phase space. These relations are, of course, well known, but here they are derived through Eqs. (3.13a) (3.13b), and (3.14), so that the relationship of the displaced parity operator formalism to the Wigner function formalism is clearly shown.

We also note the strong similarity between Eqs. (2.38a) and (2.38b) for the Bargmann representation of state vectors and Eqs. (4.14a) and (4.14b) for the Wigner representation of operators. One of the particular merits of the present formulation in terms of the displaced parity operators has been that it widens the Wigner formulation of quantum mechanics into a larger arena.

A particularly important application of the Wigner representation is to express a product of operators in terms of their respective Wigner representations. By making use of the group relation of Eq. (3.9b), together with the basic relation of Eq. (4.7a), we may easily derive the general expression

$$\Theta_1\Theta_2=\frac{4}{\pi^2}\int d^2z_1\int d^2z_2 W(\Theta_1;z_1)W(\Theta_2;z_2)\times\exp[2(z_1^*z_2-z_1z_2^*)]\times D(2z_1-2z_2)\ . \quad (4.15)$$

Use of Eq. (2.31) immediately yields the useful relation

$$\mathrm{Tr}(\Theta_1\Theta_2)=\int\frac{d^2z}{\pi}W(\Theta_1;z)W(\Theta_2;z)\ . \quad (4.16)$$

We return to the general theme of expressing products of operators in terms of their Wigner representations in Sec. VII, where we discuss the Moyal star product in the context of further applications of the parity operator and its variants.

In many practical applications we are interested in systems represented by mixed states rather than by pure states. A particularly interesting case is thus when one of the operators is a density operator ρ (i.e., with the properties $\rho=\rho^\dagger$ and $\mathrm{Tr}\rho=1$). An important example arises in equilibrium statistical mechanics, where the thermodynamic or statistical density operator representing a canonical ensemble of systems with Hamiltonian H at a temperature T is given by

$$\rho\equiv Z^{-1}\exp\left[-\frac{H}{k_BT}\right],\quad Z\equiv\mathrm{Tr}\left[\exp\left[-\frac{H}{k_BT}\right]\right], \quad (4.17)$$

where k_B is the Boltzmann constant. The expectation value $\langle\Theta\rangle$ of an operator Θ in the mixed state described by ρ may be computed from Eq. (4.17) as

$$\langle\Theta\rangle=\mathrm{Tr}(\rho\Theta)=\int\frac{d^2z}{\pi}P_W(z)W(\Theta;z)=\int\int\frac{dp\,dq}{2\pi\hbar}P_W(p,q)W(\Theta;p,q)\ , \quad (4.18)$$

where $P_W(z)\equiv W(\rho;z)$ or, equivalently, $P_W(p,q)\equiv W(\rho;p,q)$ is the Wigner representation of the density operator, or simply the Wigner function. Equations (4.14a)–(4.14c) show clearly in this case how $P_W(z)\equiv P_W(p,q)$ has the interpretation of a quasiprobability distribution in the phase space $\mathcal{W}$. Although Eq. (4.13a) immediately implies that the Wigner representation of any Hermitian operator, and hence $P_W(z)$ in particular, is a real quantity, it is not necessarily positive definite as would be a classical probability distribution in phase space.

Our previous discussion in terms of the displaced parity operator $U(z)$ now allows us to give a very natural explanation and physical interpretation for such negative values of $P_W(z)$. Thus, as explained in Sec. III, $U(z)$ is a Hermitian operator with eigenvalues -1 and $+1$, with corresponding projection operators $\Pi_1(z)$ and $\Pi_2(z)$. We denote the associated probabilities that an observation of $U(z)$ for a system in a state described by the density operator ρ will yield the values -1 and $+1$ by $P_1(\rho;z)$ and $P_2(\rho;z)$ respectively. Thus

$$P_1(\rho;z)=\mathrm{Tr}[\rho\Pi_1(z)]\ ,\quad P_2(\rho;z)=\mathrm{Tr}[\rho\Pi_2(z)]\ ,\quad P_1(\rho;z)+P_2(\rho;z)=1\ . \quad (4.19)$$

Furthermore, the mean value of the measurement is clearly seen to be given from Eq. (3.6) as

$$P_2(\rho;z)-P_1(\rho;z)=\mathrm{Tr}[\rho U(z)]=\tfrac{1}{2}W(\rho;z)\equiv\tfrac{1}{2}P_W(z)\ , \quad (4.20)$$

where we have used Eq. (4.7b). We thus see immediately from Eq. (4.20) that $P_W(z)$ satisfies the inequality

$$-2\le P_W(z)\le 2 \quad (4.21)$$

and that the probabilities $P_1(\rho;z)$ and $P_2(\rho;z)$ are given in terms of the Wigner function as

$$P_1(\rho;z)=\tfrac{1}{2}[1-\tfrac{1}{2}P_W(z)]\ ,\quad P_2(\rho;z)=\tfrac{1}{2}[1+\tfrac{1}{2}P_W(z)]\ . \quad (4.22)$$

V. PARITY AND FOURIER OPERATORS

Just as the elementary parity operator U_0 is seen from Eq. (2.10) to be a square root of the identity operator, it is interesting to note that it is itself the square of the unitary Fourier operator F_0, defined as

$$F_0\equiv\exp(\tfrac{1}{2}i\pi a^\dagger a)=\sum_{n=0}^{\infty}i^n|n\rangle\langle n|\ . \quad (5.1)$$

The reason for the choice of nomenclature for this operator will become clear in the ensuing discussion. The definition immediately yields the properties

$$F_0F_0^\dagger=I=F_0^\dagger F_0\ ,\quad F_0^2=U_0,\quad F_0^4=I\ . \quad (5.2)$$

Similarly, the mode of action of F_0 on the basic oscillator operators a and $a^\dagger$ is readily proven to be given by

$$F_0aF_0^\dagger=-ia,\quad F_0a^\dagger F_0^\dagger=ia^\dagger\ . \quad (5.3)$$

From Eq. (2.2) we thus have (again in units where $\hbar=m=\omega=1$)

$$F_0\hat{x}F_0^\dagger=\hat{p}\ ,\quad F_0\hat{p}F_0^\dagger=-\hat{x} \quad (5.4)$$

and hence also the relations

$$F_0|\alpha\rangle_x=|\alpha\rangle_p\ ,\quad F_0|\alpha\rangle_p=|-\alpha\rangle_x\ . \quad (5.5)$$

Equations (5.5) also lead to the alternative expressions for F_0,

$$F_0=\int_{-\infty}^{\infty}d\alpha\ |\alpha\rangle_{p\,x}\langle\alpha|=\int_{-\infty}^{\infty}d\alpha\ |\alpha\rangle_{x\,p}\langle-\alpha|\ . \quad (5.6)$$

We note that similar operators to F_0 have been considered elsewhere within the context of finite Hilbert spaces [34], whereas here it has been introduced for the infinite-dimensional Hilbert space $\mathcal{H}$ of the harmonic oscillator. We also remark that the trace of the operator $\exp[(\frac{1}{2}i\pi+\delta)a^\dagger a]$, where δ is a real infinitesimal, is easily found to be given by $[1-\exp(\frac{1}{2}i\pi+\delta)]^{-1}$ in the case $\delta<0$. For $\delta>0$, the trace diverges. In the case $\delta=0$, which is of interest to us here in connection with the Fourier operator, the trace may be defined by the one-sided limit and is given by

$$\mathrm{Tr}F_0=\tfrac{1}{2}(1+i) \; . \tag{5.7}$$

For an arbitrary ket vector $|g\rangle$ in $\mathcal{H}$, we define its Fourier transform $|g_F\rangle$ as

$$|g_F\rangle \equiv F_0|g\rangle \; . \tag{5.8}$$

By making use of Eqs. (5.6) and (2.14) we see that this definition immediately yields the relations

$$\begin{aligned} {}_x\langle\alpha|g_F\rangle &= {}_p\langle -\alpha|g\rangle=(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\beta\, e^{i\alpha\beta}\,{}_x\langle\beta|g\rangle \; , \\ {}_p\langle\alpha|g_F\rangle &= {}_x\langle\alpha|g\rangle=(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\beta\, e^{i\alpha\beta}\,{}_p\langle\beta|g\rangle \; . \end{aligned} \tag{5.9}$$

We thus observe that the mode of action of the Fourier operator F_0 on an arbitrary state vector in $\mathcal{H}$ is to generate its usual Fourier transform.

We may also define the Fourier transform Θ_F of an arbitrary operator Θ in $\mathcal{H}$ as

$$\Theta_F \equiv F_0 \Theta F_0^\dagger \; . \tag{5.10}$$

Thus, for a general operator $\Theta=\Theta(\hat{x},\hat{p})$, we have from Eq. (5.4) that

$$\Theta=\Theta(\hat{x},\hat{p}) \Longrightarrow \Theta_F=\Theta(\hat{p},-\hat{x}) \; . \tag{5.11}$$

As already noted in Sec. III, this very general functional definition of the Fourier transform of an arbitrary operator is quite distinct from the rather more specialized parametric Fourier transform defined in Sec. III for a class of operators defined in terms of a complex parameter. For example, we readily see that

$$D_F(z)\equiv F_0 D(z) F_0^\dagger = D(iz) \tag{5.12a}$$

or, equivalently,

$$F_0 D(z) = D(iz) F_0 \; . \tag{5.12b}$$

We may also define the displaced Fourier operator $F(z)$ by analogy with the displaced parity operator as

$$\begin{aligned} F(z) &\equiv D(z) F_0 D^\dagger(z) \\ &= \exp[\tfrac{1}{2}i\pi(a^\dagger - z^* I)(a-zI)] \; . \end{aligned} \tag{5.13}$$

In terms of the displaced number eigenstates $|n;z\rangle$ of Eq. (3.1) it clearly has the representation

$$F(z)=\sum_{n=0}^{\infty} i^n |n;z\rangle\langle n;z| \; . \tag{5.14}$$

We may also use Eq. (5.12b) together with the group relation of Eq. (3.9a) to express $F(z)$ in the forms

$$\begin{aligned} F(z) &= \exp(i|z|^2) F_0 D[-(1+i)z] \\ &= \exp(i|z|^2) D[(1-i)z] F_0 \; . \end{aligned} \tag{5.15}$$

It is clear that the displaced Fourier operator is related to the displaced parity operator through the relation

$$U(z)=F^2(z) \tag{5.16}$$

and that it also satisfies the relation

$$F^4(z)=I \; . \tag{5.17}$$

The relationship between the parity and Fourier operators studied in this section demonstrates another interesting aspect of the quasiclassical Wigner-Moyal approach to quantum mechanics. We return to this point later in this paper.

VI. INCLUSION OF SQUEEZING

We introduce next the unitary squeezing operators [24–27] $\{S(\rho);\ \rho\in\mathbb{C}\}$, defined as

$$\begin{aligned} S(\rho) &\equiv \exp[\tfrac{1}{4}\rho a^2 - \tfrac{1}{4}\rho^* a^{\dagger 2}] \; , \\ \rho &\equiv re^{i\theta} \; , \quad r>0 \; , \quad -\pi<\theta\le\pi \; . \end{aligned} \tag{6.1}$$

Their mode of action on the basic destruction and creation operators is to induce the homogeneous Bogoliubov transformation

$$\begin{aligned} S(\rho)aS^\dagger(\rho) &= a_\rho \equiv \cosh(\tfrac{1}{2}r)a + e^{-i\theta}\sinh(\tfrac{1}{2}r)a^\dagger \; , \\ S(\rho)a^\dagger S^\dagger(\rho) &= a_\rho^\dagger \equiv e^{i\theta}\sinh(\tfrac{1}{2}r)a + \cosh(\tfrac{1}{2}r)a^\dagger \; . \end{aligned} \tag{6.2}$$

Equation (6.2) immediately gives that the mode of action of the squeezing operators on the Weyl displacement operators is given by

$$S(\rho)D(z)S^\dagger(\rho)=D(z_{-\rho}) \Longrightarrow S(\rho)D(z)=D(z_{-\rho})S(\rho) \; , \tag{6.3}$$

where, by analogy with Eq. (5.2), we define

$$z_\rho \equiv \cosh(\tfrac{1}{2}r)z + e^{-i\theta}\sinh(\tfrac{1}{2}r)z^* \tag{6.4a}$$

and hence

$$z_{-\rho} = \cosh(\tfrac{1}{2}r)z - e^{-i\theta}\sinh(\tfrac{1}{2}r)z^* \; . \tag{6.4b}$$

We note that the inverse relation to the canonical transformation of Eq. (6.4a) is

$$z=\cosh(\tfrac{1}{2}r)z_\rho - e^{-i\theta}\sinh(\tfrac{1}{2}r)z_\rho^* = (z_\rho)_{-\rho} \; . \tag{6.4c}$$

We also consider the displaced and squeezed number eigenstates

$$|n;z,\rho\rangle \equiv S(\rho)|n;z\rangle = S(\rho)D(z)|n\rangle \; . \tag{6.5}$$

Clearly, although the squeezing and displacement operators do not commute, Eq. (6.3) shows that the order of the operations can be inverted if an appropriate adjustment of the displacement parameter is made at the same time. We may use these states to generalize further the operators Π_1 and Π_2 into their displaced and squeezed counterparts

$$\Pi_1(z,\rho)\equiv S(\rho)\Pi_1(z)S^\dagger(\rho) = \sum_{n=0}^{\infty} |2n+1;z,\rho\rangle\langle 2n+1;z,\rho| ,$$
$$\Pi_2(z,\rho)\equiv S(\rho)\Pi_2(z)S^\dagger(\rho) = \sum_{n=0}^{\infty} |2n;z,\rho\rangle\langle 2n;z,\rho| , \quad (6.6)$$

with the obvious properties

$$\Pi_1(z,\rho)+\Pi_2(z,\rho)=I ,$$
$$\Pi_i(z,\rho)\Pi_j(z,\rho)=\Pi_i(z,\rho)\delta_{ij} , \quad i,j=1,2 . \quad (6.7)$$

This procedure thus further extends Eq. (3.3) to provide the decomposition of the Hilbert space

$$\mathcal{H}=\mathcal{H}_1(z,\rho)\oplus\mathcal{H}_2(z,\rho) , \quad (6.8)$$

where $\mathcal{H}_1(z,\rho)$ and $\mathcal{H}_2(z,\rho)$ are the Hilbert subspaces spanned, respectively, by the odd and even displaced and squeezed number eigenstates. The squeezed and displaced parity operator is similarly defined as

$$U(z,\rho)\equiv S(\rho)U(z)S^\dagger(\rho) = \exp[i\pi(a_\rho^\dagger - zI)(a_\rho - z^* I)] = \Pi_2(z,\rho)-\Pi_1(z,\rho) . \quad (6.9)$$

It has the simply proven properties

$$U^2(z,\rho)=I , \quad U(z,\rho)=U^\dagger(z,\rho) . \quad (6.10)$$

As an example of this formalism, we consider the equilibrium state of a canonical ensemble of displaced and squeezed harmonic oscillators in contact with a heat bath at some nonzero temperature T. The Hamiltonian of the displaced and squeezed oscillator has as its lowest eigenstate a squeezed state. In this sense it clearly plays a fundamental role in the study of squeezed states and their important experimental realizations and applications. It is perhaps inappropriate and unnecessary in the present context to discuss in any detail the various experimental realizations of squeezed (e.g., two-photon coherent) states, since much literature already exists. We simply point out that in practice such states will always be produced with some unavoidable admixture of thermal noise. Two cases can be considered, namely, where the noise is or is not itself also squeezed. The example considered here refers to the former case. Thus we consider the mixed state described by the thermodynamic or statistical density operator $\rho\to\rho_{z,\rho}=\rho_{z,\rho}(T)$, as defined by Eq. (4.17) with $H\to H_{z,\rho}$, with

$$H_{z,\rho}\equiv S(\rho)D(z)H_0D^\dagger(z)S^\dagger(\rho) = \hbar\omega(f_0a^\dagger a+f_1a^2+f_1^*a^{\dagger 2}+f_2a+f_2^*a^\dagger+f_3) , \quad (6.11)$$

where the coefficients are defined explicitly as

$$f_0\equiv\cosh r , \quad f_1\equiv\tfrac{1}{2}e^{i\theta}\sinh r ,$$
$$f_2\equiv-\cosh(\tfrac{1}{2}r)z^*-e^{-i\theta}\sinh(\tfrac{1}{2}r)z , \quad (6.12)$$
$$f_3\equiv|z|^2+\cosh r .$$

We note that the density operator $\rho_{z,\rho}(T)$ has the simple expansion

$$\rho_{z,\rho}(T)=\sum_{n=0}^{\infty} p_n(T)|n;z,\rho\rangle\langle n;z,\rho| , \quad (6.13)$$

in terms of the displaced and squeezed number eigenstates of Eq. (6.5), and where $p_n(T)$ is the Planck distribution

$$p_n(T)=(1-e^{-\beta})e^{-n\beta} , \quad \beta\equiv\frac{\hbar\omega}{k_BT} . \quad (6.14)$$

At zero temperature, the density operator reduces to the projector onto the pure displaced and squeezed vacuum state

$$\rho_{z,\rho}(T)\xrightarrow[T\to 0]{}|z,\rho\rangle\langle z,\rho| , \quad (6.15)$$

where $|z,\rho\rangle$ is just the squeezed coherent state

$$|z,\rho\rangle\equiv|0;z,\rho\rangle=S(\rho)D(z)|0\rangle=S(\rho)|z\rangle . \quad (6.16)$$

The Weyl representation of $\rho_{z,\rho}$ may now be derived from Eqs. (4.5), (6.3), and (6.12) as

$$\tilde{W}(\rho_{z,\rho};w)=\pi\,\mathrm{Tr}[D(\tfrac{1}{2}iw)S(\rho)\rho_{z,0}S^\dagger(\rho)] = \pi\,\mathrm{Tr}[S(-\rho)D(\tfrac{1}{2}iw)S^\dagger(-\rho)\rho_{z,0}] = \pi\,\mathrm{Tr}[D(\tfrac{1}{2}iw_{-\rho})\rho_{z,0}] , \quad (6.17)$$

where, in the last line, we have used Eq. (6.3). By making further use of Eq. (3.9a) we find

$$\tilde{W}(\rho_{z,\rho};w)=\pi\,\mathrm{Tr}[D(-z)D(\tfrac{1}{2}iw_{-\rho})D(z)\rho_{0,0}] = \pi\exp(iz\cdot w_{-\rho})\mathrm{Tr}[D(\tfrac{1}{2}iw_{-\rho})\rho_{0,0}] . \quad (6.18)$$

Finally, the trace in Eq. (6.18) is readily evaluated in the number eigenstate representation of Eq. (2.4). By making use of Eq. (6.13) we readily find

$$\tilde{W}(\rho_{z,\rho};w)=\pi\exp[-\tfrac{1}{8}\coth(\tfrac{1}{2}\beta)|w_{-\rho}|^2+iz\cdot w_{-\rho}] . \quad (6.19)$$

The Fourier transform of Eq. (6.19) then yields the analogous Wigner representation of $\rho_{z,\rho}$, from Eq. (3.22), as

$$W(\rho_{z,\rho};\zeta)=2\tanh(\tfrac{1}{2}\beta)\exp[-2\tanh(\tfrac{1}{2}\beta)|z-\zeta_\rho|^2] . \quad (6.20)$$

The thermodynamic average of an operator Θ in the mixed state represented by the density operator $\rho_{z,\rho}(T)$ is thus given by Eqs. (4.16) and (6.20) as

$$\langle\Theta\rangle\equiv\langle\Theta\rangle(z,\rho;T)=\mathrm{Tr}[\Theta\rho_{z,\rho}(T)] , \quad (6.21)$$

$$\langle\Theta\rangle=\int\frac{d^2\zeta}{\pi}W(\Theta;\zeta)2\tanh(\tfrac{1}{2}\beta)\times\exp[-2\tanh(\tfrac{1}{2}\beta)|z-\zeta_\rho|^2]\ . \tag{6.22}$$

By changing the integration variable from ζ to ζ_ρ, and by making use of Eqs. (6.4a) and (6.4c), we find

$$\langle\Theta\rangle(z,\rho;T)=\exp[\tfrac{1}{8}\coth(\tfrac{1}{2}\beta)\Delta_z]W(\Theta;z_{-\rho})\ , \tag{6.23}$$

where Δ_z is the usual two-dimensional Laplacian operator

$$\Delta_z\equiv4\frac{\partial^2}{\partial z\partial z^*}=\frac{\partial^2}{\partial z_R^2}+\frac{\partial^2}{\partial z_I^2}\ . \tag{6.24}$$

In the proof of Eq. (6.23) we have also made use of the general relation

$$\int\frac{d^2z'}{\pi}f(z',z'^*)ke^{-k|z-z'|^2}=\exp\left[\frac{1}{4k}\Delta_z\right]f(z,z^*)\ ,\quad k>0\ , \tag{6.25}$$

which is valid for an arbitrary function $f(z,z^*)\equiv f(z)$ of the complex variable z and its conjugate z^*, provided that the integral exists. Equation (6.25) is most easily proven by taking the Fourier transform of both sides of the relation. We note that in the zero-temperature limit, Eq. (6.15) shows that Eq. (6.23) reduces to the relation

$$\langle z,\rho|\Theta|z,\rho\rangle=\exp(\tfrac{1}{8}\Delta_z)W(\Theta;z_{-\rho})\ , \tag{6.26a}$$

which itself reduces in the limit $\rho\to0$ of zero squeezing to the well-known relation between the Q and W representations

$$Q(\Theta;z)=\exp(\tfrac{1}{8}\Delta_z)W(\Theta;z)\ . \tag{6.26b}$$

By making further use of Eq. (4.10), Eq. (6.23) provides the following relations for the thermodynamic expectation values of Weyl-ordered functions of either the operators a and $a^\dagger$ or the operators $\hat{x}$ and $\hat{p}$:

$$\langle\{f(a,a^\dagger)\}_W\rangle(z,\rho;T)=\exp[\tfrac{1}{8}\coth(\tfrac{1}{2}\beta)\Delta_z]f(z_{-\rho},z^*_{-\rho})\ , \tag{6.27a}$$

$$\langle\{f(\hat{x},\hat{p})\}_W\rangle(z,\rho;T)=\exp[\tfrac{1}{8}\coth(\tfrac{1}{2}\beta)\Delta_z]\times f\left[\left[\frac{2\hbar}{m\omega}\right]^{1/2}\mathrm{Re}z_{-\rho}\ ,\sqrt{2m\hbar\omega}\,\mathrm{Im}z_{-\rho}\right]\ , \tag{6.27b}$$

where in the latter equation we have reverted to units where the Planck constant $\hbar$ and the oscillator parameters m and ω have not been set to unity (as elsewhere in this paper). In particular, if we write z in the usual phase space form

$$z\equiv\left[\frac{mw}{2\hbar}\right]^{1/2}q+\frac{i}{(2m\hbar\omega)^{1/2}}p\ , \tag{6.28}$$

one easily shows from Eqs. (6.27b) and (6.4b) that the density operator $\rho_{z,\rho}(T)$ describes a mixed state localized around the phase space point

$$\begin{aligned}\langle\hat{x}\rangle\equiv\langle\hat{x}\rangle(z,\rho;T)&=[\cosh(\tfrac{1}{2}r)-\cos\theta\sinh(\tfrac{1}{2}r)]q+\sin\theta\sinh(\tfrac{1}{2}r)\frac{p}{m\omega}\ ,\\ \langle\hat{p}\rangle\equiv\langle\hat{p}\rangle(z,\rho;T)&=[\cosh(\tfrac{1}{2}r)+\cos\theta\sinh(\tfrac{1}{2}r)]p+\sin\theta\sinh(\tfrac{1}{2}r)m\omega q\ .\end{aligned} \tag{6.29}$$

Furthermore, the associated uncertainties are also readily derived as

$$\begin{aligned}(\Delta x)^2\equiv\sigma_{xx}&\equiv\langle(\hat{x}-\langle\hat{x}\rangle)^2\rangle(z,\rho;T)\\&=\frac{\hbar}{2m\omega}\coth(\tfrac{1}{2}\beta)[\cosh r-\cos\theta\sinh r]\ ,\\(\Delta p)^2\equiv\sigma_{pp}&\equiv\langle(\hat{p}-\langle\hat{p}\rangle)^2\rangle(z,\rho;T)\\&=\tfrac{1}{2}\hbar m\omega\coth(\tfrac{1}{2}\beta)[\cosh r+\cos\theta\sinh r]\ ,\\\sigma_{px}&\equiv\langle\tfrac{1}{2}(\hat{p}\hat{x}+\hat{x}\hat{p})-\langle\hat{p}\rangle\langle\hat{x}\rangle\rangle(z,\rho;T)\\&=\tfrac{1}{2}\hbar\coth(\tfrac{1}{2}\beta)\sin\theta\sinh r\ .\end{aligned} \tag{6.30}$$

We have also derived these relations by a different means in an earlier publication [32], in which we have also discussed the generalized uncertainty relation for the mixed squeezed coherent states in terms of the so-called uncertainty determinant or variance determinant. This exhibits in a particularly revealing fashion the combination of both a minimal quantum-mechanical (Heisenberg) uncertainty factor with a corresponding thermodynamic factor which vanishes in the pure state limit as $T\to0$. Similarly, Eq. (6.27a) also allows us, for example, to calculate the photon-number expectation value for the mixed state described by the density operator $\rho_{z,\rho}(T)$.

VII. MOYAL STAR PRODUCT AND CLASSICAL MAPPINGS

We return now to the representation of products of operators in terms of their individual W representations. In the first place, the use of Eqs. (3.9d) and (3.7) in Eq. (4.15) leads easily to the important relation

$$W(\Theta_1\Theta_2;z)=\frac{4}{\pi^2}\int d^2z_1\int d^2z_2\,W(\Theta_1;z_1+z)W(\Theta_2;z_2+z)\times\exp[2(z_1^*z_2-z_1z_2^*)] \tag{7.1}$$

for the Wigner representation of the product of two operators in terms of their respective Wigner representations, where we have used the fundamental definition of Eq. (4.7b). The analogous relation in terms of the Weyl representations is given by

$$\tilde{W}(\Theta_1\Theta_2;w)=\int\frac{d^2w_1}{(2\pi)^2}\tilde{W}(\Theta_1;\tfrac{1}{2}w+w_1)\tilde{W}(\Theta_2;\tfrac{1}{2}w-w_1)\times\exp[\tfrac{1}{8}(w_1w^*-w_1^*w)]\ , \tag{7.2a}$$

$$W(\Theta_1\Theta_2;z)=\int \frac{d^2w_1}{(2\pi)^2}\int \frac{d^2w_2}{(2\pi)^2}\tilde{W}(\Theta_1;w_1)\tilde{W}(\Theta_2;w_2)\times\exp[-i(w_1+w_2)\cdot z+\tfrac{1}{8}(w_1w_2^*-w_1^*w_2)] \;. \tag{7.2b}$$

The bijective mapping $\Theta\mapsto W(\Theta)$ between a well-defined class of operators $\{\Theta\}$ in the Hilbert space $\mathcal{H}$ and a corresponding class $\{W(\Theta)\}$ of functions defined explicitly in Sec. IV is the basis of the so-called Weyl [5] (or Weyl-Wigner [6]) correspondence. The functions $W(\Theta;z)$, which are functions of the complex number z and its conjugate z^*, are formally defined on the dual $\mathcal{A}^*\equiv\mathcal{W}$ of the Heisenberg algebra, and this is just the classical phase space. Under the Weyl correspondence a product of two operators is mapped into the so-called *Moyal star product* [8] of the mapped functions

$$\Theta_1\Theta_2\mapsto W(\Theta_1\Theta_2;z)\equiv W(\Theta_1)*W(\Theta_2)(z) \;, \tag{7.3}$$

where the star product is thus given explicitly by Eq. (7.1).

An alternative form for the star product is easily obtained by first expanding one of the two Wigner representations in the integrand of Eq. (7.1) about the point z, by using the Taylor theorem. In this way we find

$$W(\Theta_1\Theta_2;z)=\sum_{m,n=0}^{\infty}\frac{(-1)^n}{m!n!}\left[\frac{i}{4}\right]^{m+n}\frac{\partial^{m+n}W(\Theta_1;z)}{\partial z_R^m\partial z_I^n}\times\frac{\partial^{m+n}W(\Theta_2;z)}{\partial z_R^n\partial z_I^m} \;. \tag{7.4a}$$

Equation (7.4a) may also formally be written in the form

$$\begin{aligned}W(\Theta_1\Theta_2;z)&=W(\Theta_1;z)\exp\left[\frac{i}{4}\left[\frac{\overleftarrow{\partial}}{\partial z_R}\frac{\overrightarrow{\partial}}{\partial z_I}-\frac{\overleftarrow{\partial}}{\partial z_I}\frac{\overrightarrow{\partial}}{\partial z_R}\right]\right]\times W(\Theta_2;z)\\&=W(\Theta_1;z)\exp\left[\frac{1}{2}\left[\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial z^*}-\frac{\overleftarrow{\partial}}{\partial z^*}\frac{\overrightarrow{\partial}}{\partial z}\right]\right]\times W(\Theta_2;z) \;,\end{aligned} \tag{7.4b}$$

where, as usual, the arrows indicate the direction in which the derivatives act. One may also show that the Moyal star product is associative

$$W(\Theta_1)*\{W(\Theta_2)*W(\Theta_3)\}=\{W(\Theta_1)*W(\Theta_2)\}*W(\Theta_3) \;. \tag{7.5}$$

By making use of Eq. (6.28) we also note that Eq. (7.4a) becomes a representation of the Moyal star product as a formal power series in the Planck constant $\hbar$,

$$W(\Theta_1\Theta_2;p,q)=\sum_{m,n=0}^{\infty}\frac{(-1)^n}{m!n!}\left[\frac{i\hbar}{2}\right]^{m+n}\times\frac{\partial^{m+n}W(\Theta_1;p,q)}{\partial q^m\partial p^n}\times\frac{\partial^{m+n}W(\Theta_2;p,q)}{\partial q^n\partial p^m} \;. \tag{7.6}$$

Another form for the Moyal star product may be found by using the easily derived expression

$$\exp\left[-\frac{i}{2}(w_1^*Z+w_1Z^\dagger)\right]e^{-iw_2\cdot z}=e^{-i(w_1+w_2)\cdot z}\exp[\tfrac{1}{8}(w_1w_2^*-w_1^*w_2)] \;, \tag{7.7}$$

where the scalar products in the exponents are defined as in Eq. (3.19) and the Bopp operators [35] Z and $Z^\dagger$ are given as

$$Z\equiv z+\frac{1}{2}\frac{\partial}{\partial z^*} \;,\quad Z^\dagger\equiv z^*-\frac{1}{2}\frac{\partial}{\partial z} \;. \tag{7.8}$$

Substitution of Eq. (7.7) into Eq. (7.2b) then yields the alternative form

$$W(\Theta_1\Theta_2;z,z^*)=\{W(\Theta_1;Z,Z^\dagger)\}_W W(\Theta_2;z,z^*) \;, \tag{7.9}$$

where we explicitly write $W(\Theta;z)\equiv W(\Theta;z,z^*)$ and where the first term on the right-hand side of Eq. (7.9) is the Weyl-ordered operator, as in Eqs. (4.10) and (4.11).

We hope to have persuaded the reader of the relevance and usefulness of the parity operator in the context of the Wigner representation and hence to the Moyal star product. Thus it should be clear that the displaced parity operator plays an absolutely central role in the whole of this section through Eq. (4.7b), although for simplicity of notation we have not made the dependence explicit. Other similar applications of the parity operator and its generalizations are also now suggested in this framework. We first note that there is a venerable tradition in quantum mechanics of developing alternatives to the usual Hilbert space approach to the quantization procedure. Thus, apart from the well-known Feynman path-integral method, the Moyal star product, based as shown here on the Weyl correspondence, provided an early and notable autonomous route to the quantization of a classical mechanics or field theory. It has since been extensively discussed by a number of other authors [36–38].

Furthermore, from a modern vantage point the Moyal star product is a particular example of *deformation theory* [39,40], wherein the algebra of functions on the classical phase space of a classical mechanics system is deformed so as to obtain consistency with its quantized counterpart. For example, in the Moyal star product quantization procedure, the quantum-mechanical commutator of two operators $\hat{\Theta}_1$ and $\hat{\Theta}_2$ is mapped into the so-called *Moyal bracket*

$$[\hat{\Theta}_1,\hat{\Theta}_2]\mapsto[\Theta_1,\Theta_2]^M\equiv W(\Theta_1)*W(\Theta_2)-W(\Theta_2)*W(\Theta_1) \;. \tag{7.10}$$

The Heisenberg equation of motion for an operator is thereby mapped into an equation of similar form for the corresponding mapped function given by its Wigner representation, with the commutator replaced by the Moyal bracket. Functions on the phase space thus generally have a similar role in deformation theory to the operators in the quantum-mechanical Heisenberg picture; the definition of the star product is thereby equivalent to the specification of a particular candidate for the quantum-mechanical counterpart to the classical starting point.

Deformation theory generalizes the above concepts for more complicated symplectic structures and Lie algebras of functions. Broadly speaking, one introduces the deformation parameter $\hbar$, and in terms of it deforms both the ordinary product and the Poisson bracket. In the latter context, one observes from Eq. (7.6) that

$$W(\Theta_1\Theta_2;p,q)\xrightarrow[\hbar\to 0]{} W(\Theta_1;p,q)W(\Theta_2;p,q)+\frac{1}{2}i\hbar\left[\frac{\partial W(\Theta_1)}{\partial q}\frac{\partial W(\Theta_2)}{\partial p}-\frac{\partial W(\Theta_1)}{\partial p}\frac{\partial W(\Theta_2)}{\partial q}\right]$$
$$-\frac{1}{4}\hbar^2\left[\frac{1}{2}\frac{\partial^2 W(\Theta_1)}{\partial q^2}\frac{\partial^2 W(\Theta_2)}{\partial p^2}-\frac{\partial^2 W(\Theta_1)}{\partial p\partial q}\frac{\partial^2 W(\Theta_2)}{\partial p\partial q}+\frac{1}{2}\frac{\partial^2 W(\Theta_1)}{\partial p^2}\frac{\partial^2 W(\Theta_2)}{\partial q^2}\right]+O(\hbar^3)\ . \qquad (7.11)$$

More generally in deformation theory we deform in terms of some specified star product as follows:

$$fg\mapsto f*g=fg+O(\hbar)\ , \qquad (7.12a)$$

$$i\hbar\{f,g\}\mapsto f*g-g*f=i\hbar\{f,g\}+O(\hbar^2) \qquad (7.12b)$$

for functions f and g in the original classical phase space and where the notation implies the usual assumption of the classical limit as $\hbar\to 0$. A special class of functions (or observables), namely, the so-called preferred functions (or preferred observables), satisfies Eq. (7.12b) with zero second-order term, as is the case of the Moyal star product of Eq. (7.11). Thereafter, all other functions are built from the preferred functions and can be interpreted as forming the universal enveloping algebra of the Lie group generated by the preferred functions.

In our own example, the Heisenberg algebra necessarily leads to the Moyal star product as the only feasible solution with the required properties and the Wigner representation thereby occupies a special role. We remark that in the case of more complicated symplectic structures than those considered here, we expect a comparable analysis to that of the present paper to be useful and with a central role again played by the analogous parity operator. We also note that deformation theory has obvious connections with the theory of quantum groups, which are structures formed by so-called quantum deformations of Lie groups. We again speculate that some of the unifying features of the present analysis might also be generalized for use in this arena. We hope to report on such extensions in the future.

VIII. DISCUSSION AND FURTHER EXTENSIONS

In this paper we have extended earlier work on the displaced parity operators. In particular we have explored the deep relationships that exist with the Wigner representation. We have also discussed the interesting connections between the parity and Fourier operators. Furthermore, special emphasis has been placed on the role of the displaced parity operator in the various classical mappings of quantum mechanics typified by the Weyl correspondence and the Moyal star product, which we have discussed in detail.

In the latter context we also note that other recent work [41–44] has led to quite distinct exact classical mappings of quantum (many-particle or field) theories, which are not subspecies of deformation theory. These are based on methods which we refer to generically as independent-cluster method (ICM) techniques and which include the configuration-interaction method, the normal coupled-cluster method (NCCM), and the extended coupled-cluster method (ECCM). Each ICM leads to a new star product [44], which, while sharing many of the properties of the Moyal star product, is conceptually quite distinct from it and has some important differences. Nevertheless, the temporal evolution of the (many-particle or many-mode) multiconfigurational amplitudes $\{x_I,\tilde{x}_I\}$, which completely characterize each ICM description of a quantum theory, is described by classical canonical equations of motion in terms of some well-defined classical Hamiltonian functional $\bar{H}[x_I,\tilde{x}_I]$. Each ICM may also be regarded as performing a definite "bosonization" [41] (and, more generally, also a "fermionization" [43]) of the original (many-body or field-theoretic) Hilbert space $\mathcal{H}$ onto the subset of coherent states in a larger bosonized and fermionized space $\mathcal{B}^{\mathcal{H}}$ in which $\mathcal{H}$ is imbedded. The ideal bosons and fermions in $\mathcal{B}^{\mathcal{H}}$ carry the same multiconfigurational index labels I as the ICM subsystem amplitudes and in each case a one-to-one correspondence exists between them.

Furthermore, each mapped classical Hamiltonian functional is defined in some classical phase space $\Gamma^H_{\rm phys}$, which is itself a submanifold of the infinite-dimensional space Γ^H spanned by all points $(x_I,\tilde{x}_I)$. It has also been emphasized [42] that the states generated by each ICM may be viewed as a well-defined set of *supercoherent states* in $\mathcal{H}$ associated with each point $(x_I,\tilde{x}_I)$ in Γ^H. While the usual coherent or squeezed coherent states are typically used to provide a *quasiclassical approximation* for the quantal system, these supercoherent generalizations provide an exact, yet similarly classicized, description. In this way, each ICM provides a supercoherent map of $\mathcal{H}$ and the quantum-mechanical Hamiltonian operator H onto the target classical phase space $\Gamma^H_{\rm phys}$ and its associated classical Hamiltonian $\overline{H}=\overline{H}[x_I,\tilde{x}_I]$ for a set of multiconfigurational fields $\{x_I,\tilde{x}_I\}$.

It has been shown explicitly [44] how the expectation values of products of operators can be computed in each case in the mapped classical phase space Γ^H in terms of a new ICM star product of the expectation values of the individual operators, in direct analogy with the Moyal star product involving Wigner representations discussed here. In this way the ICM classical map of an arbitrary quantum theory provides an equivalent classical mechanics based on the traditional concepts of phase space, Hamiltonian and flow, with the ICM star product incorporating the essentially quantum-mechanical features. The new ICM star products have the feature that the star commutators are *precisely* (generalized) Poisson brackets defined in terms of the full set of multiconfigurational amplitudes $\{x_I,\tilde{x}_I\}$. In this way commutators in $\mathcal{H}$ are mapped into Poisson brackets in Γ^H, which again thereby acquires a symplectic structure. For this reason we again expect a comparable analysis to that of the present paper, with the parity operator playing a key role, to be of benefit in a further analysis of the ICM techniques.

In the same context we note that Arponen and Bishop [42] have given a detailed analysis of the ICM techniques for an arbitrary single-mode bosonic field theory (which includes the important example of the anharmonic oscillator) within the Bargmann or holomorphic representation discussed here. They have shown explicitly how both the NCCM and the ECCM may be viewed as providing definite supercoherent or multicoherent states, in the sense that the usual Glauber (or one-photon) coherent states which underpin the holomorphic representation may be extended not only to the squeezed (or two-photon) coherent states discussed in Sec. V but also to the "hypersqueezed" (or n-photon, with $n>2$) multicoherent states. These authors showed very precisely how such hypersqueezed states, although having infinite norm within the Hilbert space $\mathcal{H}$, can be used to obtain finite and perfectly well-defined expressions for energies and expectation values of other operators.

Indeed, the introduction of such states within the coupled-cluster approaches provides a systematic means, within the broader context of many-body quantum mechanics or quantum field theory, to go beyond the Gaussian or Bogoliubov approximation implied by the restriction to the coherent or squeezed coherent states in the corresponding mapped phase space Γ^H. It is our belief that use of these coupled-cluster techniques could also profitably be made in quantum optics and allied fields, especially when cojoined with the unifying framework provided by the present analysis and its extensions envisaged here. In a similar context, an alternative approach to n-photon states has recently been presented [45] in terms of analytic representations in the multisheeted complex plane and multisheeted unit disk. The ideas discussed in the present paper are readily generalizable in this context also.

Finally, we note that the analysis presented here has been restricted to the phase space of a single oscillator or, equivalently, of a one-mode field. However, the experimental realization of squeezed states in quantum optics, for example, usually involves two modes. A general discussion of two-mode squeezed states has been given by the present authors [46]. The most general description is now in terms of the Bogoliubov transformation between the creation and destruction operators of the two independent modes and the associated algebra is Sp(4,$\mathbb{R}$). In particular, we considered in detail the states $\hat{\theta}|z_1,z_2\rangle$ where the ten-parameter family of operators $\{\hat{\theta}\}$ is associated with a unitary representation of the group Sp(4,$\mathbb{R}$) and the two-mode coherent states $|z_1,z_2\rangle\equiv|z_1\rangle|z_2\rangle$ are the joint eigenstates of the destruction operators a_1 and a_2 for the two independent modes. The operators $\hat{\theta}\equiv\hat{\theta}(\omega,\psi;\sigma,\phi;r_1,\theta_1,\lambda_1;r_2,\theta_2,\lambda_2)$ take the form

$$\hat{\theta}\equiv\mathcal{W}(\omega,\psi)\mathcal{V}(\sigma,\phi)\mathcal{U}^{(1)}(r_1,\theta_1,\lambda_1)\mathcal{U}^{(2)}(r_2,\theta_2,\lambda_2)\ , \qquad (8.1)$$

where

$$\begin{aligned}
\mathcal{W}(\omega,\psi)&\equiv\exp(\tfrac{1}{2}\omega e^{i\psi}a_1a_2^\dagger-\tfrac{1}{2}\omega e^{-i\psi}a_1^\dagger a_2)\ ,\\
\mathcal{V}(\sigma,\phi)&\equiv\exp(\tfrac{1}{2}\sigma e^{i\phi}a_1a_2-\tfrac{1}{2}\sigma e^{-i\phi}a_1^\dagger a_2^\dagger)\ ,\\
\mathcal{U}^{(j)}(r_j,\theta_j,\lambda_j)&\equiv\exp(\tfrac{1}{4}r_je^{i\theta_j}a_j^2-\tfrac{1}{4}r_je^{-i\theta_j}a_j^{\dagger 2})\\
&\quad\times\exp(i\lambda_ja_j^\dagger a_j)\ ,
\end{aligned} \qquad (8.2)$$

where all ten parameters are real.

The operators $\{\mathcal{U}^{(j)}(r_j,\theta_j,\lambda_j)\}\equiv\{S^{(j)}(\rho_j)\times\exp(i\lambda_ja_j^\dagger a_j)\}$ are just the usual unitary representations of the SU(1,1) subgroups associated with squeezing of the two modes $j=1,2$ independently. We have explained elsewhere [27] that this is just the so-called $(\frac{1}{4},\frac{3}{4})$ representation of SU(1,1). The operators $\mathcal{V}$ and $\mathcal{W}$ are similarly associated with two different types of two-mode squeezing. Thus the operators $\{\mathcal{V}(\sigma,\phi)\exp[i\mu(a_1^\dagger a_1+a_2^\dagger a_2+1)]\}$ lead to the so-called discrete series representation of another SU(1,1) subgroup [47], while the operators $\{\mathcal{W}(\omega,\psi)\exp[i\nu(a_1^\dagger a_1-a_2^\dagger a_2)]\}$ lead to the Schwinger representation of the SU(2) subgroup of Sp(4,$\mathbb{R}$). Both of these two-mode types of squeezing have found applications. Examples include the study of interferometers [48] and other applications in quantum optics [49]. It should be clear that all of the various types of two-mode squeezing can readily be incorporated within our formulation by simply working in the phase space of the two modes, taken as the direct product of their individual phase spaces. The formalism presented here is thus straightforward to extend. More generally, the extension to n-mode squeezing associated with the group Sp($2n$,$\mathbb{R}$) is also, in principle, not difficult. We hope to report on such extensions elsewhere.

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