Understanding the bifurcation to traveling waves in a class-B laser using a degenerate Ginzburg-Landau equation

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(Received 5 May 1994)

We analyze the Risken-Nummendal–Graham-Haken instability occurring in a homogeneously broadened two-level unidirectional ring laser modeled by the Maxwell-Bloch equations. We investigate the class-B limit of these equations and derive partial differential equations describing the evolution of traveling waves. In particular, we obtain a degenerate Ginzburg-Landau equation with higher-order nonlinearities. We then investigate this equation and determine periodic traveling wave solutions. We discuss their stability in terms of the laser parameters and predict unusual properties for the long-time behavior of the laser.

PACS number(s): 42.50.−p, 42.55.−f, 42.60.Mi

I. INTRODUCTION

We consider the case of a homogeneously broadened two-level unidirectional ring laser. When the laser is pumped to high inversion levels and when the cavity mode spacing is sufficiently small, sideband modes become excited and their nonlinear interaction results in pulsations that travel around the ring cavity. This has been called the Risken-Nummendal–Graham-Haken instability after Risken and Nummendal [1,2] and Graham and Haken [3], who simultaneously derived the conditions for which the continuous-wave (cw) output becomes unstable. The former also carried out a numerical investigation of the self-pulsing phenomena and discussed transient effects.

Haken and Ohno [4,5] obtained an equation for the critical bifurcating mode, and found a periodic solution coexisting with the stable steady state. They determine a bifurcation equation for the mode appearing at the Hopf bifurcation but the complexity of the coefficients prevents an analysis of the effects of the particular laser parameters. Using a combination of analytical and numerical techniques, Fu [6] has specifically analyzed the traveling wave solution in the class-B limit. He obtains a bifurcation equation and discusses conditions for a supercritical or subcritical bifurcation, which are simpler than those described by Haken and Ohno.

In a previous paper [7], we reexamined the Hopf bifurcation formulated by Fu [6]. Using singular perturbation methods we derived the bifurcation equation and determined the direction of bifurcation analytically. We found that the bifurcation is supercritical (subcritical) for all wave numbers greater than (less than) that characterizing the minimum of the neutral stability curve. A higher-order analysis was necessary to fully resolve a singularity, known as a vertical bifurcation, that occurs at the minimum of the neutral stability curve. We will briefly review these results as a guide to the analysis that will be presented here.

In this paper, we consider the limit of a very large cavity length. In this limit, the spectrum of bifurcating modes becomes continuous. Thus, as the pump intensity is raised beyond the second threshold, there is a band, containing an infinite number of modes, that becomes unstable. This causes a modulation of the order parameter and is commonly referred to as "sideband" instability. We use the method of multiple scales to derive partial differential equations (PDE's) describing the space-time evolution of the order parameter that results from the sideband instability.

The use of multiscale methods to study the formation of patterns, which in the present case are the pulses traveling around the ring laser, was pioneered by Newell and Whitehead [8] and Segal [9] in the area of hydrodynamic instabilities, and was later generalized in form and application by Newell [10]. For a steady bifurcation with nonzero wave number, the canonical evolution equation is the real-Ginzburg Landau-equation given by

$$A_T = m_1 A_{ZZ} + m_2 A + m_3 |A|^2 A \ , \tag{1}$$

where A is the order parameter, T is a slow time, Z is a long space variable and the coefficients m_i are real.

An important feature of our problem is the change of the direction of the bifurcation at the minimum of the neutral stability curve. As a result, we obtain a different and more complicated equation of the form

$$A_T = m_1 A_{ZZ} - iB_1 |A|^2 A_Z + iB_2 A^2 A_Z^* + m_2 A + m_3 |A|^2 A + m_4 |A|^4 A \ , \tag{2}$$

This is called a "degenerate" evolution equation in Ref. [15]. Degenerate evolution equations have been of recent interest due to the simultaneous occurrence of periodic

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and steady states in experiments involving binary fluid convection [11,12]. Theoretical investigations have been performed by a number of authors [13–16] where various combinations of nonlinear terms and real and imaginary coefficients have been considered. In all these studies, the coefficients of the degenerate evolution equation have not been determined in terms of the parameters of the original problem. Our analysis of the laser problem provides an example of a degenerate equation which is derived from the original laser equations. We have found the explicit dependence of the coefficients upon the physical parameters. Furthermore, we determine periodic traveling wave solutions and determine their stability. In contrast to (1), we find that the stability boundary (called Eckhaus boundaries) *does* depend upon the parameters of the problem [16].

This paper is organized as follows: In Sec. II we review the results obtained in Ref. [7]; in particular, we rewrite the multimode laser equations in the class-B limit. In Sec. III we derive the PDE that describe the slow-time and space evolution of bifurcating solutions, while in Sec. IV we simplify these results. In Sec. V we look for traveling wave solutions and determine conditions for their existence and stability. Finally, Sec. VI is a discussion of the results.

II. REVIEW

A. Formulation

Risken and Nummendal [1] have normalized the Maxwell-Bloch equations so that the uniform steady-state solution (cw solution) is unity. Their equations are in terms of the real amplitudes of the electric field E, the polarization field P, and the population inversion D (subscripts indicate partial derivatives):

$$E_t + cE_r + \kappa E = \kappa P \ , \tag{3}$$

$$P_t + \gamma_\perp P = \gamma_\perp ED \ , \tag{4}$$

$$D_t + \gamma_\parallel D = \gamma_\parallel(\lambda + 1 - \lambda EP) \ . \tag{5}$$

$$E(r+L,t) = E(r,t) \ , \quad P(r+L,t) = P(r,t) \ ,$$
$$D(r+L,t) = D(r,t) \ , \tag{6}$$

In these equations, L is the length of the laser cavity, t is time, and r is space measured in the direction of light propagation. $\gamma_\parallel$, $\gamma_\perp$, and κ are the decay constants for the population inversion, polarization, and, cavity, respectively, and c is the speed of light in the host material. λ is the bifurcation parameter and measures the strength of the pumping. It is defined as $\lambda = (D_0/D_{\text{cw}}) - 1$, where D_{cw} denotes the steady-state uniform value of the population inversion and D_0 is the pump rate.

We now rescale (3)–(6) into a form that is more suitable for our asymptotic analysis of the class-B limit. This corresponds to the limit $\epsilon \to 0$, where ϵ is given by

$$\epsilon = \sqrt{\gamma_\parallel / \kappa} \ll 1 \ . \tag{7}$$

This is a singular limit since the problem loses one equation if $\epsilon = 0$. Using singular perturbation techniques we may eliminate this difficulty by introducing new variables defined by

$$E^2 = 1 + y \ , \tag{8}$$

$$D = 1 + \epsilon \left[\frac{\lambda}{2} \right]^{1/2} x \ , \tag{9}$$

$$\frac{P}{E} = 1 + \epsilon \left[\frac{\lambda}{2} \right]^{1/2} z \ , \tag{10}$$

$$t = \frac{1}{\epsilon\kappa\sqrt{2\lambda}} s \ , \quad r = \frac{1}{\epsilon\kappa\sqrt{2\lambda}} \frac{c}{a} \xi \ . \tag{11}$$

The new dependent variables are deviations from the cw solution. The rescaling allows us to treat all the dependent variables as $O(1)$ quantities and is instrumental in the success of our analysis. These new variables are motivated by the $\epsilon \to 0$ limit of the linear stability results of Ref. [1] (see Carr [17]).

After inserting (8)–(11) into (3)–(6), we obtain a weakly perturbed conservative system of equations similar to those studied by Erneux, Baer, and Mandel [18]

$$x_s = -y - \frac{\epsilon}{\delta}[x + I_0(1+y)z] \ , \tag{12}$$

$$y_s + a y_\xi = (1+y)z \ , \tag{13}$$

$$z_s = \frac{1}{\epsilon\delta} \left[d(x - z) - y_s \frac{\left[1 + \epsilon \frac{\delta}{2} z \right]}{(1+y)} \right] , \tag{14}$$

where the parameters d, δ, and I_0 are defined by

$$d = \frac{\gamma_\perp}{\kappa} \ , \quad \delta = \sqrt{2I_0} \ , \quad I_0 = \lambda \ . \tag{15}$$

a is defined as the wave number whose value is restricted by the periodic boundary conditions. The variable ξ is a nondimensional space variable that without loss of generality we allow to vary between 0 and 2π. Thus, we have 2π-periodic boundary conditions of the form

$$x(\xi + 2\pi, s) = x(\xi, s) \ , \quad y(\xi + 2\pi, s) = y(\xi, s) \ ,$$
$$z(\xi + 2\pi, s) = z(\xi, s) \ . \tag{16}$$

These conditions imply that the wave number a takes only discrete values; from the relation $\xi = (\epsilon\delta\kappa L a)/c = 2\pi n$ we have

$$a = \frac{2\pi c}{\epsilon\kappa\delta L} n \quad (n = 0, \pm 1, \pm 2, \ldots) \ . \tag{17}$$

In this paper, we will be considering the large cavity limit, $L \to \infty$, for which the spectrum of bifurcating modes is approximately continuous.

The multimode laser equations given by (12)–(16) are now in a form that is convenient for the ϵ small asymptotic limit.

B. Linear stability analysis

We examine the linear stability of the cw solution $(x,y,z) = 0$ by using a perturbation expansion valid for small ϵ. This study will serve to motivate the nonlinear analysis. Equations (12)–(14) are linearized and we seek a solution of the form $(x,y,z)=(u,v,w)e^{(i\xi+\sigma s)}$. The variables u, v, and w, as well as the growth rate σ are then expanded as power series in ϵ.

By examining the $O(1)$ and $O(\epsilon)$ problem, we find that $\sigma=i\omega+\epsilon\sigma_1$, where ω and σ_1 satisfy

$$\left[1+\frac{1}{d}\right]\omega^2+a\omega-1=0 , \quad (18)$$

$$\left[\frac{\omega^4}{d^2}2I_0-\frac{\omega^2}{d}3I_0+1+I_0\right]\frac{1}{\sqrt{2I_0}} = -\left[1+\omega^2\left[1+\frac{1}{d}\right]\right]\sigma_1 . \quad (19)$$

Equation (18) is the linear dispersion relation between ω and a. Using ω as a parameter, we can determine the neutral stability curve $I_0=I_0(a)$ by substituting Eq. (18) into Eq. (19). The neutral stability curve separates the values of I_0 and a corresponding to a stable steady-state solution from those for which the steady state is unstable. This curve is shown in Ref. [7] for different values of d. The minimum of the curve is given by

$$I_0=I_m=8 , \quad \omega=\omega_m=\pm\sqrt{3d/4} , \qquad a=a_m=\frac{(1-3d)}{4\omega_m} . \quad (20)$$

In the limit $L\to\infty$ the critical value at which the cw solution becomes unstable occurs at (I_m,a_m).

C. The Hopf bifurcation

We now assume that the first instability of the uniform solution corresponds to a simple eigenvalue, i.e., there is a unique n such that $a(n)$ becomes unstable at $I_0(a(n))$. We are interested in determining periodic traveling wave solutions of these equations. Specifically, we seek 2π-periodic solutions in the characteristic variable $Z=\xi+\omega s$, where ω is the Hopf bifurcation frequency to be determined. We then analyze the resulting system of ordinary differential equations (ODE's) by expanding the dependent variables in ϵ.

The $O(1)$ problem remains *nonlinear* but is found to be conservative. It admits a one-parameter family of periodic solutions and has a first integral. We would like to know how the amplitude of the periodic solution and the frequency vary as a function of the bifurcation parameter I_0. This is accomplished by using the fact that the period is equal to 2π and analyzing the higher-order problem. To prevent unbounded solutions in the $O(\epsilon)$ problem, we determine a solvability condition given by

$$I_0=\frac{\left[a+\left[1+\frac{1}{d}\right]\omega\right]^2}{(a+\omega)\left[-a-\left[1-\frac{1}{d}\right]\omega\right]} \quad (21)$$

This is the bifurcation equation and relates the frequency ω to the bifurcation parameter I_0. Together with the 2π-periodicity condition, which provides a relation between ω and the amplitude of the periodic solution, we may determine how the amplitude changes with I_0. In general, this 2π-periodicity condition cannot be found analytically, but we determine an analytical approximation based upon the small-amplitude limit.

A small-amplitude analysis for arbitrary wave number determines that there is a vertical bifurcation at the minimum of the neutral stability curve, i.e., the bifurcation equation is singular when $a=a_m$. To resolve this singularity we refine our analysis by specifically examining about a_m. To this end, we define a new small parameter $\eta\ll 1$ as $a(\eta)=a_m+\eta^2 a_2$ $(a_2=\pm 1)$, as a measure of the detuning the wave number from the minimum.

We then employ the small-amplitude Poincaré-Lindstedt method to solve the $O(1)=O(\epsilon^0)$ as a power series in η. We determine the amplitude dependence of the frequency as

$$\omega_0=\omega_m=\pm\sqrt{3d/4} , \qquad \omega_2=-\frac{\omega d}{(7+3d)}|x_0|^2-\frac{3d}{(7+3d)}a_2 , \quad (22)$$

where $\omega=\omega_0+\eta^2\omega_2$. Note that ω_2 depends on the wave number a_2, and on the amplitude x_0, which is the leading approximation to x expanded in ϵ.

We obtain the bifurcation equation by substituting (22) into the solvability condition for the $O(\epsilon)$ problem, Eq. (21). We obtain

$$\frac{I_0-8}{(\eta^4 72)}=m_1|x_0|^4+m_2a_2|x_0|^2+m_3a_2^2 , \quad (23)$$

where the coefficients m_1, m_2, and m_3 are all positive for $d>\frac{1}{3}$ and given by

$$m_1=\left[\frac{(3d-1)d}{4(7+3d)}\right]^2 , \quad m_2=\frac{2\sqrt{3d}\,(3d-1)d}{(7+3d)^2} , \qquad m_3=\frac{48d}{(7+3d)^2} . \quad (24)$$

The left-hand side indicates that we are considering small deviations from the minimum of the neutral stability curve where $I_m=8$. The right-hand side indicates that for positive deviations in the wave number, $a_2>0$, the bifurcation is supercritical and stable. At the minimum, $a_2=0$, the bifurcation is also supercritical and stable. For deviations in the wave number below the minimum, $a_2<0$, the limit point of the small-amplitude subcritical bifurcation is found. This determines a hysteresis loop in which the basic state will jump up to large amplitude solutions at the bifurcation point and jump down to the basic state at the limit point. Note that for $d=\frac{1}{3}$ the result is again singular because both m_1 and m_2 equal 0; this limit is discussed in Ref. [7].

III. AMPLITUDE EQUATIONS

We now consider the case of a laser with a sufficiently large cavity so that the continuous approximation for the spectrum of eigenvalues is appropriate. As the pump is increased beyond the second threshold, a band of modes becomes unstable. In this section, equations describing the evolution of the amplitude of the most unstable mode will be derived from the laser equations.

Specifically, we consider the multimode laser equations (12)–(14) previously derived in Sec. II C. We look for a power series solution in ϵ by expanding x, y, and z, e.g., $x=x_0+\epsilon x_1+O(\epsilon^2)$. From the linear stability analysis we know that the solution decays with a decay constant proportional to ϵ; this motivates defining an additional time scale $\bar{\tau}=\epsilon s$. We find that the polarization variables z_0 and z_1 can be eliminated to yield equations only for x and y. The space variable has been rescaled so that the wave number does not appear explicitly in the equations. Note that the problem is now on an infinite domain, and we require only that the solution remain bounded. The laser equations are then given by

$$x_{0s}+y_0=0\ ,$$

$$\left[1+\frac{1}{d}\right]y_{0s}+y_{0\xi}=(1+y_0)x_0\ , \tag{25}$$

and

$$x_{1s}+y_1=-\frac{1}{\delta}\left[x_0(1+I_0)+I_0x_0y_0-\frac{I_0}{d}y_{0s}\right]-x_{0\bar{\tau}}\ ,$$

$$(1+1/d)y_{1s}+y_{1\xi}-x_1-(x_0y_1+y_0x_1)$$

$$=\frac{\delta}{d}\left[(1+y_0)y_0-\frac{1}{d}\frac{y_{0s}^2}{1+y_0}+\frac{1}{d}y_{0ss}-\frac{1}{2}x_0y_{0s}\right]$$

$$-\left[1+\frac{1}{d}\right]y_{0\bar{\tau}}\ . \tag{26}$$

We use the method of multiple scales to determine solutions in the vicinity of the neutral stability curve minimum. The appropriate slow scales are determined by expanding the neutral stability conditions. We also take into account the fact that the bifurcation is known to be vertical at the minimum [7]. The dependent variables are allowed to be a function of multiple time and space scales, which are treated as independent variables. They are given by

$$t_0=s\ ,\quad t_2=\eta^2 s\ ,\quad \tau=\eta^4\bar{\tau}\ ,\quad r_0=\xi\ ,\ \text{and}\ r_2=\eta^2\xi\ , \tag{27}$$

where η is a small parameter defined by

$$I_0=8+\eta^4 J_4\ ,\quad \eta\ll 1\ . \tag{28}$$

Thus, the bandwidth of unstable wave numbers is $O(\eta^2)$ as I_0 is raised beyond the second threshold.

A. The leading problem for x_0 and y_0

We look for a small amplitude solution to the leading problem (25) by expanding x_0 and y_0 as

$$x_0(s,\xi;\eta)\approx\eta x_{01}(t_0,t_2,\tau,r_0,r_2)+\eta^2x_{02}(t_0,t_2,\tau,r_0,r_2)\cdots\ ,$$

$$y_0(s,\xi;\eta)\approx\eta y_{01}(t_0,t_2,\tau,r_0,r_2)+\eta^2y_{02}(t_0,t_2,\tau,r_0,r_2)\cdots\ , \tag{29}$$

The expansions (29) are substituted into (25) and lead to a succession of linear problems for the coefficients of each power of η. The problem for x_{01} and y_{01} is given by

$$\frac{\partial x_{01}}{\partial t_0}+y_{01}=0\ ,$$

$$\left[1+\frac{1}{d}\right]\frac{\partial y_{01}}{\partial t_0}+\frac{\partial y_{01}}{\partial r_0}-x_{01}=0\ . \tag{30}$$

This problem admits a traveling wave solution of the form

$$\begin{bmatrix}x_{01}\\ y_{01}\end{bmatrix}=\alpha(t_2,\tau,r_2)\begin{bmatrix}1\\ -i\omega\end{bmatrix}e^{i(\omega t_0+ar_0)}+\text{c.c.}\ , \tag{31}$$

where c.c. means complex conjugate. α is an unknown amplitude which is a function of the slow time and space variables. ω satisfies the dispersion relation

$$\left[1+\frac{1}{d}\right]\omega^2+a\omega-1=0\ . \tag{32}$$

The next problem for x_{02} and y_{02} is

$$\frac{\partial x_{02}}{\partial t_0}+y_{02}=0\ ,$$

$$\left[1+\frac{1}{d}\right]\frac{\partial y_{02}}{\partial t_0}+\frac{\partial y_{02}}{\partial r_0}-x_{02}=x_{01}y_{01}\ . \tag{33}$$

Note that (33) is an inhomogeneous system. The homogeneous equations are identical to (30) and admit a periodic traveling wave solution. Therefore, the right-hand side must satisfy a solvability condition. Specifically, the right-hand side must be orthogonal to the homogeneous adjoint solution when integrated over the domain of the independent variables. Since the homogeneous solution is periodic it is sufficient to integrate over the respective periods. Designating the right-hand sides of the first and second equation of (33) as (R_1,R_2), respectively the solvability condition is formulated is

$$\int_0^{2\pi/\omega}\int_0^{2\pi/a}(R_1,R_2)\cdot[(1,i\omega)e^{-i(\omega t_0+ar_0)}+\text{c.c.}]dr_0dt_0=0\ . \tag{34}$$

The solvability condition is identically satisfied because the right-hand side contains only second-harmonic terms. We can then solve for x_{02} and y_{02}.

Since we still do not know α, we consider the $O(\eta^3)$ problem, given by

$$\frac{\partial x_{03}}{\partial t_0}+y_{03}=\frac{\partial x_{01}}{\partial t_2} ,$$

$$\left[1+\frac{1}{d}\right]\frac{\partial y_{03}}{\partial t_0}+\frac{\partial y_{03}}{\partial r_0}-x_{03}$$

$$=x_{02}y_{01}+x_{01}y_{02}-\left[1+\frac{1}{d}\right]\frac{\partial y_{01}}{\partial t_2}-\frac{\partial y_{01}}{\partial r_2} . \quad (35)$$

Using the solvability condition (34), we obtain a first-order partial differential equation for α

$$\left[1+\left[1+\frac{1}{d}\right]\omega^2\right]\frac{\partial \alpha}{\partial t_2}+\omega^2\frac{\partial \alpha}{\partial r_2}+\frac{i\omega^3}{3}|\alpha|^2\alpha=0 . \quad (36)$$

Analogous to (22) of our previous Hopf bifurcation analysis, (36) is a relation between the frequency correction and the amplitude, i.e., it is a PDE form of the nonlinear dispersion relation.

Higher-order corrections are required [up to $O(\eta^5)$] in order to solve the next problem for x_1 and y_1 in (26). We omit the details of this additional analysis.

B. The effect of damping

Equation (36) does not provide a relation between the amplitude of the solution and the bifurcation parameter and thus, motivates the analysis of (26). Since x_{0s} and y_{0s} satisfy the homogeneous problem for x_1 and y_1, and (x_0,y_0) is a periodic traveling wave in the variables t_0 and r_0, a new solvability condition must be satisfied. This condition is

$$\int_0^{2\pi/\omega}\int_0^{2\pi/a}(R_1,R_2)\cdot(x_1^{A*},y_1^{A*})dr_0dt_0=0 , \quad (37)$$

where R_1 and R_2 are the right-hand sides of (26) and the solution to the homogeneous adjoint problem is

$$x_1^A=x_0 ,\quad y_1^A=\frac{y_0}{(1+y_0)} . \quad (38)$$

We now solve (37) by using the expansions of x_0, y_0, and I_0 in powers of η. To this end, we also need to use the chain rule for y_{0s}, which appears in R_1 and R_2. This leads to a series of integral conditions, which are easily solved.

The $O(\eta)$ condition is

$$\frac{16\omega^4}{d^2}-\frac{24\omega^2}{d}+9=0 . \quad (39)$$

Equation (39) matches the condition (19) of the linear stability analysis, which allows us to determine ω. Using (32) and (39), we find $\omega=\omega_m$ and a_m. We use the negative value of ω, which gives a positive value for a_m when $d>\frac{1}{3}$.

The $O(\eta^2)$, $O(\eta^3)$, and $O(\eta^4)$ conditions are identically satisfied. Note that the $O(\eta^3)$ condition is satisfied because of the choice of the slow variables, the definition of η, and the fact that the bifurcation is vertical at $a=a_m$. We find that the $O(\eta^5)$ condition is the first condition that is nonzero. We have

$$(7+3d)\frac{\partial \alpha}{\partial \tau}=\frac{28}{d}\frac{\partial^2\alpha}{\partial t_2^2}-\left[i6\sqrt{3d}\,|\alpha|^2+\frac{20}{d}\frac{1}{\alpha^*}\frac{\partial \alpha^*}{\partial t_2}\right]\frac{\partial\alpha}{\partial t_2}$$

$$+\frac{J_4}{8}\alpha-\frac{9d^2}{16}|\alpha|^4\alpha . \quad (40)$$

This is a second evolution equation for α. It contains the dissipation time scale and the bifurcation parameter. Equations (36) and (40) will now be the subject of a more detailed analysis.

IV. THE SIMPLIFIED AMPLITUDE EQUATION

Equations (36) and (40) are two separate contributions to the general evolution equation. This is because they come from the $O(1)$ and $O(\epsilon)$ problems resulting from the first expansion of the solution in powers of ϵ. In order to more clearly understand their respective contributions to both the frequency correction and the amplitude modulation, it is worthwhile to combine (36) and (40) into a single equation. Moreover, by rescaling all of the variables, we may reduce the number of independent coefficients. These new variables are defined by

$$\alpha=\left[\frac{4(7+3d)}{3d(3d-1)}\right]^{1/2}\tilde{\alpha} ,$$

$$r_2=\frac{12\sqrt{3d}}{7+3d}r ,\quad t_2=\frac{12}{\sqrt{3d}}t ,\quad \tau=(7+3d)T ,$$

$$\lambda=\frac{J_4}{8} . \quad (41)$$

Furthermore, (40) can be rewritten in terms of the slow space variable using (36). The resulting equation is then multiplied by ϵ and added to (36) to give

$$\tilde{\alpha}_t+\tilde{\alpha}_r+\epsilon\tilde{\alpha}_T$$

$$=iB_0|\tilde{\alpha}|^2\tilde{\alpha}+\epsilon\left[\frac{7}{12}\tilde{\alpha}_{rr}-\frac{5}{12}\frac{\tilde{\alpha}_r^*\tilde{\alpha}_r}{\tilde{\alpha}^*}-iB_1|\tilde{\alpha}|^2\tilde{\alpha}_r\right.$$

$$\left.+iB_2\tilde{\alpha}^2\tilde{\alpha}_r^*+\lambda\tilde{\alpha}-|\tilde{\alpha}|^4\tilde{\alpha}\right] . \quad (42)$$

$$B_0=\frac{8}{3d-1} ,\quad B_1=-2+\tfrac{1}{4}B_0 ,\quad B_2=-\tfrac{3}{4}B_0 . \quad (43)$$

Note that this equation is valid only to $O(\epsilon)$ and we ignore all $O(\epsilon^2)$ effects.

Equation (42) is now a modified complex Ginzburg-Landau equation. It describes the bifurcation to periodic traveling waves, which has been called "type I_O: oscillatory periodic" by Cross and Hohenberg [20]. This equation differs from other Ginzburg-Landau equations in several ways. First, it is an equation for the amplitude of a traveling wave solution propagating in one direction only. This is the result of the particular form of the original multimode laser equations. Second, we note the term $(\tilde{\alpha}_r^*\tilde{\alpha}_r)/\tilde{\alpha}^*$, which results from the nonlinear frequency correction. Finally, we observe the presence of the fifth-order nonlinearity, which results from the vertical bifurcation at $a=a_m$.

We may simplify (42) by introducing a moving reference frame and removing the nonlinear dispersion. Letting

$$\tilde{\alpha}=e^{i\Omega t}A(T,Z)\ , \quad \text{and } Z=r-t\ , \tag{44}$$

we determine Ω to be

$$\Omega=B_0|\tilde{\alpha}|^2\ , \tag{45}$$

so that the resulting evolution equation for $A(T,Z)$ is

$$A_T=\tfrac{7}{12}A_{ZZ}-\frac{5}{12}\frac{A_Z^*A_Z}{A^*}-iB_1|A|^2A_Z+iB_2A^2A_Z^* +\lambda A-|A|^4A\ . \tag{46}$$

By introducing the characteristic variable Z and removing the nonlinear dispersion, we have effectively solved (36) and applied its result, the introduction of traveling waves, to (40). It is common in the study of evolution equations to propose equations such as (46) directly from the Landau equation. However, the coefficients B_1 and B_2 and the particular structure of the diffusion term cannot be predicted easily.

Equation (46) is similar to the degenerate evolution equation studied by Doelman and Eckhaus [16] except for the diffusion term

$$\frac{7}{12}A_{ZZ}-\frac{5}{12}\frac{A_Z^*A_Z}{A^*}\ . \tag{47}$$

These terms correspond to

$$\frac{28}{d}\frac{\partial^2\alpha}{\partial t_2^2}-\frac{20}{d}\frac{1}{\alpha^*}\frac{\partial\alpha^*}{\partial t_2}\frac{\partial\alpha}{\partial t_2} \tag{48}$$

in (40). The term multiplied by the coefficient $20/d$, which we will refer to as the "diffusion split" term, is unexpected. However, we have verified that it is still a contribution of the linearized theory. We find that the sum of the coefficients $(28/d+20/d=48/d)$, as well as the time derivative and linear term coefficients, is predicted by expanding the neutral stability curve about the minimum. These results are consistent with the general theory of evolution equations [19]. The diffusion split term results from the fact that the leading-order problem for $\epsilon\rightarrow 0$ is nonlinear. This is in contrast to most other derivations in which the hierarchy of problems, including the leading-order problem, are all linear.

Before proceeding, we mention that (46) is consistent with the stability results of the basic state and the single-mode bifurcation. We demonstrate this for the latter case. For the single-mode bifurcation, we seek a solution of the form

$$A(T,Z)=R(T)e^{ia_2Z}\ , \tag{49}$$

and assume that there are no sideband modes to affect stability. We obtain the Landau equation

$$R_T=R(\lambda-a_2^2-2a_2R^2-R^4)=-\frac{\partial V(R)}{\partial R}\ , \tag{50}$$

which describes the evolution of arbitrary initial conditions to the single-mode solution. We have thus determined the stability of the bifurcating mode. Note that the right-hand side of (50) is a scaled version of the bifurcation equation derived in Ref. [7]. This equation is similar to the results of Haken and Ohno [4,5], who interpret the right-hand side in terms of a potential V. Their coefficients are much more complex and do not make apparent the importance of the wave number in determining the direction of the bifurcation.

V. PERIODIC TRAVELING WAVES AND THEIR STABILITY

In this section, we determine stationary wave solutions (SW) to (46) in terms of the characteristic variable Z. In terms of the variables r and t, these correspond to traveling waves with phase velocity equal to one. Since the nonlinear dispersion has been removed by introducing Ω, the coefficients of (46) are all real. Therefore, similar to the real Ginzburg-Landau equation, we will refer to the Eckhaus boundary as separating regions of stable and unstable solutions.

We look for a SW solution of the form,

$$A=A_0e^{ia_2Z} \tag{51}$$

Since Z is proportional to the slow variables r_2 and t_2, a_2 is proportional to the deviation of the wave number from the minimum. Specifically,

$$a-a_m=\eta^2a_2\ . \tag{52}$$

Introducing (51) into (46), we obtain a condition for the existence of SW solutions

$$\lambda=(A_0^2+a_2)^2\ . \tag{53}$$

Equation (53) is the bifurcation equation. If $A_0=0$, (53) is the neutral stability curve (NS). For $a_2>0$ ($a_2<0$) the bifurcation is supercritical (subcritical). The limit point for $a_2<0$, defined by $\partial\lambda/\partial A_0=0$, is $\lambda=0$. The region in the (a,λ) plane between the limit-point curve and the NS curve defines the region of bistability. This is shown in Fig. 1. Referring to Doelman and Eckhaus [16], our case corresponds to $B=B_1+B_2=-2$, which implies $S=B^2/4-1=0$ in their classification. They do not explicitly treat the case $S=0$ but we found that the results are similar to the case $S>0$.

We now examine the stability of the SW solution by adding a small perturbation R to the basic solution. R depends on the slow scales T and Z, and takes the form of a single mode with arbitrary wave number

$$A=[A_0+R(Z,T)]e^{ia_2Z}\ ,$$
$$R(Z,T)=R_1e^{\sigma T+ikZ}+R_2e^{\sigma^*T-ikZ}\ , \tag{54}$$

Upon substituting (54) into (46) and linearizing, we obtain a system of algebraic equations for R_1 and R_2. From this system we determine that the SW solutions are stable if the real part of σ is negative. We find that the following conditions must be satisfied:

$$\lambda<A_0^4(2-cBH)+a_2A_0^2(2cH-B)\ ,$$
$$B=B_1+B_2\ ,\quad H=\tfrac{1}{4}(B_1-B_2)\ ,\quad c=\frac{12}{7} \tag{55}$$

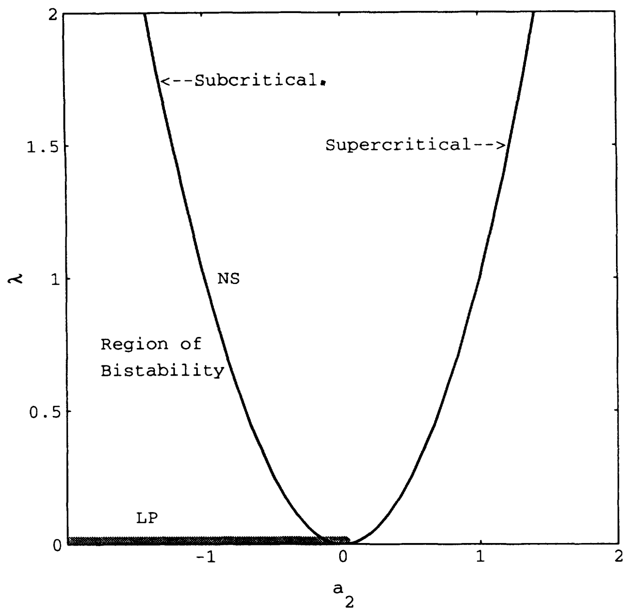

FIG. 1. The neutral stability curve (NS) in the (λ, a_2) plane, where a_2 is proportional to the deviation of the wave number from a_m. The location of the limit point for each value of a is designated by (LP). The region of bistability lies between the subcritical NS and LP.

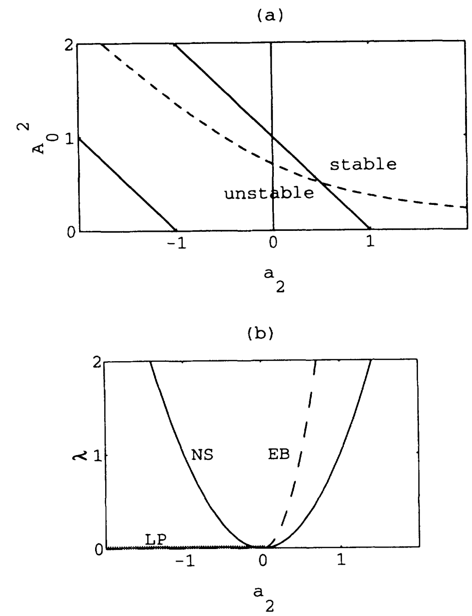

FIG. 2. $\frac{1}{3} < d < d_z = \frac{77}{15}$: (a) The solid lines represent the A_0^2 as a function of a_2, when $\lambda = 1$. For $a_2 > 0$ $(a_2 < 0)$ the bifurcation is supercritical (subcritical). Amplitudes greater than (less than) the dashed line correspond to stable (unstable) solutions. The intersection point is the Eckhaus boundary. (b) The Eckhaus boundary (EB) and neutral stability curve (NS) are shown as dashed and solid lines, respectively. λ is the bifurcation parameter. For $a_2 > 0$ the EB is the dashed line, while for $a_2 < 0$ the EB runs along the limit-point curve (LP).

H has been called the "hidden parameter" in Ref. [16] since it characterizes the part of the coefficients that cannot be derived from the Landau equation. Our result is analogous to (2.14) of Ref. [16] except for the parameter c, which results from the diffusion split term.

Our analysis of (53) and (55) is now similar to that in Refs. [15] and [16]. For any particular λ, (53) defines a pair of lines in the (a_2, A_0^2) [shown as solid lines in Figs. 2(a)–4(a)]. These lines represent the amplitude of the solution as a function of a_2 for $\lambda > 0$. Equation (55) defines a hyperbola for $\lambda > 0$, and is represented by a dashed line in Figs. 2(a)–4(a). The portion of the solid line above the hyperbola corresponds to stable solutions. The intersection point is the Eckhaus boundary.

Figures 2(b)–4(b) show the complete Eckhaus curve in the (λ, a_2) plane. The left half of the Eckhaus boundary is the limit-point curve $\lambda = 0$. Understanding how the Eckhaus boundary depends upon the parameter d is made easier by first using Figs. 2(a)–4(a). This is because the coefficients in (55) may change sign depending on the value of the parameters, thus making the inequality difficult to interpret.

Using the results of the previous section, we can derive the relationship between the hidden parameter H, and the physical parameter d. It is given by $H = (7-3d)/[2(3d-1)]$. In addition, we determine an explicit form for the Eckhaus curve in terms of the parameter d:

$$\lambda = C(d) a_2^2$$

where

$$C(d) = \frac{4(35+3d)^2}{(-77+15d)^2} . \tag{56}$$

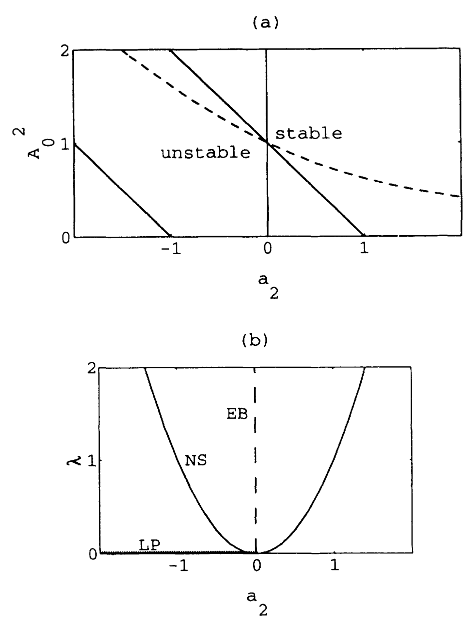

FIG. 3. $d = d_z = \frac{77}{15}$ (see also Fig. 2): (a) The intersection of the solid and dashed line occurs at $a_2 = 0$. In (b), the Eckhaus boundary is the solid dashed line together with the limit-point curve (LP). Only solutions with $a_2 < 0$ are stable.

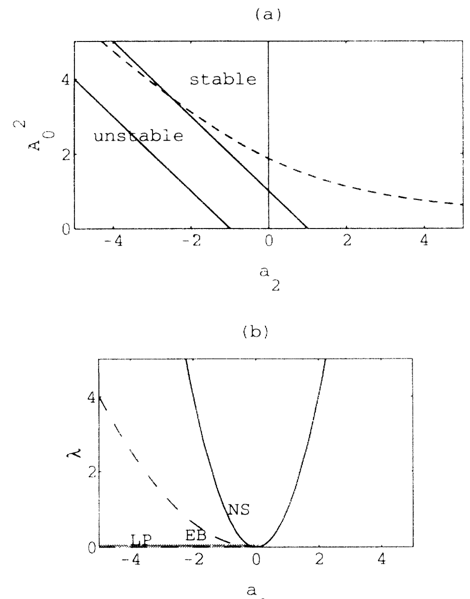

FIG. 4. $d \to \infty$, $\lambda \sim \frac{4}{25}a^2$ (see also Fig. 2): (a) The intersection of the solid and dashed line occurs at $a_2 < 0$. In (b) the Eckhaus boundary (EB) is the dashed line together with the limit-point curve (LP). Note that for a fixed value of the wave number a_2 there is an upper bound to the value of the bifurcation parameter λ for which the solution is stable. When $d = d_c = \frac{49}{3}$ the Eckhaus boundary exactly coincides with the left side of the neutral stability curve (NS). For $d > d_c$, the EB lies outside the NS as shown.

Our result is valid provided that d is larger than $\frac{1}{3}$ (if $d \to \frac{1}{3}, B_0 \to \infty$ in (43) and our expansion of the solution becomes singular [7]). However, by examining $C(d)$ we see that for $\frac{1}{3} < d < d_z = \frac{77}{15}$ the Eckhaus boundary lies in the right-half plane of the (a, λ) parameter space [see Fig. 2(b)]. Consequently, there exists a band of stable traveling wave solutions for all $a_2 < a_{\max} > 0$, where $a_{\max} = (\lambda/C)^{1/2}$. If $d > d_z$, the band of stable modes becomes more restricted and we discuss this important change of the stability digram in the next section.

VI. DISCUSSION

Class-*B* lasers are characterized by a vertical-Hopf bifurcation located at the minimum of the neutral stability curve [7]. By taking advantage of this particular feature, we use asymptotic methods to obtain a degenerate Ginzburg-Landau equation for the long-time behavior of traveling wave solutions. Degenerate evolution equations have been proposed recently in order to describe phenomena that are not explained by the classical Ginzburg-Landau equation; namely, the presence of a subcritical bifurcation for certain wave numbers. At the end of Sec. IV, we discussed the similarities and differences between our equation and equations previously analyzed [16].

Of particular physical interest is the fact that the stability of the periodic traveling wave solutions now depends on the laser parameters. At the end of Sec. V, we determine the Eckhaus boundary that separates stable and unstable traveling waves [see (56)]. We note that if $d \geq d_z = \frac{77}{15}$ all stable solutions are characterized by $a \leq a_m$; see Fig. 3. If the initial periodic wave is characterized by a wave number $a > a_m$, we expect a gradual change to a stable traveling wave with $a \leq a_m$. If $d > d_z$, the Eckhaus boundary lies to the left of the neutral stability curve and a stable traveling wave characterized by $a < a_m$ will become unstable if the bifurcation parameter is *raised* above the Eckhaus boundary (this is shown in Fig. 4 for the case when $d > \frac{49}{3}$, which is discussed below). In the classical Ginzburg-Landau equation an initially stable traveling wave becomes unstable only if the bifurcation parameter is *lowered* below the Eckhaus boundary.

If d is further increased and surpasses $d_c = \frac{49}{3}$, the Eckhaus curve will be located to the left of the neutral stability curve as shown in Fig. IV. There are no stable traveling wave solutions having a wave number inside the neutral stability curve. In this case, we expect that the initial wave will progressively change to a stable regime characterized by a low value of the wave number.

ACKNOWLEDGMENTS

This work was supported by the U.S. Air Force Office of Scientific Research under Grant No. AFSOR 93-1-0084 and the National Science Foundation under Grant No. DMS 9308009. This work was also supported by the Fonds National de la Recherche Scientifque (Belgium) and the Interuniversity Attraction Pole of the Belgian government.

[1] H. Risken and K. Nummendal, J. Appl. Phys. **39**, 4662 (1968).

[2] H. Risken, in *Optical Instabilities, Cambridge Studies in Modern Optics,* edited by R. W. Boyd, M. G. Raymer, and L. N. Narducci (Cambridge University Press, Cambridge, 1986), Vol. 4.

[3] R. Graham and H. Haken, Z. Phys. **213**, 420 (1968).

[4] H. Haken and H. Ohno, Opt. Commun. **16**, 205 (1976).

[5] H. Ohno and H. Haken, Phys. Lett. **59A**, 261 (1976).

[6] H. Fu, Phys. Rev. A **40**, 1868 (1989).

[7] T. W. Carr and T. Erneux, Phys. Rev. A **50**, 724 (1994).

[8] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).

[9] L. A. Segal, J. Fluid Mech. **38**, 203 (1969).

[10] A. C. Newell, Lectures Appl. Math. (Springer, New York, 1974), Vol. 15.

[11] P. Kolodner, D. Bennon, and C. M. Surko, Phys. Rev. Lett. **60**, 1723 (1990).

[12] J. J. Niemela, B. Ahlers, and D. D. Cannel, Phys. Rev. Lett. **64**, 1365 (1990).

[13] H. R. Brand and R. J. Deissler, Phys. Rev. A **45**, 3732 (1992).

[14] W. van Saarloos and P. C. Hohenberg, Physica D **56**, 303 (1992).
[15] W. Eckhaus and G. Iooss, Physica D **39**, 124 (1989).
[16] A. Doelman and W. Eckhaus, Physica D **53**, 249 (1991).
[17] T. W. Carr, Ph.D. thesis, Northwestern University (1993).
[18] T. Erneux, S. M. Baer, and P. Mandel, Phys. Rev. A **35**, 1165 (1987).
[19] A. C. Newell, *Lectures in the Sciences of Complexity* (Addison-Wesley, Reading, MA, 1989), Vol. I.
[20] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).