Continuum bound states

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We compare three procedures for constructing bound states whose energies are embedded in the continuous part of the energy spectrum. We use methods based on the Gel'fand-Levitan and Marchenko equations to generate new potentials which look superficially similar to potentials supporting continuum bound states, but which instead produce isolated energies embedded in the continuum for which no physically acceptable wave function exists. Two examples are used to illustrate these results.

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I. INTRODUCTION

Recently there has been much interest in the construction of local potentials which support continuum bound states, that is, bound states with energies embedded in the continuous spectrum. Such states were first proposed by von Neumann and Wigner in 1929 [1]. They suggested that certain spatially oscillating potentials with amplitude decreasing with distance could support positive-energy bound states by means of coherent diffractive scattering. At one particular momentum determined by the wavelength of the oscillations of the potential, destructive interference of the reflected waves can cause the amplitude of the wave function to vanish at large distances, thereby producing a normalizable state.

Inspired by the von Neumann–Wigner results, Stillinger [2] and Herrick [3] proposed that superlattices consisting of ultrathin semiconductor layers might be used to construct potentials supporting such bound states. Capasso *et al.* [4] used the Bragg reflection conditions to carefully construct such a superlattice for which they report the remarkable observation of an electronic bound state with no classical turning points. Infrared absorption measurements revealed a narrow isolated transition from a bound state within a quantum well to a bound state at an energy greater than the barrier height. Weber [5] has studied a solvable model inspired by the ideas of Stillinger and Herrick and by the Capasso *et al.* experiments.

Recent observations of narrow positron-electron coincidence peaks in heavy-ion collisions [6,7] have also spurred interest in continuum bound states. The measured widths of 30–60 keV are close to the experimental resolution and should therefore be taken as upper limits on the actual widths. In a theoretical effort stimulated by these experiments, Spence and Vary [8] found zero width resonances by solving the e^+e^- scattering problem using three different relativistic two-body wave equations. They concluded that these are bound states in the continuum, although we have shown [9] that they cannot be continuum bound states of the von Neumann–Wigner type. Weber and Hammer [10] have constructed a local model potential—not of the von Neumann–Wigner type—for the Klein-Gordon equation, which supports multiple bound states with energies that match those of the observed peaks.

In this paper, we confine our attention to the investigation of local potentials which support bound states with energies embedded in the continuous spectrum of the nonrelativistic radial Schrödinger equation. In addition to the von Neumann–Wigner procedure, two other methods exist for constructing such states. Moses and Tuan [11], and later Meyer-Vernet [12], used the Gel'fand-Levitan equation to produce potentials supporting one or more bound states in the continuum. More recently, Pappademos, Sukhatme and Pagnamenta [13] have used methods associated with supersymmetric quantum mechanics to find potentials which support continuum bound states. The purpose of this paper is to compare the results of these three procedures.

We shall use units in which $\hbar=1$ and $2m=1$. For simplicity, we shall confine our attention to the case of zero angular momentum; the generalization to nonzero angular momentum is straightforward. All three methods begin with a radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+V_c(r)\right]\varphi(r)=k^2\varphi(r)\ ,\quad 0\le r<\infty\ , \tag{1}$$

using a "comparison potential" $V_c(r)$ chosen to support a continuous energy eigenvalue spectrum (possibly in addition to a number of bound states). We assume that $V_c(r)$ satisfies conditions sufficient for the continuum wave functions to be normalized so that their asymptotic form is

$$\varphi(r,k)\to\sin[kr+\delta(k)]\ ,\quad r\to\infty\ . \tag{2}$$

For simplicity we assume that $V_c(r)$ can be expanded in a Laurent series and is no more singular than r^{-1} at the origin. If this is so, physically acceptable wave functions $\varphi(r)$ must satisfy $\varphi(0)=0$ and $0<|\varphi'(0)|<\infty$. The wave function of the continuum state with energy k^2 will be denoted by $\varphi(r,k)$. One particular continuum state, with energy k_1^2, wave function $\varphi(r,k_1)\equiv\varphi_1(r)$, and scattering phase shift $\delta(k_1)\equiv\delta_1$, will be singled out for special attention. The goal of all three procedures is to construct a new potential $V(r)$ such that the new Schrödinger equation supports the same continuum spectrum as Eq. (1)

with the exception of the energy k_1^2, which now belongs to a bound state embedded in the continuum.

In Sec. II we show some of the variety of potentials and bound states obtained by the method of von Neumann and Wigner. In Sec. III we use the formalism developed in the preceding paper [16] to discuss the Meyer-Vernet [12] treatment of continuum bound states. Section IV uses the formalism of Ref. [16] to remove the continuum bound state created by the von Neumann–Wigner method. This allows us to construct potentials analogous to those created by the von Neumann–Wigner and Meyer-Vernet methods, but which do not support continuum bound states. These results are illustrated by two examples in Sec. V. In Sec. VI we consider the Pappademos-Sukhatme-Pagnamenta [13] method based on supersymmetric quantum mechanics and show that it is exactly equivalent to the Meyer-Vernet method. In Sec. VII we summarize our results, evaluate the relative merits of the procedures we have studied, and discuss some questions arising out of the earlier literature on isospectral Hamiltonians.

II. THE von NEUMANN–WIGNER PROCEDURE

The von Neumann–Wigner method consists of constructing a normalizable wave function and then using the Schrödinger equation with fixed positive energy to find the potential. Consider the radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+V\right]\psi=k^2\psi\ ,\quad 0\le r<\infty\ . \tag{3}$$

We assume that the effective potential V goes to zero for large r. Then $E=k^2$. Only those solutions of Eq. (3) which vanish at the origin and are bounded as $r\to\infty$ are physically acceptable.

We now choose

$$\psi(r)=f(r)\varphi_1(r)\ , \tag{4}$$

where f is a function of r yet to be specified and $\varphi_1(r)\equiv\varphi(r,k_1)$ is the solution of Eq. (1) with energy $E_1=k_1^2$. If f is reasonably behaved at the origin and decreases sufficiently rapidly as $r\to\infty$, we obtain a bound state. By solving Eq. (3) for the potential we find

$$V(r)=V_c(r)+2\frac{\varphi_1'(r)}{\varphi_1(r)}\frac{f'(r)}{f(r)}+\frac{f''(r)}{f(r)}\ , \tag{5}$$

where the prime indicates derivatives with respect to r.

In order that the potential remains finite at the zeros of φ_1, we take f to be a function of r through the variable s defined by

$$s(r)=\int_0^r[\varphi_1(\xi)]^2d\xi\ . \tag{6}$$

If $V_c(r)$ is sufficiently well behaved at the origin for

$$\varphi_1(r)\underset{r\to0}{\to}O(r)\ , \tag{7}$$

then

$$s(r)\underset{r\to0}{\sim}O(r^3)\ . \tag{8}$$

From Eqs. (5) and (6) it follows that

$$V(r)=V_c(r)+4\varphi_1'(r)\varphi_1(r)\frac{f_s(s)}{f(s)}+\varphi_1^4(r)\frac{f_{ss}(s)}{f(s)}\ , \tag{9}$$

where derivatives with respect to s are denoted by subscripts. Also, by virtue of Eqs. (4) and (6),

$$\int[\psi(r)]^2dr=\int[f(s)]^2ds\ . \tag{10}$$

We first consider constraints on $f(s)$ as r approaches zero. It is not sufficient for $\psi(r)$ to be normalizable: in order that the energy spectrum associated with Eq. (3) be bounded below, $\psi(r)$ must approach zero as $r\to0$ at least as fast as $r^{1/2}$ (see Ref. [17]). Since $\varphi_1(r)$ is of order r as $r\to0$, it follows that $f(s)$ must satisfy

$$\lim_{r\to0}r^{1/2}f(s)\le O(1)\ . \tag{11}$$

We next consider constraints on $f(s)$ as $r\to\infty$. For this, it is convenient to write

$$f(s)=e^{-g(s)}\ . \tag{12}$$

Then

$$V(r)-V_c(r)=-4g_s(s)\varphi_1'(r)\varphi_1(r)+[g_s^2(s)-g_{ss}(s)]\varphi_1^4(r)\ . \tag{13}$$

The condition that $\psi(r)$ be normalizable requires that

$$\lim_{s\to\infty}\frac{g(s)}{\ln(s)}>\frac{1}{2}\ , \tag{14}$$

while the requirement that $V(r)$ be bounded as $r\to\infty$ implies that

$$\lim_{s\to\infty}\frac{g(s)}{s}<\infty\ . \tag{15}$$

It follows that there exist constants α, β, γ, and r_0, such that

$$\gamma\ln(2s)<|g(s)|<\beta(2s)^\alpha\ ,\quad r>r_0\ , \tag{16}$$

with

$$0<\alpha\le1\ , \tag{17}$$

$$\beta>0\ , \tag{18}$$

and

$$\gamma>\tfrac{1}{2}\ . \tag{19}$$

There exist infinitely many possible asymptotic forms for $g(s)$ which fit between the bounds given by Eq. (16). As examples we cite forms such as $(2s)^\alpha[\ln(2s)]^{-\sigma}$ and $(2s)^\alpha\{\ln[\ln(2s)]\}^{-\sigma}$ with $\sigma>0$, $(2s)^\rho[\ln(2s)]^\sigma$ and $(2s)^\rho\{\ln[\ln(2s)]\}^\sigma$ with $0<\rho<\alpha$, and $[\ln(2s)]^\sigma$ with $\sigma>1$. We ignore these complications and restrict our discussion to the two bounds given by Eq. (16), which we combine into a single asymptotic form for $g(s)$ of the

form

$$g(s) \underset{s\to\infty}{\to} \beta(2s)^{\alpha}+\gamma \ln(2s)-\ln(A)+R(s) , \tag{20}$$

$$\lim_{s\to\infty} R(s)=0 . \tag{21}$$

The lower bound for $|g(s)|$ in Eq. (16) corresponds to setting $\beta=0$ and $\gamma>\frac{1}{2}$ in Eq. (20), while the upper bound corresponds to $\beta>0$ and $0<\alpha\leq 1$, with no restrictions on γ. Since

$$s(r) \underset{r\to\infty}{\to} \tfrac{1}{2}r+O(1) , \tag{22}$$

the asymptotic form of the continuum bound-state wave function is

$$\psi(r) \underset{r\to\infty}{\to} Ar^{-\gamma}\exp(-\beta r^{\alpha})\sin(k_1 r+\delta_1) , \tag{23}$$

while the asymptotic form of $V(r)$ as $r\to\infty$ is

$$\begin{aligned} V(r)-V_c(r) \underset{r\to\infty}{\to} & -4(\alpha\beta r^{\alpha-1}+\gamma r^{-1}) \\ & \times k_1\sin[2(k_1 r+\delta_1)] \\ & +4\beta^2\alpha^2 r^{2(\alpha-1)}\sin^4(k_1 r+\delta_1) \\ & +o(r^{-1}) . \end{aligned} \tag{24}$$

If $\beta=0$, the continuum bound state decreases with increasing r according to a power law rather than showing the exponential decrease normal for a conventional bound state. The supporting potential $V(r)$ oscillates with wave number $2k_1$ and with an amplitude which decreases as r^{-1} with increasing r. If $\beta\neq 0$, the continuum bound state decreases exponentially as $r\to\infty$, but the exponent is determined by the parameter α rather than by the energy of the state, as would be the case for a conventional bound state. Furthermore, if $\beta\neq 0$ the potential for large r oscillates with an amplitude which decreases more slowly than r^{-1} as $r\to\infty$, while if α is as large as 1 the amplitude of the oscillations for large r remains constant.

The von Neumann–Wigner procedure provides a wide range of possible potentials which support continuum bound states. However, it provides no method other than numerical integration for constructing the complete energy spectrum and the associated energy eigenfunctions. Nevertheless, if $\beta>0$ and $0<\alpha<\frac{1}{2}$, or if $\beta=0$, we can at least demonstrate the existence of scattering solutions satisfying the asymptotic condition of Eq. (2) for all $k\neq k_1$. For this, we rely on an analysis by Atkinson [14]. If $V(r)$ satisfies the conditions

$$\left|\int_r^{\infty} d\xi\, V(\xi)\right| < \infty , \tag{25}$$

$$\begin{aligned} I_c(r) &\equiv \left|\int_r^{\infty} d\xi\, V(\xi)\cos(2k\xi)\right| < \infty , \\ I_s(r) &\equiv \left|\int_r^{\infty} d\xi\, V(\xi)\sin(2k\xi)\right| < \infty , \end{aligned} \tag{26}$$

and

$$\int_0^{\infty} d\xi |V(\xi)I_c(\xi)| < \infty , \quad \int_0^{\infty} d\xi |V(\xi)I_s(\xi)| < \infty , \tag{27}$$

then every nontrivial solution of Eq. (3) has an asymptotic form given by Eq. (2) (Theorem 2.1 of Ref. [14]). For this argument, we assume that the potential of Eq. (13) has no singularities for finite r. Provided $k\neq k_1$ and either $\beta=0$ or else $\beta\neq 0$ and $0<\alpha<\frac{1}{2}$, the conditions of Eqs. (25)–(27) are satisfied by a potential with asymptotic form Eq. (24). This is sufficient to prove the existence of a scattering state for any positive energy k^2 provided $k\neq k_1$, but does not give any procedure for constructing analytic expressions for the scattering states. However, Atkinson's theorem does not settle the question of whether or not bound states exist. In particular, we do not know whether any bound states of Eq. (1) survive as bound states of Eq. (3), possibly with a shifted energy. If it were not for these difficulties, one could construct potentials with multiple bound states of positive energy by repeating the von Neumann–Wigner procedure. For example, one could use the potential supplying a bound state at $E_1=k_1^2$ as the comparison potential for the introduction of a bound state at $E_2=k_2^2$. The methods to be discussed in Secs. III and IV will overcome this difficulty, but at the cost of restricting the potentials to a small subset of those produced by the von Neumann–Wigner method.

The possible existence of a continuum bound state is associated with the oscillating long-range tail of the potential. Can one use a potential with an arbitrarily weak asymptotic form and still create a continuum bound state at an arbitrarily chosen energy? This question is partially answered by Eq. (24). Provided $\beta\neq 0$, the coefficient of the leading term in the asymptotic expansion of $V(r)$ can be made arbitrarily small, independently of the value of k or $E=k^2$. If $\gamma=0$ in addition to $\beta\neq 0$, then the coefficient of the next leading term is small if that of the leading term is small. Furthermore, if $\beta\to 0$ but $\beta\neq 0$, then Eq. (23) shows that the wave-function tail increases in spatial extent. On the other hand, if $\beta=0$, so that we require $\gamma>\frac{1}{2}$ for normalizability, then in order to create a continuum bound state with energy $E=k^2$ one must use a potential whose leading asymptotic term has a coefficient at least equal to $2k$. For the original example of von Neumann and Wigner [1], with $\beta=0$ and $\gamma=2$, the coefficient of the leading asymptotic term in $V(r)$ is $8k$. These remarks do not contradict the assertion made by Vary *et al.* [18] that for a von Neumann–Wigner continuum bound state inserted at an arbitrarily chosen energy, the value of the potential *at a fixed value of* r can be made arbitrarily small by a suitable choice of the parameter.

Does an arbitrary potential with asymptotic form given by Eq. (24) necessarily support a continuum bound state at energy k_1^2? On the one hand, there are infinitely many comparison potentials $V_c(r)$ which support scattering states such that the scattering phase shift at energy k_1^2 has the value δ_1, and any one of these could be used as the starting point of the construction. Then again, the function $f(s)$ is largely arbitrary. These arguments might suggest that almost any potential with the asymptotic form given by Eq. (24) will support a bound state with energy k_1^2. However, bound states are always associated with standing waves, which can exist only for cer-

tain discrete values of the energy. It would seem highly improbable that the existence of a bound state of energy $E_1=k_1^2$ could be inferred from the asymptotic form of the potential alone, regardless of how the potential behaves at smaller radii. For example, suppose $V(r)$ has the asymptotic form Eq. (24) with $\beta=0$, $\gamma>\frac{1}{2}$. Then any solution of Eq. (3) with $k=k_1$ must be a linear combination of solutions $\psi_\pm(r,k_1)$ having asymptotic forms

$$\psi_\pm(r,k_1)\to A_\pm r^{\pm\gamma}\sin(k_1 r+\delta_1+\tfrac{1}{4}\pi\pm\tfrac{1}{4}\pi) . \tag{28}$$

For a physically acceptable wave function $\psi(r,k_1)$, the coefficients of $\psi_+(r,k_1)$ and $\psi_-(r,k_1)$ are determined by the condition $\psi(0,k_1)=0$. In addition, the wave function of a bound state with energy k_1^2 must be square integrable. This will not be so unless the coefficient of $\psi_+(r,k_1)$ is zero. One would expect this to be true only for a small subset of potentials with asymptotic form Eq. (24). We conclude that while there are infinitely many potentials which do support a continuum bound state at energy $E_1=k_1^2$, there are infinitely many more which do not do so, even though they appear to have the appropriate asymptotic behavior for large r. We shall return to this question in Sec. IV, where we shall give an explicit procedure for constructing potentials which do not support continuum bound states.

III. THE MEYER-VERNET METHOD

The Gel'fand-Levitan equation has the form

$$K(r,r')=g(r,r')-\int_0^r d\xi\, K(r,\xi)g(\xi,r') , \tag{29}$$

where the kernel $g(r,r')$ is constructed from solutions $\varphi(r,E)$ of Eq. (1) with the boundary condition $\varphi(0,E)=0$. In the context of the present paper, $g(r,r')$ is chosen to be [15,16]

$$g(r,r')=\sum_{i=1}^{n}\lambda_i^{-1}\varphi_i(r)\varphi_i(r') , \tag{30}$$

where for convenience we write $\varphi(r,E_i)\equiv\varphi_i(r)$. The parameters λ_i must be such that the $n\times n$ matrix $\Delta(r)$ defined by

$$\Delta_{ij}(r)=\lambda_i\delta_{ij}+\int_0^r d\xi\,\varphi_i(\xi)\varphi_j(\xi) , \quad i,j=1,\ldots,n \tag{31}$$

is nonsingular, but are otherwise arbitrary. In the preceding paper [16], we have shown that Eq. (29) with $g(r,r')$ defined by Eq. (30) has the unique solution

$$K(r,r')=\sum_{i,j=1}^{n}\varphi_i(r)[\Delta^{-1}(r)]_{ij}\varphi_j(r') . \tag{32}$$

The function $\psi(r,E)$ defined by

$$\begin{aligned}\psi(r,E)&=\varphi(r,E)-\int_0^r d\xi\, K(r,\xi)\varphi(\xi,E)\\&=\varphi(r,E)-\sum_{i,j=1}^{n}\varphi_i(r)[\Delta^{-1}(r)]_{ij}\\&\quad\times\int_0^r d\xi\,\varphi_j(\xi)\varphi(\xi,E) ,\end{aligned} \tag{33}$$

is the solution corresponding to energy E of a new Schrödinger equation with potential $V(r)$ given by

$$V(r)=V_c(r)-2\frac{d}{dr}K(r,r)=V_c(r)-2\frac{d^2}{dr^2}\ln|\det\Delta(r)| . \tag{34}$$

Clearly $\psi(0,E)=0$ and E belongs to the physical spectrum of the new Hamiltonian if $\psi(r,E)$ is bounded as $r\to\infty$. As shown in Ref. [16], $\psi(r,E)$, $E\neq E_i$, corresponds to a bound state or a continuum state with energy E of the new Schrödinger equation according to whether $\varphi(r,E)$ corresponds to a bound state or a continuum state of the comparison Schrödinger equation. If, however, $E=E_i$ for some i, then

$$\psi_i(r)\equiv\psi(r,E_i)=\sum_{j=1}^{n}\varphi_j(r)[\Delta^{-1}(r)]_{ji}\lambda_i . \tag{35}$$

Conditions for $\psi_i(r)$ to be normalizable are discussed in Ref. [16] In particular, it is sufficient that $\lim_{r\to\infty}\Delta_{ii}(r)=\infty$.

Meyer-Vernet's approach to continuum bound states [12] is easily described in the present formalism. If some or all of the E_i are embedded in the continuous spectrum of the comparison Hamiltonian, the corresponding wave functions $\varphi_i(r)$ are not normalizable even though they are bounded as $r\to\infty$. The constraint that $\Delta(r)$ be nonsingular then requires that $\lambda_i>0$. Since the $\varphi_i(r)$ are continuum eigenstates, $\lim_{r\to\infty}\Delta_{ii}(r)=\infty$ and therefore the functions $\psi_i(r)$ given by Eq. (35) are normalizable and correspond to continuum bound states of the Hamiltonian with potential $V(r)$ given by Eq. (34).

As an illustration of the method, we choose $n=1$ and $\lambda_1\equiv\lambda>0$. Then the (normalized) wave function for the continuum bound state with energy $E_1\equiv k_1^2>0$ is

$$\psi_1(r)\equiv\psi(r,k_1)=\frac{\lambda^{1/2}\varphi_1(r)}{\lambda+\int_0^r d\xi|\varphi_1(\xi)|^2}=\frac{\lambda^{1/2}\varphi_1(r)}{\lambda+s(r)} , \tag{36}$$

while the potential which supports this continuum bound state is

$$\begin{aligned}V(r)&=V_c(r)-2\frac{d}{dr}\left[\frac{\varphi_1^2(r)}{\lambda+s(r)}\right]\\&=V_c(r)-2\frac{d^2}{dr^2}\ln|\lambda+s(r)| .\end{aligned} \tag{37}$$

It is apparent that this is just a special case of the von Neumann–Wigner procedure discussed in Sec. II with

$$f(s)=\frac{\lambda^{1/2}}{[\lambda+s(r)]} . \tag{38}$$

The asymptotic forms of $\psi(r)$ and $V(r)$ for large r are given by Eqs. (23) and (24), respectively, with $\alpha=0$, $\gamma=1$, and $A=2\lambda^{1/2}$.

In the more general case, one sees from Eq. (31) that $\Delta(r)$ approaches a diagonal form in the limit as $r\to\infty$. Hence

$$\lim_{r\to\infty}\left[\frac{\det\Delta(r)}{\prod_{i=1}^{n}\Delta_{ii}(r)}\right]=1\ , \tag{39}$$

so that the asymptotic form of the potential generated by the Meyer-Vernet method is given by

$$V(r)-V_c(r)\underset{r\to\infty}{\sim}-2\sum_{i=1}^{n}\frac{d^2}{dr^2}\ln\left\{\lambda_i+\int_0^r d\xi[\varphi_i(\xi)]^2\right\}$$
$$\underset{r\to\infty}{\sim}-4r^{-1}\sum_{i=1}^{n}k_i\sin[2(k_ir+\delta_i)]+O(r^{-2})\ , \tag{40}$$

while the (normalized) continuum bound-state wave functions have the asymptotic forms

$$\psi_i(r)\underset{r\to\infty}{\sim}2\lambda^{1/2}r^{-1}\sin(k_ir+\delta_i)+O(r^{-2})\ . \tag{41}$$

Whereas one cannot use the von Neumann–Wigner method to construct potentials by analytic methods which are guaranteed to possess more than one continuum bound state, doing so is easy with the Meyer-Vernet approach as we have presented it. However, this flexibility is gained at the cost of obtaining a less general class of potentials supporting continuum bound states.

IV. REMOVAL OF von NEUMANN–WIGNER STATES

In Ref. [16] we showed that one can remove bound-state energies from the spectrum of a Hamiltonian using methods based on either the Gel'fand-Levitan or the Marchenko equation. We gain more insight into the von Neumann–Wigner potentials by using these methods to delete the continuum bound state. These procedures will not change any of the other energy eigenvalues associated with the original von Neumann–Wigner potential. In general, however, they will not recover the comparison potential $V_c(r)$. Instead, for the Gel'fand-Levitan approach we find

$$\bar{V}(r)=V(r)-2\frac{d^2}{dr^2}\ln\left[\int_r^\infty d\xi\,\psi^2(\xi)\right]\ , \tag{42}$$

while for the Marchenko method we find

$$\tilde{V}(r)=V(r)-2\frac{d^2}{dr^2}\ln\left[\int_0^r d\xi\,\psi^2(\xi)\right]\ , \tag{43}$$

where $\psi(r)$ and $V(r)$ are defined by Eqs. (4) and (5), respectively. As is clear from Eqs. (42) and (43), $\bar{V}(r)$ and $\tilde{V}(r)$ are not the same. By Eq. (10),

$$\int_r^\infty d\xi[\psi(\xi)]^2=\int_s^\infty d\xi[f(\xi)]^2\equiv J(s) \tag{44}$$

and

$$\int_0^r d\xi[\psi(\xi)]^2=\int_0^s d\xi[f(\xi)]^2\equiv I(s)\ . \tag{45}$$

Hence

$$\bar{V}(r)=V_c(r)+4\varphi_1'(r)\varphi_1(r)\left\{\frac{f_s(s)}{f(s)}+\frac{[f(s)]^2}{J(s)}\right\}$$
$$+[\varphi_1(r)]^4\left\{\frac{f_{ss}(s)}{f(s)}+\frac{4f_s(s)f(s)}{J(s)}+\frac{2[f(s)]^4}{[J(s)]^2}\right\} \tag{46}$$

and

$$\tilde{V}(r)=V_c(r)+4\varphi_1'(r)\varphi_1(r)\left\{\frac{f_s(s)}{f(s)}-\frac{[f(s)]^2}{I(s)}\right\}$$
$$+[\varphi_1(r)]^4\left\{\frac{f_{ss}(s)}{f(s)}-\frac{4f_s(s)f(s)}{I(s)}+\frac{2[f(s)]^4}{[I(s)]^2}\right\}\ . \tag{47}$$

We consider the asymptotic forms of $J(s)$ and $I(s)$ in order to study the asymptotic forms of $\bar{V}(r)$ and $\tilde{V}(r)$. From Eqs. (12), (20), and (44)

$$J(s)\underset{s\to\infty}{\sim}A^2\int_s^\infty d\xi(2\xi)^{-2\gamma}\exp[-2\beta(2\xi)^\alpha]\ . \tag{48}$$

The leading term, obtained by an integration by parts, is

$$J(s)\underset{s\to\infty}{\sim}\begin{cases}\frac{1}{4}A^2(\alpha\beta)^{-1}(2s)^{1-\alpha-2\gamma}\exp[-2\beta(2s)^\alpha], & \beta\neq0\\ \frac{1}{2}A^2(2\gamma-1)^{-1}(2s)^{-2\gamma+1}\ , & \beta=0\ .\end{cases}$$

From Eqs. (44), (45), and (49)

$$I(s)=\int_0^\infty d\xi[f(s)]^2-J(s)$$
$$\underset{s\to\infty}{\sim}\begin{cases}N^2+O(s^{1-\alpha-2\gamma}\exp[-2\beta(2s)^\alpha])\ , & \beta\neq0\\ N^2+O(s^{-2\gamma+1})\ , & \beta=0\ ,\end{cases} \tag{50}$$

where

$$N^2=\int_0^\infty d\xi[f(\xi)]^2\ . \tag{51}$$

We shall first consider the asymptotic form of $\tilde{V}(r)$ for large r, postponing discussion of $\bar{V}(r)$ until later. From Eqs. (46) and (50),

$$\tilde{V}(r)-V(r)=-4\varphi_1'(r)\varphi_1(r)\frac{[f(s)]^2}{I(s)}$$
$$-2[\varphi_1(r)]^4\left[\frac{2f_s(s)f(s)}{I(s)}-\frac{[f(s)]^4}{[I(s)]^2}\right]\ ,$$
$$\underset{r\to\infty}{\sim}O(r^{-2\gamma}\exp[-2\beta r^\alpha])\ . \tag{52}$$

Because $I(0)=0$, this potential is singular at $r=0$. If

$$f(s)\underset{s\to0}{\sim}O(s^\rho)\ , \tag{53}$$

where $\rho>-\frac{1}{6}$ by Eqs. (8) and (11), it can be shown that

$$\lim_{r\to0}\{r^2[\tilde{V}(r)-V(r)]\}=6(2\rho+1)>0\ . \tag{54}$$

Since this is positive, $\tilde{V}(r)$ is a physically acceptable po-

tential. From Eq. (52) it follows that $\tilde{V}(r)$ is indistinguishable from the von Neumann–Wigner potential $V(r)$ for sufficiently large r, although if $\beta=0$ the approach of $\tilde{V}(r)$ to $V(r)$ as $r\to\infty$ may be quite slow.

The wave function with energy k_1^2 constructed by the Marchenko procedure to satisfy the Schrödinger equation with potential $\tilde{V}(r)$ is

$$\tilde{\chi}(r,k_1)=\frac{N^2}{\int_0^s d\xi[f(\xi)]^2}f(s)\varphi(r,k_1)\underset{r\to\infty}{\sim}\psi(r,k_1)\ . \tag{55}$$

As anticipated, the function $\tilde{\chi}(r,k_1)$ is not normalizable, because it diverges at $r=0$. We denote the solution of the Schrödinger equation with energy k_1^2 and which vanishes at the origin by $\chi(r,k_1)$. Then the Wronskian of $\tilde{\chi}(r,k_1)$ and $\chi(r,k_1)$ is a constant. Since $\lim_{r\to\infty}\tilde{\chi}(r,k_1)=0$, it follows that $\chi(r,k_1)$ must be unbounded in the same limit. Hence there does not exist any physically acceptable solution of the Schrödinger equation with potential $\tilde{V}(r)$ corresponding to the energy $E_1=k_1^2$.

We now return to the consideration of $\bar{V}(r)$. Because the asymptotic forms of $J(s)$ are quite different for $\beta\neq0$ and $\beta=0$, we consider these two cases separately.

If $\beta\neq0$, then

$$\frac{[f(s)]^2}{J(s)}\underset{s\to\infty}{\sim}4\alpha\beta(2s)^{\alpha-1} \tag{56}$$

and

$$\frac{4f_s(s)f(s)}{J(s)}+\frac{2[f(s)]^4}{[J(s)]^2}=2\frac{[f(s)]^2}{J(s)}\left\{\frac{[f(s)]^2}{J(s)}-2g_s(s)\right\}\underset{r\to\infty}{\sim}O(r^{\alpha-2})\ . \tag{57}$$

Hence

$$\bar{V}(r)-V_c(r)\underset{r\to\infty}{\sim}4k_1(\alpha\beta r^{\alpha-1}-\gamma r^{-1})\sin[2(k_1r+\delta_1)]+4\beta^2\alpha^2r^{2(\alpha-1)}\sin^4(k_1r+\delta_1)+O(r^{\alpha-2})\ . \tag{58}$$

Since $\beta\neq0$ implies $\alpha>0$, this potential decreases with increasing r less rapidly than r^{-1}. However, from Eqs. (17), (18), and (24), the sign of the term proportional to $4k_1\alpha\beta r^{-1}$ is wrong for Eq. (58) to be the asymptotic form of a von Neumann–Wigner potential.

We next consider $\beta=0$. In this case,

$$\frac{[f(s)]^2}{J(s)}\underset{r\to\infty}{\sim}2(2\gamma-1)r^{-1}\ , \tag{59}$$

while

$$\frac{4f_s(s)f(s)}{J(s)}+\frac{2[f(s)]^4}{[J(s)]^2}\underset{r\to\infty}{\sim}O(r^{-2})\ . \tag{60}$$

Therefore

$$\bar{V}(r)-V_c(r)\underset{r\to\infty}{\sim}4(\gamma-1)k_1r^{-1}\sin[2(k_1r+\delta_1)]+O(r^{-2})\ . \tag{61}$$

Unless $\gamma=1$, we are faced with a potential which oscillates with an amplitude which decreases at large r only as slowly as r^{-1}. Since $\beta=0$, it follows from Eqs. (19) and (24) that Eq. (61) cannot be the asymptotic form of a von Neumann–Wigner potential.

By using the Gel'fand-Levitan method we have destroyed the von Neumann–Wigner continuum bound state, but it is conceivable that the wave function $\bar{\chi}(r,k_1)$ with $\bar{\chi}(0,k_1)=0$ created by the procedure might be a scattering state. From Eqs. (4), (35) and (10),

$$\bar{\chi}(r,k_1)=\frac{N^2}{\int_s^\infty d\xi[f(\xi)]^2}f(s)\varphi(r,k_1)\ . \tag{62}$$

From Eqs. (12), (20), (44), and (49),

$$\bar{\chi}(r,k_1)\underset{r\to\infty}{\sim}\begin{cases}O(s^{\alpha+\gamma-1}\exp[\beta(2s)^\alpha])\ , & \beta\neq0\\ O(s^{\gamma-1})\ , & \beta=0\ .\end{cases} \tag{63}$$

Hence if either $\beta\neq0$ or $\beta=0$ and $\gamma>1$, the wave function $\bar{\chi}(r,k_1)$ is unbounded as $r\to\infty$, so that $E_1=k_1^2$ is neither a bound-state energy nor a continuum-state energy. If $\beta=0$ and $\gamma=1$, then $\bar{\chi}(r,k_1)$ represents a scattering state. If $\beta=0$ and $\frac{1}{2}<\gamma<1$, then $\bar{\chi}(r,k_1)$ represents neither a bound state nor a scattering state. The function $\bar{\chi}(r,k_1)$ is bounded and satisfies the boundary condition at the origin, so that it is a physically acceptable wave function; however, it oscillates with a decreasing amplitude for large r, so that it is not a conventional scattering state; but neither is it normalizable, so that it does not represent a true bound state.

It follows from this discussion that corresponding to any von Neumann–Wigner potential $V(r)$ whose asymptotic form is given by Eq. (24) and which supports a bound state in the continuum with energy $E_1=k_1^2$ there always exists at least one potential $\bar{V}(r)$ whose asymptotic forms contain oscillatory terms with amplitude decreasing no faster than r^{-1} and which does not support any physically acceptable state with energy E_1. Indeed whatever may be the values of the parameters α, β, and γ, the asymptotic form of $\bar{V}(r)$ is indistinguishable from that of $V(r)$, so that $\bar{V}(r)$ cannot be distinguished from a von Neumann–Wigner potential on the basis of its asymptotic form. Unless $\beta=0$ and $\gamma=1$, there exists another potential $\bar{V}(r)$ whose asymptotic form includes oscillating terms with an amplitude decreasing no faster than r^{-1} as $r\to\infty$ and which supports a physically acceptable state with energy E_1 only if $\beta=0$ and $\frac{1}{2}<\gamma<1$, but when this state exists it is neither a bound state nor a conventional scattering state; one might term it an imperfectly localized state. If $\beta=0$ and $\gamma=1$, then for large r the difference $\bar{V}(r)-V_c(r)$ decreases at least as fast as r^{-2} and therefore $\bar{V}(r)$ supports a conventional scattering solution with energy E_1.

V. TWO EXAMPLES

In this section we present two examples designed to illustrate the theory presented in the preceding section. For both examples, we shall choose

$$V_c(r)=0 \tag{64}$$

and

$$\varphi_1(r)=2k_1^{1/2}\sin(k_1r)\ , \tag{65}$$

so that

$$s\equiv\int_0^r d\xi[\varphi_1(r)]^2=2k_1r-\sin(2k_1r)\ . \tag{66}$$

Example 1. For the first example, we choose

$$f(s)=(1+s)^{-1/4}\exp[1-(1+s)^{1/2}]\ , \tag{67}$$

so that

$$J(s)\equiv\int_s^\infty d\xi|f(\xi)|^2=\exp[2-2(1+s)^{1/2}] \tag{68}$$

and

$$I(s)=1-\exp[2-2(1+s)^{1/2}]\ . \tag{69}$$

The von Neumann–Wigner potential is given by

$$\begin{aligned}V(r)=&-4k_1^2[2(1+s)^{-1/2}+(1+s)^{-1}]\\&\times\sin(k_1r)\cos(k_1r)\\&+k_1^2[4(1+s)^{-1}+8(1+s)^{-3/2}+5(1+s)^{-2}]\\&\times\sin^4(k_1r)\ ,\end{aligned} \tag{70}$$

while the von Neumann–Wigner bound-state wave function is

$$\psi(r)=2k_1^{1/2}(1+s)^{-1/4}\exp[1-(1+s)^{1/2}]\sin(k_1r)\ . \tag{71}$$

If this state is now removed by the Gel'fand-Levitan method, the new potential is

$$\begin{aligned}\bar{V}(r)&=V(r)-2\frac{d^2}{dr^2}\ln[J(r)]\\&=4k_1^2[2(1+s)^{-1/2}-(1+s)^{-1}]\sin(k_1r)\cos(k_1r)+k_1^2[4(1+s)^{-1}-8(1+s)^{-3/2}+5(1+s)^{-2}]\sin^4(k_1r)\ .\end{aligned} \tag{72}$$

This differs from Eq. (70) only in the signs of two terms. The wave function $\psi(r)$ of the von Neumann–Wigner state has been transformed into

$$\bar{\chi}(r)=\frac{\psi(r)}{J(s)}=2k_1^{1/2}(1+s)^{-1/4}\exp[(1+s)^{1/2}-1]\sin(k_1r)\ , \tag{73}$$

which clearly oscillates with a diverging amplitude as $r\to\infty$. Since $\bar{\chi}(r)$ satisfies the physical boundary condition at the origin and no other solution can do so, the Schrödinger equation with potential $\bar{V}(r)$ has no physically acceptable solutions with energy $E_1=k_1^2$.

If the von Neumann–Wigner state is removed using the Marchenko method, the potential becomes

$$\begin{aligned}\tilde{V}(r)&=V(r)-2\frac{d^2}{dr^2}\ln[I(r)]\\&=-4k_1^2\{2(1+s)^{-1/2}\coth[(1+s)^{1/2}-1]+(1+s)^{-1}\}\sin(k_1r)\cos(k_1r)\\&\quad+k_1^2\{8(1+s)^{-1}\coth^2[(1+s)^{1/2}-1]-4(1+s)^{-1}\\&\qquad+8(1+s)^{-3/2}\coth[(1+s)^{1/2}-1]+5(1+s)^{-2}\}\sin^4(k_1r)\ .\end{aligned} \tag{74}$$

This potential is readily seen to satisfy Eq. (54) with $\rho=0$. The Marchenko procedure transforms the von Neumann–Wigner wave function $\psi(r)$ into

$$\tilde{\chi}(r)=\frac{\psi(r)}{I(s)}=\frac{k_1^{1/2}(1+s)^{-1/4}\sin(k_1r)}{\sinh[(1+s)^{1/2}-1]}\ . \tag{75}$$

Since s is of order r^3 for small r, the function $\tilde{\chi}(r)$ diverges like r^{-2} as $r\to0$ and $\tilde{\chi}(r)$ is not normalizable. For very large r, $\tilde{\chi}(r)$ behaves like $r^{-1/4}\exp(-r^{1/2})\sin(k_1r)$. The second solution of the Schrödinger equation with potential $\tilde{V}(r)$ must behave for large r like $r^{1/4}\exp(r^{1/2})\cos(k_1r)$ in order that the Wronskian of the two solutions be a constant. Hence the Schrödinger equation with potential $\tilde{V}(r)$ has no physically acceptable solutions with energy $E_1=k_1^2$. The three potentials $V(r)$, $\bar{V}(r)$, and $\tilde{V}(r)$ of example 1 are compared in Fig. 1.

Example 2. For the second example, we choose

$$f(s)=(2n-1)^{1/2}(1+s)^{-n}\ ,\quad n>\tfrac{1}{2}\ , \tag{76}$$

so that

$$J(s)\equiv\int_s^\infty d\xi[f(\xi)]^2=\frac{1}{(1+s)^{2n-1}} \tag{77}$$

and

$$I(s)=1-\frac{1}{(1+s)^{2n-1}}\ . \tag{78}$$

Then the von Neumann–Wigner potential is

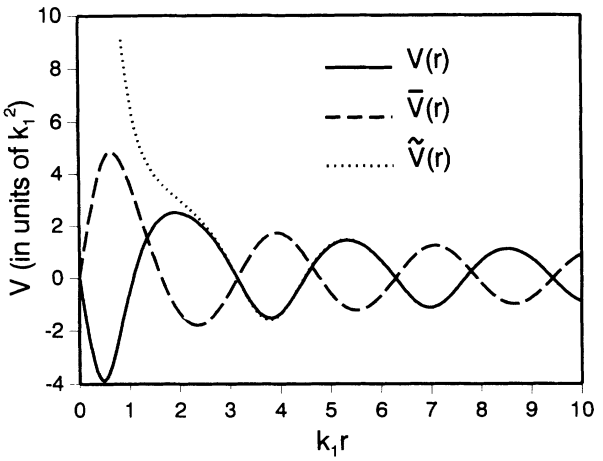

FIG. 1. Comparison of three potentials for example 1. The von Neumann–Wigner potential is $V(r)$, the potential that results from eliminating the continuum bound state using the Gel'fand-Levitan method is $\bar{V}(r)$, and the potential that results from eliminating the continuum bound state using the Marchenko method is $\tilde{V}(r)$. Unlike $V(r)$, neither $\bar{V}(r)$ nor $\tilde{V}(r)$ supports a continuum bound state.

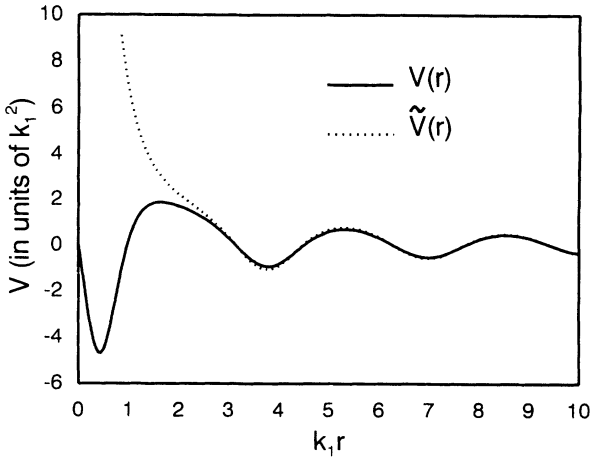

FIG. 2. Comparison two potentials for example 2. The von Neumann–Wigner potential is $V(r)$ and the potential that results from eliminating the continuum bound state using the Marchenko method is $\tilde{V}(r)$. Unlike $V(r)$, $\tilde{V}(r)$ does not support a continuum bound state.

$$V(r)=16nk_1^2\left[-\frac{1}{1+s}\sin(k_1r)\cos(k_1r)+\frac{(n+1)}{(1+s)^2}\sin^4(k_1r)\right] \tag{79}$$

and the von Neumann–Wigner wave function is

$$\psi(r)=2(2n-1)^{1/2}k_1^{1/2}\frac{\sin(k_1r)}{(1+s)^n}\ . \tag{80}$$

Use of the Gel'fand-Levitan method to remove the von Neumann–Wigner state yields the potential

$$\bar{V}(r)=16(n-1)k_1^2\left[\frac{1}{1+s}\sin(k_1r)\cos(k_1r)+\frac{(n-2)}{(1+s)^2}\sin^4(k_1r)\right]\ . \tag{81}$$

This vanishes as it should if $n=1$, for then the same potential and wave function could be obtained by the Meyer-Vernet method. The von Neumann–Wigner wave function is transformed into

$$\bar{\chi}(r)=2(2n-1)^{1/2}k_1^{1/2}(1+s)^{n-1}\sin(k_1r)\ . \tag{82}$$

If $n=1$, then (apart from a constant factor) this reduces as it should to $\varphi_1(r)$ of Eq. (65). If $n>\frac{1}{2}$, as is required according to the work of Sec. II, then $\bar{\chi}(r)$ is not normalizable. If, however, $\frac{1}{2}<n<1$, the function $\bar{\chi}(r)$ represents neither a bound state nor a conventional scattering state: instead, $\bar{\chi}(r)$ oscillates with an amplitude which decreases monotonically as r increases, but not fast enough for it to be normalizable. This illustrates the material of Sec. IV.

We illustrate the use of the Marchenko method to eliminate the von Neumann–Wigner state for the special case $n=1$. The new potential is

$$\tilde{V}(r)=16k_1^2\left\{-\frac{1}{s}\sin(k_1r)\cos(k_1r)+\frac{2}{s^2}\sin^4(k_1r)\right\}\ . \tag{83}$$

This potential behaves like $6r^{-2}$ as $r\to 0$, as required by Eq. (54) with $\rho=0$. The solution of the Schrödinger equation with this potential, for energy $E_1=k_1^2$, is

$$\tilde{\chi}(r)=2k_1^{1/2}\frac{\sin(k_1r)}{s}\ , \tag{84}$$

which clearly diverges at $r=0$. However, any other solution with this energy behaves like $r\cos(k_1r)$ for large r, which is unacceptable for a physical solution. Hence once again we have a potential which has no physically acceptable solutions for energy k_1^2. The two potentials $V(r)$ and $\tilde{V}(r)$ for example 2 are compared in Fig. 2.

VI. METHODS BASED ON SUPERSYMMETRY

Supersymmetry in nonrelativistic quantum mechanics was first considered by Witten [19]. Fairly soon, it was recognized that the techniques of supersymmetrical quantum mechanics are a direct application of a theorem due to Darboux [20]. We refer to Luban and Pursey [21] for a simple exposition of the Darboux procedure and to Dutt, Khare, and Sukhatme [22] for an elementary introduction which emphasizes supersymmetry. The cited references include extensive bibliographies.

In supersymmetric quantum mechanics with unbroken supersymmetry, the ground state is nondegenerate, but all of the remaining energy eigenvalues are doubly degenerate. The state space can be divided into two sectors,

sometimes misleadingly referred to as the boson and fermion sectors. Transitions between the sectors are achieved using the generators of the supersymmetry. For convenience we shall refer to sectors 1 and 2, with sector 1 containing the ground state. Within each sector, the dynamics is described by a conventional nonrelativistic Schrödinger equation. If the potential in the sector containing the nondegenerate ground state is $V_1(r) \equiv V_c(r)$, then the potential in the other sector is

$$V_2(r) = V_1(r) - 2\frac{d^2}{dr^2}[\ln\varphi_0(r)] , \tag{85}$$

where $\varphi_0(r)$ is the ground-state wave function. The second sector potential may be regarded as the result of using the Darboux method to eliminate the ground state in the first sector. The first sector potential may be recovered from $V_2(r)$ by using the Darboux method to reintroduce a ground state, but as noted by Nieto [23] this process is ambiguous: more than one potential in sector 1 can be a supersymmetric partner to the same potential in sector 2. As shown by Pursey [24], the different sector 1 potentials are related by "renormalization" of the ground state. Exactly the same effect may be achieved by use of the Darboux method to first delete the ground state and then reintroduce a new ground state or using either the Gel'fand-Levitan-Abraham-Moses approach [21] or the Marchenko equation [25] to directly renormalize the ground state.

At first sight, it would seem that this procedure could be applied only to the ground state since the potential $V_2(r)$ defined by Eq. (85) has double poles at the zeros of the wave function $\varphi(r)$. However, Pappademos, Sukhatme, and Pagnamenta [13] discovered that these singularities vanish when the deleted state is replaced in a second step by one with the same energy. When the two-step process is applied to the scattering state with wave function $\varphi_1(r) \equiv \varphi(r,k_1)$, this function is replaced by

$$\psi_1(r) \equiv \psi(r,k_1) = \frac{\varphi_1(r)}{\lambda + \int_0^r d\xi |\varphi_1(\xi)|^2} , \tag{86}$$

[Eq. (4) of Ref. [13], translated into the notation of the present paper] while the new potential is

$$V(r) = V_c(r) - 2\frac{d^2}{dr^2}\ln\left[\lambda + \int_0^r d\xi |\varphi_1(\xi)|^2\right] \tag{87}$$

[Eq. (2) of Ref. [13]]. Apart from the normalization of $\psi_1(r)$, these are identical to Eqs. (36) and (37) of this paper, respectively. Hence the Pappademos-Sukhatme-Pagnamenta method for creating a single continuum bound state is exactly equivalent to the Meyer-Vernet method. This should be no surprise in view of the analogous result for the ground state [21].

Pappademos, Sukhatme, and Pagnamenta introduce several continuum bound states by applying their technique a number of times in succession. As we have just demonstrated, this must be equivalent to successive applications of the Meyer-Vernet procedure for $n=1$. However, we have shown in Ref. [16] that m Meyer-Vernet transformations with $n=1$ applied in succession are equivalent to a single transformation with $n=m$. Hence the supersymmetry procedure for constructing continuum bound states developed by Pappademos, Sukhatme, and Pagnamenta is in every respect equivalent to the Meyer-Vernet procedure and the decision of which formalism to use is a matter of personal preference only.

VII. CONCLUSION

The von Neumann–Wigner [1] method described in Sec. II allows the construction of a wide range of potentials which support a single continuum bound state, but provides no analytic procedure for constructing the complete energy spectrum or the energy eigenfunctions for states other than the continuum bound state. For many of the von Neumann–Wigner potentials, a theorem due to Atkinson [14] demonstrates that scattering solutions exist for any $k \neq k_1$, so that the potential supports only the one continuum bound state.

This conclusion may be generalized. Suppose that we are given a potential with asymptotic form

$$V(r) \underset{r\to\infty}{\sim} \sum_{i=1}^{n} A_i r^{\alpha_i - 1}\sin(k_i r + \delta_i) + o(r^{-1}) ,$$
$$0 \le \alpha_i < \tfrac{1}{2} . \tag{88}$$

Then Atkinson's theorem shows that this potential supports scattering solutions for $k \neq k_i$, $i=1,\ldots,n$, and therefore can support at most n continuum bound states. However, we are unable to determine whether or not the potential does indeed support as many as n continuum bound states or even if it supports any at all. Indeed in Secs. IV and V we were able to construct potentials whose asymptotic form is given by Eq. (88) either with $n=1$ or with $\alpha=0$, but which do not support either a scattering state or a bound state with energy k_1^2.

The problems encountered with the von Neumann–Wigner procedure are overcome either by the Meyer-Vernet [12] method based on the Gel'fand-Levitan equation and discussed above in Sec. III or by the exactly equivalent method of Pappademos, Sukhatme, and Pagnamenta [13] based on the Darboux transformation in the guise of supersymmetric quantum mechanics and discussed in Sec. VI. These equivalent procedures generate a restricted class of von Neumann–Wigner potentials. Starting from a comparison Schrödinger equation whose solutions are completely known, potentials may be constructed which support an arbitrary number of bound states at arbitrary energies embedded in the continuum with no additional changes in the energy spectrum of the comparison Hamiltonian, while the methods yield explicit analytic expressions for the complete set of energy eigenfunctions. These advantages make these methods particularly suitable for formal investigations. However, the range of von Neumann–Wigner potentials which can be constructed in this way is quite limited. While the methods of Meyer-Vernet and Pappademos, Sukhatme, and Pagnamenta are exactly equivalent, the compact notation for the Meyer-Vernet method, which we develop in Ref. [16], offers advantages for formal investigations. For

practical calculations involving just one continuum bound state, however, the advantage may lie with the supersymmetry method. Both methods rapidly become unwieldy for practical calculations when the number of continuum bound states is increased.

In a different context, Pursey [26] showed that more complex "renormalizations" of the ground state (assumed bound) could be achieved by using any one of the Abraham-Moses procedure, any analogous procedure based on the Marchenko equation, or the Darboux transformation to eliminate the original ground state, and then using a different one of these three procedures to reintroduce a new ground state. Can similar mixed procedures be used to construct new kinds of continuum bound states? The answer is no. The Gel'fand-Levitan-Abraham-Moses procedure cannot be used as in Sec. III to eliminate a scattering state without replacing it by a continuum bound state by the Meyer-Vernet procedure. From Ref. [16], this would require the choice $\lambda_i = -\int_0^\infty d\xi[\phi_i(\xi)]^2 = -\infty$ since scattering states are not normalizable. The Marchenko equation with an *ansatz* analogous to Eq. (30) can be used instead of the Gel'fand-Levitan equation only for wave functions $\phi_i(r)$, which satisfy $\lim_{r\to\infty} r^{1/2}\phi_i(r)=0$, eliminating the use of scattering solutions. Thus a procedure based on the Marchenko equation cannot even be used to "renormalize" a scattering state into a continuum bound state. The only remaining option is to use the supersymmetry approach to eliminate the scattering state, then attempt to use either the Gel'fand-Levitan equation or the Marchenko equation to introduce the continuum bound state. For the latter step, one must use in $g(r,r')$ a function

$$\bar{\varphi}_i(r)=[\varphi_i(r)]^{-1}\left\{\lambda_1+\lambda_2\int_0^r d\xi[\varphi_i(\xi)]^2\right\}, \tag{89}$$

where the parameters λ_1 and λ_2 must be chosen such that $\int_0^r d\xi[\bar{\varphi}_i(\xi)]^2$ exists (for the Gel'fand-Levitan equation) or $\int_r^\infty d\xi[\bar{\varphi}_i(\xi)]^2$ exists (for the Marchenko equation). Neither of these conditions can be met if $\varphi_i(r)$ possesses zeros in the range $0<r<\infty$, as is true if $\varphi_i(r)$ corresponds to a scattering state. We conclude that mixed procedures, which can generate new classes of isospectral Hamiltonians when applied to the ground state, cannot be used to generate Hamiltonians supporting bound states embedded in the continuum.

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