ARTICLES

Formulations of certain Gel'fand-Levitan and Marchenko equations

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We present a succinct and compact formalism for the solutions of certain Gel'fand-Levitan and Marchenko equations associated with the theories of isospectral Hamiltonians and of continuum bound states.

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I. INTRODUCTION

In this paper we develop a succinct and compact form of the solution for certain special but interesting cases of the Gel'fand-Levitan [1] and Marchenko [2] equations. These are best known in the context of inverse scattering theory [3]. However, Abraham and Moses [4] applied the Gel'fand-Levitan equation to the problem of isospectral Hamiltonians for quantum mechanics in one dimension, while Moses and Tuan [5] rediscovered continuum bound states while using the Gel'fand-Levitan equation to study potentials with zero scattering phase shifts. In a pedagogically oriented work, Meyer-Vernet [6] used the Gel'fand-Levitan equation to generate continuum bound states, recreating the Moses-Tuan work, but with a different emphasis. Luban and Pursey [7–9] have published an extensive treatment of isospectral Hamiltonians in one-dimensional nonrelativistic quantum mechanics and their work is readily generalized to apply to the radial equation of the three-dimensional problem. Pursey [8] used the Marchenko equation in the study of isospectral Hamiltonians. The special cases of the Gel'fand-Levitan and Marchenko equations that we consider belong to the latter contexts.

In the next section we shall develop the notational conventions to be used throughout the paper. Section III will develop the solution of the Gel'fand-Levitan equation. The Marchenko equation is treated in less detail in Sec. IV since our development closely parallels that of the Gel'fand-Levitan equation. In Sec. V we consider the effect of iterating the Gel'fand-Levitan-Abraham-Moses procedure and demonstrate the result to be equivalent to a single application of the procedure. The results presented here are used in the study of continuum bound states in the following paper [10].

II. NOTATIONAL CONVENTIONS

We shall consider the radial equation of nonrelativistic single-particle quantum mechanics with a spherically symmetric potential. We choose units such that $2m=1$ and $\hbar=1$. For simplicity, we shall consider only states with zero angular momentum; however, the theory is readily generalized to states with arbitrary angular momentum. The time-independent radial Schrödinger equation can be written

$$\left[-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+V(r)-E\right]R(r,E)=0 \tag{1}$$

or

$$\left[-\frac{d^2}{dr^2}+V(r)-E\right]\psi(r,E)=0 , \tag{2}$$

with

$$\psi(r,E)\equiv rR(r,E) . \tag{3}$$

We assume that $V(r)$ possesses a Laurent expansion about $r=0$ with the coefficient of r^{-2} greater than $-\frac{1}{4}$ and that

$$\lim_{r\to\infty} r|V(r)|<\infty . \tag{4}$$

Then physically acceptable solutions of Eq. (2) must satisfy

$$\lim_{r\to 0} r^{-1/2}|\psi(r,E)|<\infty \tag{5}$$

and

$$\lim_{r\to\infty} |\psi(r,E)|<\infty . \tag{6}$$

One can always find a solution of Eq. (2) which satisfies either Eq. (5) or (6), but not both unless E belongs to the physical energy eigenvalue spectrum. If E is a bound state energy, then

$$\int_0^\infty d\xi[\psi(\xi,E)]^2<\infty , \tag{7}$$

which implies that

$$\lim_{r\to\infty} r^{1/2}|\psi(r,E)|=0 . \tag{8}$$

We denote a solution of Eq. (2) which satisfies Eq. (5), but does not necessarily satisfy Eq. (6) by $\psi(r,E)$, and a solution which satisfies Eq. (8), but not necessarily Eq. (5) by $\tilde{\psi}(r,E)$.

In the forms in which we shall use them, both the Gel'fand-Levitan and Marchenko equations relate the solutions of Eq. (2) to those of a Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+V_c(r)-E\right]\varphi(r,E)=0 , \tag{9}$$

using a comparison potential $V_c(r)$. We assume that the solutions of Eq. (9) are known for arbitrary E. We use $\varphi(r,E)$ and $\tilde{\varphi}(r,E)$ to denote the solution of Eq. (9) which satisfy the same boundary conditions as $\psi(r,E)$ and $\tilde{\psi}(r,E)$, respectively.

III. THE GEL'FAND-LEVITAN EQUATION

The Gel'fand-Levitan equation as used in inverse scattering theory [3] may be written as

$$K(r,r')=g(r,r')-\int_0^r d\xi\, K(r,\xi)g(\xi,r') . \tag{10}$$

The kernel $g(r,r')$ has the form of a Stieltjes integral

$$g(r,r')=\int dh(E)\varphi(r,E)\varphi(r',E) , \tag{11}$$

where the functions $\varphi(r,E)$ are solutions of Eq. (9) with the boundary condition $\varphi(0,E)=0$. The weight function $h(E)$ is related to the scattering data (that is, scattering phase shifts together with "norming constants" of bound states) associated with the Schrödinger equations of Eqs. (2) and (9), where

$$V(r)=V_c(r)-2\frac{d}{dr}K(r,r) . \tag{12}$$

In the context of inverse scattering theory, the solutions of Eq. (9) are assumed to be fully known, but the potential $V(r)$ is unknown. If the scattering data are known (possibly from experiment) for the Schrödinger equation with the unknown potential $V(r)$, then $h(E)$ can be constructed, Eq. (10) solved for $K(r,r')$, and the previously unknown potential found from Eq. (12).

The solution $\psi(r,E)$ of Eq. (2) corresponding to the energy E is expressed in terms of the solution to Eq. (9) by

$$\psi(r,E)=\varphi(r,E)-\int_0^r dr' K(r,r')\varphi(r',E) . \tag{13}$$

Clearly $\psi(0,E)=0$ and E belongs to the physical spectrum of the new Hamiltonian if $\psi(r,E)$ is bounded as $r\to\infty$. Even if $\varphi(r,E)$ is bounded as $r\to\infty$, the new wave function $\psi(r,E)$ need not be, so that even if E belongs to the spectrum of the comparison Hamiltonian with potential $V_c(r)$, it need not be part of the spectrum of the new Hamiltonian. Likewise even if $\varphi(r,E)$ is not bounded as $r\to\infty$, so that E is not part of the spectrum of the comparison Hamiltonian, $\psi(r,E)$ may still be bounded so that E may belong to the spectrum of the Hamiltonian with potential $V(r)$ given by Eq. (12).

In the context of quantum mechanics on the infinite line $-\infty<x<\infty$, Abraham and Moses [4] considered a special case in which $g(x,x')$, was a finite sum of products of wave functions satisfying Eq. (9). In adapting the Abraham-Moses work for the radial Schrödinger equation, we choose

$$g(r,r')=\sum_{i=1}^{n}\lambda_i^{-1}\varphi_i(r)\varphi_i(r') , \tag{14}$$

where

$$\varphi_i(r)\equiv\varphi(r,E_i) \tag{15}$$

and the parameters λ_i are arbitrary apart from certain constraints which will become clear shortly. With $g(r,r')$ given by Eq. (14), the Gel'fand-Levitan equation Eq. (10) is readily solved for the function $K(r,r')$ in terms of the symmetric $n\times n$ matrix $\Delta(r)$ defined by

$$\Delta_{ij}(r)=\lambda_i\delta_{ij}+\int_0^r d\xi\,\varphi_i(\xi)\varphi_j(\xi),\quad i,j=1,\ldots,n . \tag{16}$$

Provided that the parameters λ_i are chosen so that the matrix $\Delta(r)$ is nonsingular for $0\le r<\infty$,

$$K(r,r')=\sum_{i,j=1}^{n}\varphi_i(r)[\Delta^{-1}(r)]_{ij}\varphi_j(r') , \tag{17}$$

as may be verified by direct substitution. Alternatively, by using Eqs. (10), (13), and (14), one finds that

$$K(r,r')=\sum_{i=1}^{n}\lambda_i^{-1}\psi_i(r)\varphi_i(r') , \tag{18}$$

where $\psi_i(r)\equiv\psi(r,E_i)$. The new potential is

$$\begin{aligned}V(r)&=V_c(r)-2\frac{d}{dr}\sum_{i,j=1}^{n}\varphi_i(r)[\Delta^{-1}(r)]_{ij}\varphi_j(r)\\&=V_c(r)-2\frac{d^2}{dr^2}\ln|\det\Delta(r)| ,\end{aligned} \tag{19}$$

while the wave function of the new Schrödinger equation corresponding to energy E is

$$\psi(r,E)=\varphi(r,E)-\sum_{i,j=1}^{n}\varphi_i(r)[\Delta^{-1}(r)]_{ij}\times\int_0^r d\xi\,\varphi_j(\xi)\varphi(\xi,E) . \tag{20}$$

The normalization integral for $\psi(r,E)$ is of interest. By using the identity

$$\sum_{j,k=1}^{n}[\Delta^{-1}(r)]_{ij}\varphi_j(r)\varphi_k(r)[\Delta^{-1}(r)]_{kl}=-\frac{d}{dr}[\Delta^{-1}(r)]_{il} \tag{21}$$

and performing an integration by parts, one finds that

$$\int_0^r d\xi[\psi(\xi)]^2=\int_0^r d\xi[\phi(\xi)]^2-\sum_{i,j=1}^{n}\left[\int_0^r d\xi\,\phi(\xi)\varphi_i(\xi)\right][\Delta^{-1}(r)]_{ij}\times\left[\int_0^r d\xi\,\phi(\xi)\varphi_j(\xi)\right] , \tag{22}$$

where for clarity we have suppressed the argument E in ψ and φ. If $E\neq E_i$, $i=1,\ldots,n$, the second term on the right-hand side of Eq. (20) vanishes in the limit as $r\to\infty$. Hence $\psi(r,E)$, $E\neq E_i$, corresponds to a bound state or a continuum state with energy E of the new Schrödinger equation according to whether $\varphi(r,E)$ corresponds to a bound state or a continuum state of the comparison Schrödinger equation. If, however, $E=E_i$ for some i, then

$$\psi_i(r)\equiv\psi(r,E_i)=\sum_{j=1}^{n}\varphi_j(r)[\Delta^{-1}(r)]_{ji}\lambda_i \tag{23}$$

and

$$\int_0^r d\xi[\psi_i(\xi)]^2=\lambda_i-\lambda_i^2[\Delta^{-1}(r)]_{ii} \ . \tag{24}$$

Thus the normalizability of $\psi_i(r)$ depends on the behavior of the ii diagonal element of $\Delta^{-1}(r)$ as $r\to\infty$. A simple consideration shows that

$$\lim_{r\to\infty}\frac{[\Delta_{ij}(r)]^2}{\Delta_{ii}(r)\Delta_{jj}(r)}=0 \ , \tag{25}$$

even if one or both of $\varphi_i(r)$ and $\varphi_j(r)$ are unbounded as $r\to\infty$. It is then easy to show that

$$\lim_{r\to\infty}[\Delta^{-1}(r)]_{ii}[\Delta(r)]_{ii}=1 \ . \tag{26}$$

From this, if $\int_0^\infty\varphi_i^2(r)dr=\infty$ and $\lambda_i>0$, then $\lim_{r\to\infty}[\Delta^{-1}(r)]_{ii}=0$ and $\psi_i(r)$ is normalizable, so that E_i is a bound-state energy eigenvalue of the new Hamiltonian with potential given by Eq. (12). In particular, if E_i is a continuum energy eigenvalue of the comparison Hamiltonian, then the procedure has created a new Hamiltonian whose spectrum contains E_i as a bound-state energy. This is the basis of the Meyer-Vernet method of creating bound states embedded in the continuum. If $\varphi_i(r)$ is normalizable with

$$\int_0^\infty d\xi[\varphi_i(\xi)]^2=N_i^2 \ , \tag{27}$$

then

$$\int_0^\infty d\xi[\psi_i(\xi)]^2=\frac{\lambda_i N_i^2}{\lambda_i+N_i^2} \tag{28}$$

and $\psi_i(r)$ is normalizable provided $|2\lambda_i+N_i^2|>N_i^2$, but with a "norming constant" different from that of $\varphi_i(r)$. If Eq. (23) holds and $\lambda_i=-N_i^2$, then $\psi(r,E_i)$ is not normalizable and in general the energy eigenvalue E_i of the comparison Hamiltonian does not belong to the spectrum of the new Hamiltonian. The only exceptions occur if

$$\lim_{r\to\infty}\left|\frac{\varphi_i(r)}{\int_r^\infty d\xi[\varphi_i(\xi)]^2}\right|<\infty \ , \tag{29}$$

in which case $\psi_i(r)$ given by Eq. (23) remains bounded as $r\to\infty$. Equation (29) is not satisfied if $\varphi_i(r)$ decreases exponentially as $r\to\infty$, as would be true of a conventional bound state. However, Eq. (23) is satisfied if $\varphi_i(r)\underset{r\to\infty}{\longrightarrow}\mathcal{O}(r^{-\gamma})$ with $\frac{1}{2}<\gamma\le 1$. As we show in the following paper, such exceptional cases can arise if E_i, the energy eigenvalue associated with the normalizable function $\varphi_i(r)$ is embedded in the continuous spectrum of Eq. (9), in which case $\varphi_i(r)$ represents a continuum bound state of the kind first proposed by von Neumann and Wigner [11].

IV. THE MARCHENKO EQUATION

The form of the Marchenko equation given by Newton [3] is

$$A(r,r')=h(r+r')+\int_r^\infty d\xi\, h(r'+\xi)A(r,\xi) \ , \tag{30}$$

where $A(r,r')$ is related to the Jost solution of Eq. (2) with energy $E=k^2$ by

$$A(r,r')=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{-ikr'}[f_+(k,r)-e^{ikr}] \ , \tag{31}$$

$h(r)$ is defined in terms of the S matrix $S(k)$ associated with Eq. (2) together with any bound-state energies and bound-state norming constants for this equation, the Jost solutions of Eq. (2) are

$$f_+(k,r)=e^{ikr}+\int_r^\infty d\xi\ A(r,\xi)e^{ik\xi} \ , \tag{32}$$

and $V(r)$ is given by

$$V(r)=-2\frac{d}{dr}A(r,r) \ . \tag{33}$$

If, however, the potentials $V(r)$ and $V_c(r)$ satisfy suitable constraints, one may use the completeness relations for and orthogonality of the solutions of Eq. (9) to find an equivalent equation

$$\tilde{K}(r,r')=\tilde{g}(r,r')+\int_r^\infty d\xi\,\tilde{K}(r,\xi)\tilde{g}(\xi,r') \ , \tag{34}$$

where $\tilde{g}(r,r')$ is of the form

$$\tilde{g}(r,r')=\int d\tilde{h}(E)\tilde{\varphi}(r,E)\tilde{\varphi}(r',E) \ , \tag{35}$$

the potentials are related by

$$V(r)=V_c(r)-2\frac{d}{dr}\tilde{K}(r,r) \ , \tag{36}$$

and the solutions of Eqs. (2) and (9) are related by

$$\tilde{\psi}(r,E)=\tilde{\varphi}(r,E)+\int_r^\infty d\xi\,\tilde{K}(r,\xi)\tilde{\varphi}(r,E) \ . \tag{37}$$

Indeed if $V_c(r)=0$, then Eqs. (30)–(33) are recovered as a special case of Eqs. (34)–(37). The tilde over the symbols in the new equations indicates the importance of the boundary conditions at infinity.

It will turn out that the conditions on $V(r)$ and $V_c(r)$ required for Newton's proof of Eqs. (30)–(33) and for the details of the derivation of Eqs. (34)–(37) from Eqs. (30)–(33) are of no importance for the present work. In the context of inverse scattering theory, one would compute $h(r)$ of Eq. (30) or $\tilde{g}(r,r')$ of Eq. (35) from scattering data, solve either Eq. (30) for $A(r,r')$ or Eq. (34) for $\tilde{K}(r,r')$, and obtain $V(r)$ either from Eq. (33) or (36). However, as with the Gel'fand-Levitan equation one can construct a new Schrödinger equation with potential $V(r)$ by making any suitable choice for $\tilde{g}(r,r')$. By analogy with Eq. (14), we choose

$$\tilde{g}(r,r')=\sum_{i=1}^{n}\tilde{\lambda}_i^{-1}\tilde{\varphi}(r,E_i)\tilde{\varphi}(r',E_i)\equiv\sum_{i=1}^{n}\tilde{\lambda}_i^{-1}\tilde{\varphi}_i(r)\tilde{\varphi}_i(r') , \quad (38)$$

where for convenience we write

$$\tilde{\varphi}(r,E_i)\equiv\tilde{\varphi}_i(r) . \quad (39)$$

In this case, one may directly verify that $\tilde{\varphi}(r,E)$ given by Eq. (37) is a solution of Eq. (2) with $V(r)$ given by Eq. (36) provided $\tilde{K}(r,r')$ satisfies Eq. (34). Apart from some notation changes, Eqs. (34)–(37) are the form of the Marchenko theory which was used [8] in the context of one-dimensional quantum mechanics.

The solution of Eq. (34) is

$$\tilde{K}(r,r')=\sum_{i,j=1}^{n}\tilde{\varphi}_i(r)[\tilde{\Delta}^{-1}(r)]_{ij}\tilde{\varphi}_j(r') , \quad (40)$$

where

$$\tilde{\Delta}_{ij}(r)=\tilde{\lambda}_i\delta_{ij}-\int_r^{\infty}d\xi\,\tilde{\varphi}_i(\xi)\tilde{\varphi}_j(\xi) . \quad (41)$$

The need for stringent boundary conditions at infinity is clear from Eq. (41). The functions $\tilde{\varphi}_i(r)$ must satisfy Eq. (8) rather than the less stringent Eq. (6). In particular, none of the $\tilde{\varphi}_i(r)$ may be scattering solutions of Eq. (9). One should note, however, that there need be no restrictions on the behavior of the $\tilde{\varphi}_i(r)$ at the origin.

The remainder of the development follows closely that of Sec. III. The potential $V(r)$ is

$$V(r)=V_c(r)-2\frac{d^2}{dr^2}\ln|\det\tilde{\Delta}(r)| . \quad (42)$$

The solution of Eq. (2) constructed from $\tilde{\varphi}(r,E)$ is

$$\tilde{\psi}(r,E)=\tilde{\varphi}(r,E)+\sum_{i,j=1}^{n}\tilde{\varphi}_i(r)[\tilde{\Delta}^{-1}(r)]_{ij}\times\int_r^{\infty}d\xi\,\tilde{\varphi}_j(\xi)\tilde{\varphi}(\xi,E) \quad (43)$$

and

$$\int_r^{\infty}d\xi[\tilde{\psi}(\xi)]^2=\int_r^{\infty}d\xi[\tilde{\varphi}(\xi)]^2-\sum_{i,j=1}^{n}\int_r^{\infty}d\xi\,\tilde{\varphi}_i(\xi)\tilde{\varphi}(\xi)[\tilde{\Delta}^{-1}(r)]_{ij}\times\int_r^{\infty}d\xi\,\tilde{\varphi}_j(\xi)\tilde{\varphi}(\xi) . \quad (44)$$

Once again, if $E\neq E_i$, $i=1,\dots,n$, the function $\tilde{\psi}(r,E)$ corresponds to a bound state or a continuum state of Eq. (2) according to whether $\tilde{\varphi}(r,E)$ corresponds to a bound state or a continuum state of Eq. (9). One should note that if $\tilde{\varphi}(r,E)$ corresponds to a continuum state so that $\tilde{\varphi}(r,E)$ is asymptotically a sine wave for large r, the condition Eq. (8) satisfied by each of the $\tilde{\varphi}_i(r)$ is still sufficient to ensure convergence of the integral $\int_r^{\infty}d\xi\,\tilde{\varphi}_j(\xi)\tilde{\varphi}(\xi)$. If $E=E_i$ for some i, then

$$\tilde{\psi}_i(r)\equiv\tilde{\psi}(r,E_i)=\sum_{i,j=1}^{n}\tilde{\varphi}_i(r)[\tilde{\Delta}^{-1}(r)]_{ij}\tilde{\lambda}_j \quad (45)$$

and

$$\int_r^{\infty}d\xi[\tilde{\psi}_i(\xi)]^2=\tilde{\lambda}_i^2\left[\tilde{\Delta}^{-1}(r)\right]_{ii}-\tilde{\lambda}_i . \quad (46)$$

This time, the normalizability of $\tilde{\psi}_i(r)$ depends on the behavior of $[\tilde{\Delta}^{-1}(r)]_{ii}$ at the origin. One can easily show that $\lim_{r\to 0}[\tilde{\Delta}^{-1}(r)]_{ii}\tilde{\Delta}_{ii}(r)=1$, in analogy with Eq. (26). If $\lim_{r\to 0}[r^{1/2}\tilde{\varphi}_i(r)]\neq 0$, so that $\tilde{\varphi}_i(r)$ is not normalizable, and if also $\tilde{\lambda}_i<0$, then $\tilde{\psi}_i(r)$ is normalizable and E_i belongs to the eigenvalue spectrum of Eq. (2), although it does not belong to that of Eq. (9). If $\int_0^{\infty}d\xi[\tilde{\varphi}_i(\xi)]^2=N_i^2<\infty$ and $|2\tilde{\lambda}_i-N_i^2|>N_i^2$, then $\int_0^{\infty}d\xi[\tilde{\psi}_i(\xi)]^2=(\tilde{\lambda}_iN_i^2)/(\tilde{\lambda}_i-N_i^2)>0$ and E_i belongs to the spectra of both Schrödinger equations. If $\tilde{\lambda}_i=N_i^2$, the function $\tilde{\psi}_i(r)$ is not normalizable and although E_i belongs to the eigenvalue spectrum of Eq. (9), it does not belong to that of Eq. (2).

In Ref. [8] it was shown that the Gel'fand-Levitan-Abraham-Moses procedure and the analogous procedure based on the Marchenko equation lead to different potentials $V(r)$ (although the Hamiltonians are unitarily equivalent) when used to insert a single new energy eigenvalue or to delete a single eigenvalue from the spectrum of Eq. (9), but that both procedures yield the same $V(r)$ when used to "renormalize" a single state. This result is readily generalized in the context of the present formalism. The potentials $V(r)$ produced by the two procedures will be identical only if $\det\Delta(r)=\text{const}\times\det\tilde{\Delta}(r)$. Since the definitions of $\varphi_i(r)$ and $\tilde{\varphi}_i(r)$ differ only in the prescribed boundary conditions, a necessary condition for the existence of both $\det\Delta(r)$ and $\det\tilde{\Delta}(r)$ is $\varphi_i(r)=c_i\tilde{\varphi}_i(r)$, $i=1,\dots,n$, where the c_i are constants which may be chosen to be 1 without any loss of generality. This is not possible if the procedures are to be used to insert one or more new energy eigenvalues into the spectrum: hence the procedures must yield different potentials $V(r)$ in this case. Because of the boundary conditions satisfied by the two functions, $\varphi_i(r)=\tilde{\varphi}_i(r)$ implies their normalizability; as before, we denote the normalization integrals by N_i^2. If in addition $\tilde{\lambda}_i=\lambda_i+N_i^2$, $i=1,\dots,n$, with $|2\tilde{\lambda}_i-N_i^2|=|2\lambda_i+N_i^2|>N_i^2$, then $\tilde{\Delta}_{ij}(r)=\Delta_{ij}(r)$, $i,j=1,\dots,n$, so that the potentials produced by the two methods are identical. This corresponds to a renormalization of one or more of the bound states. If, however, $\tilde{\lambda}_i=-\lambda_i=N_i^2$ for some i, then $\tilde{\Delta}_{ii}(r)=N_i^2+\Delta_{ii}(r)$ so that the two procedures generate different potentials $V(r)$ when used to delete one or more energy eigenvalues.

V. ITERATIONS

Either of the two procedures discussed above can be iterated. We shall present the discussion only for the Gel'fand-Levitan equation; an analogous treatment works for the Marchenko equation. Starting from the Schrödinger equation Eq. (2) rather than from Eq. (9), one can form the Gel'fand-Levitan equation

$$\bar{K}(r,r')=\bar{g}(r,r')-\int_0^r d\xi\,\bar{K}(r,\xi)\bar{g}(\xi,r') \quad (47)$$

with

$$\bar{g}(r,r')=\sum_{i=1}^{m}\bar{\lambda}_i^{-1}\psi_i(r)\psi_i(r') , \tag{48}$$

where the $\psi_i(r)\equiv\psi(r,\bar{E}_i)$ are given in terms of the $\varphi(r,\bar{E}_i)$ by Eq. (20). The solution for $\bar{K}(r,r')$ in terms of the $\psi(r,\bar{E}_i)$ and a symmetric $m\times m$ matrix $\bar{\Delta}(r)$ is given by the obvious substitutions in Eqs. (16) and (17). The new potential is

$$\begin{aligned}\bar{V}(r)&=V(r)-2\frac{d^2}{dr^2}\ln|\det\bar{\Delta}(r)|\\&=V_c(r)-2\frac{d^2}{dr^2}\ln|\det\Delta(r)\det\bar{\Delta}(r)| ,\end{aligned} \tag{49}$$

while the energy eigenfunctions of the new Schrödinger equation are

$$\begin{aligned}\chi(r,E)&=\psi(r,E)-\int_0^r dr'\bar{K}(r,r')\psi(r',E)\\&=\varphi(r,E)-\int_0^r dr'\hat{K}(r,r')\varphi(r,E) ,\end{aligned} \tag{50}$$

where

$$\hat{K}(r,r')=K(r,r')+\bar{K}(r,r')-\int_{r'}^{r}d\xi\,\bar{K}(r,\xi)K(\xi,r') . \tag{51}$$

Remarkably, $\hat{K}(r,r')$ is the solution of a Gel'fand-Levitan equation

$$\hat{K}(r,r')=\hat{g}(r,r')-\int_0^r d\xi\,\hat{K}(r,\xi)\hat{g}(\xi,r') \tag{52}$$

with

$$\hat{g}(r,r')=\sum_{i=1}^{n+m-p}\hat{\lambda}_i^{-1}\varphi_i(r)\varphi_i(r') , \tag{53}$$

where p is the number of energies common to the two sets $\{E_i,\ i=1,\dots,n\}$ and $\{\bar{E}_i,\ i=1,\dots,m\}$, which we assume to be ordered so that the common energies are numbered from $i=n-p+1$ to $i=n$ and $E_{n+i}\equiv\bar{E}_{p+1}$ for $i=1,\dots,m-p$,

$$\varphi_i(r)\equiv\varphi(r,E_i),\quad i=1,\dots,n+m-p \tag{54}$$

and

$$\hat{\lambda}_i=\begin{cases}\lambda_i, & i=1,\dots,n-p\\ \lambda_i\bar{\lambda}_{i-n+p}(\lambda_i+\bar{\lambda}_{i-n+p})^{-1}, & i=n-p+1,\dots,n\\ \bar{\lambda}_{i-n+p}, & i=n+1,\dots,n+m-p .\end{cases} \tag{55}$$

Thus the result of applying two Gel'fand-Levitan-Abraham-Moses operations in succession is equivalent to a single operation, with modified λ parameters for any energies which are used in both procedures. We note that if $\bar{\lambda}_{i-n+p}+\lambda_i=0$ for some i, then $\hat{\lambda}_i=\infty$ and $\varphi_i(r)$ no longer contributes to $\hat{g}(r,r')$.

We now prove these results. Since the solutions of a Schrödinger equation are completely determined by the potential, it is sufficient to prove that

$$\det\bar{\Delta}(r)\det\Delta(r)=\text{const}\,\det\hat{\Delta}(r) . \tag{56}$$

We begin with $m=1$. In this case,

$$\det\bar{\Delta}(r)=\bar{\lambda}_1+\int_0^r d\xi[\psi(\xi,\bar{E}_1)]^2 . \tag{57}$$

If $\bar{E}_1\neq E_i$, $i=1,\dots,n$, then by Eq. (22), with $\bar{\lambda}_1\equiv\lambda_{n+1}$, and by making use of the formula for the inverse of a matrix,

$$\det\bar{\Delta}(r)=\frac{1}{\det\Delta(r)}\left[\hat{\Delta}_{n+1,n+1}(r)\det\Delta(r)-\sum_{i,j=1}^{n}\hat{\Delta}_{n+1,i}(r)M_{ij}(r)\hat{\Delta}_{j,n+1}(r)\right] , \tag{58}$$

where $M_{ij}(r)=M_{ji}(r)$ is the signed minor of $\Delta_{ij}(r)$ in $\det\Delta(r)$. A little thought now shows that

$$\det\bar{\Delta}(r)=\frac{\det\hat{\Delta}(r)}{\det\Delta(r)} , \tag{59}$$

and Eq. (56) is proved for this special case. Now let us suppose that $\bar{E}_1=E_n$. Then from Eqs. (24) and (57),

$$\det\bar{\Delta}(r)=\bar{\lambda}_1+\lambda_n-\lambda_n^2\frac{M_{nn}(r)}{\det\Delta(r)} . \tag{60}$$

If $\bar{\lambda}_1+\lambda_n=0$, then

$$\det\bar{\Delta}(r)\det\Delta(r)=-\lambda_n^2M_{nn}(r) \tag{61}$$

and $M_{nn}(r)=\det\hat{\Delta}(r)$, where $\hat{\Delta}(r)$ is now the $(n-1)\times(n-1)$ matrix obtained from $\Delta(r)$ by deleting the nth row and column. Thus the contribution of the state with energy E_n has been deleted from the Abraham-Moses procedure as a result of the iteration. Finally, if $\bar{\lambda}_1+\lambda_n\neq0$, then Eq. (60) yields

$$\det\bar{\Delta}(r)\det\Delta(r)=(\bar{\lambda}_1+\lambda_n)\left[\det\Delta(r)-\frac{\lambda_n^2}{\bar{\lambda}_1+\lambda_n}M_{nn}(r)\right] . \tag{62}$$

If we now consider the expansion of $\det\Delta(r)$ by the nth row, it is apparent that

$$\det\Delta(r)-\frac{\lambda_n^2}{\bar{\lambda}_1+\lambda_n}M_{nn}(r)=\det\hat{\Delta}(r) , \tag{63}$$

where $\Delta(r)$ and $\hat{\Delta}(r)$ differ only in that λ_n has been replaced by

$$\hat{\lambda}_n=\frac{\bar{\lambda}_1\lambda_n}{\bar{\lambda}_1+\lambda_n} . \tag{64}$$

This completes the proof for the special case $m=1$.

If $m>1$, the special case just proven allows us to replace the second Abraham-Moses procedure by a sequence of procedures performed in succession, each with $m=1$. At each application of a new $m=1$ transformation, we obtain a new matrix $\hat{\Delta}(r)$, corresponding new functions $\hat{K}(r,r')$ and $\hat{g}(r,r')$, and a new potential $V(r)$, with parameters $\hat{\lambda}_i$ given by Eq. (55). This completes the proof in the general case.

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