Most general and simplest algebraic relationship between energy eigenstates of a hydrogen atom and a harmonic oscillator of arbitrary dimensions

Gao-Jian Zeng
*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China and Department of Physics, Hunan Normal University, Hunan 410006, People's Republic of China**

Ka-Lin Su
Department of Physics, Yueyang Normal College, Yeyang, Hunan 414000, People's Republic of China

Min Li
Changde Higher College, Changde, Hunan 415003, People's Republic of China
(Received 8 October 1993)

In this paper, we work out a most general and the simplest relationship between the energy eigenstates of a d-dimensional hydrogen atom and those of a D-dimensional harmonic oscillator in terms of the su(1,1) algebra.

PACS number(s): 03.65.Fd, 11.30.Na, 02.20.−a

The relationship between the hydrogen atom and the harmonic oscillator has been studied by many authors [1–13]. Many works are based on a coordinate transformation [1–9]. For example, the so-called Kustaanheimo-Stiefel (KS) transformation was used by Boiteux, Kibler and Negadi [1–3,9], etc., to show that the bound states of the hydrogen atom correspond to the energy eigenstates of a four-dimensional harmonic oscillator subject to a constraint. Some works are based on the transformation of radical equations. For example, the general relationship between the radical function of a d-dimensional hydrogen atom and that of a D-dimensional harmonic oscillator has been given by Kostelecky *et al.* [10] and Nieto [12] etc. All these works are very valuable.

In this paper, we also study the relationship between the hydrogen atom and the harmonic oscillator, but the method is purely algebraic. We use the su(1,1) algebra as a bridge to establish a connection between the energy eigenstates of the d-dimensional hydrogen atom and those of the D-dimensional harmonic oscillator. In the question we study, the dimensions of the hydrogen atom and the harmonic oscillator are arbitrary; therefore our results are the most general. Since the algebraic method is used, our results are also the simplest.

First, we consider a real linear space of dimension d and construct the operators K_1, K_2, and K_3 in it:

$$K_1 = \tfrac{1}{2}(r \triangle + r), \tag{1}$$

$$K_2 = i\left(\frac{d-1}{2} + x_j \frac{\partial}{\partial x_j}\right), \tag{2}$$

$$K_3 = -\tfrac{1}{2}(r \triangle - r), \tag{3}$$

where $\triangle = \frac{\partial^2}{\partial x_j \partial x_j}$, $r = (x_j x_j)^{\frac{1}{2}}$, and $j = 1, 2, 3, ..., d$. It is easy to show that the operators satisfy the commutation relations

$$[K_1, K_2] = -iK_3, \tag{4}$$

$$[K_2, K_3] = iK_1, \tag{5}$$

$$[K_3, K_1] = iK_2. \tag{6}$$

These relations show that the operators K_1, K_2, and K_3 constitute the su(1,1) algebra. If we define K_+ and K_- by $K_\pm = K_1 \pm iK_2$, then one has

$$[K_3, K_\pm] = \pm K_\pm, \tag{7}$$

$$[K_+, K_-] = -2K_3. \tag{8}$$

When $d = 3$, the expressions (1)–(3) of the elements of the su(1,1) algebra have been applied in many references [14,15].

Next, we write the Hamiltonian of the d-dimensional hydrogen atom

$$H = -\tfrac{1}{2} \triangle - \frac{e^2}{r}, \tag{9}$$

where we have assumed $\hbar = \mu(\text{mass}) = 1$. Using (1) and (3), (9) may be reduced to

$$(K_1 + K_3)H = -\tfrac{1}{2}(K_1 - K_3) - e^2. \tag{10}$$

The eigenequation of the Hamiltonian of the d-dimensional hydrogen atom is given by

$$H|d, n\rangle = E_n|d, n\rangle, \tag{11}$$

where $|d, n\rangle$ is the eigenstate and E_n is the corresponding

*Address for correspondence.

eigenvalue. Kostelecky, Nieto, and Truax [10] obtained the energy eigenvalue E_n of the d-dimensional hydrogen atom; the result is [11,12]

$$E_n = -\frac{e^4}{2}\frac{1}{[n+(1/2)(d-3)]^2}. \tag{12}$$

From (10) and (11), one may obtain

$$\{-[(1/2)+E_n]K_1 + [(1/2)-E_n]K_3 - e^2\}|d,n\rangle = 0. \tag{13}$$

Defining the function θ_n by

$$\cosh\theta_n = \frac{1-2E_n}{\sqrt{-8E_n}}, \quad \sinh\theta_n = -\frac{1+2E_n}{\sqrt{-8E_n}}, \tag{14}$$

and using the relation that the elements of the su(1,1) algebra satisfy

$$e^{-iK_2\theta_n}K_3 e^{iK_2\theta_n} = K_3\cosh\theta_n + K_1\sinh\theta_n, \tag{15}$$

then the equation (13) can be rewritten as

$$\left\{e^{-iK_2\theta_n}K_3 e^{iK_2\theta_n} - \frac{e^2}{\sqrt{-2E_n}}\right\}|d,n\rangle = 0 \tag{16}$$

or

$$\left\{K_3 - \frac{e^2}{\sqrt{-2E_n}}\right\}e^{iK_2\theta_n}|d,n\rangle = 0. \tag{17}$$

It is easy to see that Eq. (17) is an eigenequation of the operator K_3. Thus we have transferred the eigenequation of the Hamiltonian of the d-dimensional hydrogen atom into the eigenequation of the operator K_3.

In order to find the relationship between the hydrogen atom and the harmonic oscillator in arbitrary dimensions, we consider a system of a D-dimensional harmonic oscillator and construct the operators

$$K_+ = \frac{1}{2}\sum_{\alpha=1}^{D}(a_\alpha^\dagger)^2, \tag{18}$$

$$K_- = \frac{1}{2}\sum_{\alpha=1}^{D}a_\alpha^2, \tag{19}$$

$$K_3 = \frac{1}{4}\sum_{\alpha=1}^{D}(a_\alpha^\dagger a_\alpha + a_\alpha a_\alpha^\dagger), \tag{20}$$

where a_α and $a_\alpha^\dagger$ ($\alpha = 1,2,3,...,D$) are annihilation and creation operators of the D-dimensional harmonic oscillator, respectively; they satisfy $[a_\alpha, a_\alpha^\dagger] = \delta_{\alpha\beta}$. It is easy to show that the operators K_+, K_-, and K_3 defined by Eqs. (18), (19), and (20) also satisfy the commutation relations (7) and (8), and therefore they also constitute the su(1,1) algebra. When $D = 1$, the expressions (18)–(20) of the elements of the su(1,1) algebra have been applied in many references [16].

It is well known that the Hamiltonian of the D-dimensional harmonic oscillator is

$$H' = \frac{\omega}{2}\sum_{\alpha=1}^{D}(a_\alpha^\dagger a_\alpha + a_\alpha a_\alpha^\dagger). \tag{21}$$

Its eigenequation can be written as

$$H'|D,N\rangle' = E'_N|D,N\rangle', \tag{22}$$

where $|D,N\rangle'$ is the eigenvector, E'_N is the corresponding eigenvalue, and

$$E'_N = \omega(N + D/2), \tag{23}$$

while $N \geq 0$ is the eigenvalue of the operator $\sum_{\alpha=1}^{D}a_\alpha^\dagger a_\alpha$. Comparing (20) with (21), then the eigenequation (22) may be rewritten as

$$K_3|D,N\rangle' = \tfrac{1}{2}(N + D/2)|D,N\rangle'. \tag{24}$$

Both Eqs. (17) and (24) are the eigenequation of the operator K_3; therefore, comparing (24) with (17), one may work out the following map:

$$|D,N\rangle' = e^{iK_2\theta_n}|d,n\rangle, \tag{25}$$

$$\tfrac{1}{2}(N + D/2) = \frac{e^2}{\sqrt{-2E_n}}, \tag{26}$$

or

$$|d,n\rangle = e^{-iK_2\theta_n}|D,N\rangle', \tag{27}$$

$$E_n = -\frac{2e^4}{(N+D/2)^2}. \tag{28}$$

Thus the relationship between the energy and the energy eigenvector of the d-dimensional hydrogen atom and those of the D-dimensional harmonic oscillator has been established by means of the map. It is the most general and simplest algebraic relationship. In the expression (25), the operator K_2 may be expressed in terms of the creation operator $a_\alpha^\dagger$ and annihilation operator a_α of the harmonic oscillator as

$$K_2 = -\frac{i}{4}\sum_{\alpha=1}^{D}[(a_\alpha^\dagger)^2 - a_\alpha^2]. \tag{29}$$

However, if the energy eigenstates are expressed in the coordinate space, then the operator K_2 should be expressed in terms of coordinates and differentials on coordinates.

It is noted that, because a harmonic oscillator of dimension D may be considered as a set of D independent harmonic oscillators of dimension 1, $|D,N\rangle'$ in practice may be written as $|D,N;n_1,n_2,...,n_D\rangle'$, where $n_\alpha = 0,1,2,...$ is the eigenvalue of the operator $a_\alpha^\dagger a_\alpha$, and $n_1 + n_2 \cdots + n_D = N$. The energy of the harmonic oscillator of dimension D depends on N; since, for a fixed N, there are generally several sets $(n_1,n_2,...,n_D)$, the energy level N of the harmonic oscillator of dimension D is generally degenerate. In the same way, the energy level n of the hydrogen atom of dimension d is also generally degenerate. We do not discuss the degeneracy problem in this paper.

Substituting (12) into (26), one may obtain

$$D = 4n - 2N + 2d - 6. \tag{30}$$

The relation (30) shows that D must be an even integer. In other words, the hydrogen atom of arbitrary dimension is only connected with the harmonic oscillator of even number dimension. If one sets $D = 2d + 2\lambda$, where λ is an integer, then (30) is reduced to the following relations:

$$D = 2d + 2\lambda, \tag{31}$$

$$N = 2n - \lambda - 3. \tag{32}$$

Noting $D \geq 2$, $N \geq 0$, and $n \geq 1$, one may obtain from (31) and (32)

$$1 - d \leq \lambda \leq -1. \tag{33}$$

When $d = 2$, λ only takes the value -1 and $D = 2$; when $d = 3$, λ may take the values $-2, -1$ and D=2,4; when $d = 4$, λ may take the values $-3, -2, -1$ and D=2,4,6, etc. In other words, the energy eigenstates of the hydrogen atom of dimension 2 can only be expressed in terms of that of the harmonic oscillator of dimension 2, while the energy eigenstates of the hydrogen atom of dimension 3 may be expressed in terms of that of the harmonic oscillator of dimension 2 or 4, the energy eigenstates of the hydrogen atom of dimension 4 may be expressed in terms of that of the harmonic oscillator of dimension 2, 4, and 6, etc. According to these facts, we also conclude that the energy eigenstates of a hydrogen atom of arbitrary dimension may always be expressed in terms of the energy eigenstates of the harmonic oscillator of dimension 2.

If one requires that the ground state ($n = 1$) of the hydrogen atom of dimension d must be expressed in terms of that ($N = 0$) of the harmonic oscillator of dimension D, then there is the following relation between the dimension d of the hydrogen atom and the dimension D of the harmonic oscillator:

$$D = 2(d - 1), \tag{34}$$

which shows that the energy eigenstates of the hydrogen atom of dimension 3 can only be expressed in terms of that of the harmonic oscillator of dimension 4, and the energy eigenstates of the hydrogen atom of dimension 4 can only be expressed in terms of that of the harmonic oscillator of dimension 6, etc.

Finally, we show that the results of this paper are not only the most general and simplest, but also will have important applications. For example, when one tries to find the energy eigenstates of a q-deformed hydrogen atom, the results in this paper will be the best way.

This work is supported by the National Science Fund of China.

[1] M. Boiteux, C. R. Acad. Sci. Ser. B **274**, 867 (1972).
[2] M. Boiteux, C. R. Acad. Sci. Ser. B **276**, 1 (1973).
[3] M. Boiteux, Physica **65**, 381 (1973).
[4] A.O. Barut, C.K.E. Schneider, and R. Wilson, J. Math. Phys. **20**, 2244 (1979).
[5] M. Kibler and T. Negadi, Lett. Nuovo Cimento **37**, 225 (1983).
[6] M. Kibler and T. Negadi, J. Phys. A **16**, 4265 (1983).
[7] F.H.J. Cornish, J. Phys. A **17**, 323 (1984).
[8] F.H.J. Cornish, J. Phys. A **17**, 2191 (1984).
[9] M. Kibler and T. Negadi, Phys. Rev. A **29**, 2891 (1984).
[10] V.A. Kostelecky, M.M. Nieto, and D.R. Truax, Phys. Rev. D **32**, 2627 (1985).
[11] A.O. Barut and I.H. Duru, Proc. R. Soc. London Ser. A **333**, 217 (1973).
[12] M.M. Nieto, Am. J. Phys. **47**, 1067 (1979).
[13] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966); **38**, 346 (1966).
[14] M. Bednar, Ann. Phys. (N.Y.) **75**, 305 (1973).
[15] B.G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
[16] T. Lisowski, J. Phys. A **25**, 1295 (1992).