Laser with injected squeezed vacuum: Phase diffusion and intensity fluctuations

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We investigate the effect of injected squeezed vacuum on the phase and intensity fluctuations of a laser. By using a stochastic simulation of the Langevin equations, we find that the phase diffusion noise of a laser with injected squeezed vacuum can be transiently reduced by a factor of 2, in agreement, with earlier predictions. Further, we find that the laser linewidth is broadened. The key observation is that the phase-diffusion rate is time dependent. We find that the steady-state intensity fluctuations are increased by the injected squeezing, in qualitative (but not quantitative) agreement with an earlier prediction. The problem of locking is examined. We find that the injected squeezed vacuum does not cause the laser to lock to any particular phase. We present an approximate solution of the Fokker-Planck equation in steady state, and a simple geometrical, vector kick model from which most of our results can be obtained.

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I. INTRODUCTION

What is the lower quantum limit of the laser linewidth? The theoretical lower limit was first obtained by Schawlow and Townes; $\Delta\omega_{ST}=C/2\bar{n}=D_0$, where D_0 is the phase-diffusion rate [1]. Here C is the empty cavity linewidth and $\bar{n}$ is the mean number of photons in the cavity. This fundamental limit arises from amplified vacuum fluctuations and spontaneous emission noise, and is due to the quantum nature of the field. Various schemes have been proposed to reduce the linewidth below this limiting value [2,3]. A few years ago, one of us [4] considered the case of squeezed vacuum rather than ordinary vacuum incident on the output port of the laser. When the phase of the squeezed field (relative to the laser field) is chosen so that the field leaking into the cavity has reduced phase noise, the phase-diffusion coefficient of the Fokker-Planck equation describing this system can be reduced by a factor of 2. As the phase-diffusion coefficient can be simply related to the linewidth for an ordinary laser above threshold, one might naively assume that a reduction in the phase-diffusion rate would narrow the linewidth by the same factor of 2. Subsequent calculations of the temporal eigenvalues of the equivalent Fokker-Planck equation suggested however that the laser linewidth could not be reduced by injected squeezing [5]. For an ordinary laser the smallest eigenvalue of the Fokker-Planck equation is doubly degenerate, and equal to the laser linewidth $\Delta\omega_{ST}$. The results of Ginzel and co-workers [5] indicate that as the amount of squeezing is increased, this degeneracy is broken, with one eigenvalue becoming smaller, and the other larger, than $\Delta\omega_{ST}$. The eigenvalue that initially decreases with increased squeezing was shown to reach a minimum much greater than $\Delta\omega_{ST}/2$ and then increase as the squeezing is increased further. The implication was that the laser linewidth could not be lowered to the value of $\Delta\omega_{ST}/2$, or substantially reduced in any meaningful way.

To shed some light on the noise properties of a laser with injected squeezed vacuum, we have taken the Fokker-Planck equation for this system, and have formed an equivalent set of Ito stochastic differential equations. These were then directly simulated on the computer. We find that the laser phase-diffusion rate, defined as

$$D(t)=\lim_{r\to 0}\frac{\langle[\phi(t+\tau)-\phi(t)]^2\rangle}{\tau}, \tag{1}$$

can indeed be made as small as $D_0/2$, for times small compared to $(1/\Delta\omega_{ST})e^{-2r}\equiv(1/D_0)e^{-2r}$, which we refer to as the relaxation time τ_R. Here r is the usual squeezing parameter. This transient reduction in the phase-diffusion rate is not immediately obvious because one might expect that the phase-diffusion rate could not in any case be smaller than the smallest eigenvalue calculated by Ginzel and co-workers [5]. For times comparable to the relaxation time, and for large r, the average phase-diffusion rate increases to a value of $(\Delta\omega_{ST}/4)e^{2r}=(D_0/4)e^{2r}$. Here we refer to the time averaged phase-diffusion rate as the average phase-diffusion rate. This average phase-diffusion rate is one half of the value obtained when the squeezed light is phased such that increased phase noise is injected into the cavity.

As the intensity of the laser satisfies an equation of the form

$$\dot{I}=(A-C)I-BI^2 \tag{2}$$

in the semiclassical theory, it seems possible that the laser would quench the large amplitude fluctuations that it must see when quiet phase light is injected. The laser would then seem to prefer the state where it sees reduced amplitude fluctuations, and hence increased phase fluctuations. From this naive picture, one might think that intensity fluctuations are reduced in the steady state, but they are actually enhanced [6]. According to Marte, Ritsch, and Walls [3], this is a stable steady state, whereas the steady state with reduced phase noise is unstable. Marte, Ritsch, and Walls have considered the effects of squeezed pumps as well as injected squeezing, and it is the latter work that is relevant here. Our results, however, do not support this interpretation. This state is not stable as the laser phase suffers large phase kicks in this state, and eventually the laser phase explores all possible angles. The steady-state fluctuations in amplitude (as well as phase) are an average of the "quiet" and "noisy" values. In the steady state, there is no corresponding decrease in amplitude noise. The laser phase is uniformly distributed over 2π, but the phase-diffusion rate is dependent on the relative instantaneous phase of the laser and injected squeezing. This phase, defined as ϕ_{LS}, is constantly changing. Thus the average phase-diffusion rate in the steady state will be the average over all possible phases weighted by the probability that the laser field occupies that phase. This is in general larger than when no squeezing is present.

We also present results of a direct numerical solution of the Fokker-Planck equation, which can be described for large laser intensities by a fairly simple approximate form. The results of this work are consistent with those obtained from the simulation of the Ito stochastic differential equations. Finally, we present a simple semiclassical vector kick model that reproduces results obtained by the more rigorous treatments described above.

Overall, we find that the laser with injected squeezed vacuum behaves as a laser with one half the phase-diffusion noise of a normal laser, but that this behavior is transitory. However, we stress that the time over which the phase noise is reduced is indeed a useful one, and that the essence of earlier predictions is correct [4]. The phase-diffusion rate is reduced for a time short compared to the integration time necessary to resolve the laser linewidth ($1/\Delta\omega_{ST}$); after this times it becomes significantly greater than in the unsqueezed case, resulting in an increased linewidth. We should note that the transient nature of the noise reduction is to be expected in all practical applications of squeezing, and not just in the present model. Unless special precautions are taken to ensure that the squeezed light and local oscillator drift in phase together, the noise will only be reduced for a range of times where the relative phase has the correct value. At other times, the phase mismatch will in general result in increased noise.

II. PHYSICAL MODEL

The system under consideration is depicted in Fig. 1. The master equation for the laser with injected squeezing is [4,6]

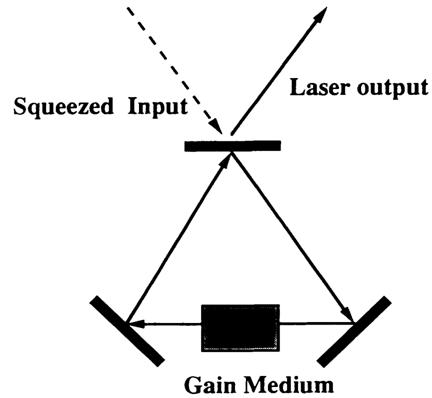

FIG. 1. Schematic of the physical system under consideration, a single-model ring laser with squeezed vacuum injected into the output port.

$$\begin{aligned}\dot{\rho}=&-\frac{C}{2}(N+1)[a^{\dagger}a\rho-a\rho a^{\dagger}+\text{H.c.}]\\&-\frac{(A+CN)}{2}[aa^{\dagger}\rho-a^{\dagger}\rho a+\text{H.c.}]\\&+\frac{B}{8}[\rho(a^{\dagger}a)^2+3aa^{\dagger}\rho aa^{\dagger}-4a^{\dagger}\rho aa^{\dagger}a+\text{H.c.}]\\&+\frac{MC}{2}[e^{i\phi_{LS}(0)}(a^2\rho-2a\rho a+\rho a^2)+\text{H.c.}]\ ,\end{aligned}\tag{3}$$

where A is the linear gain, C is the loss rate, and B is the saturation factor as in the equation for the laser intensity in standard third-order laser theory, Eq. (2). The squeezed reservoir is described by N and M. Here $N=\sinh^2 r$, $M=\cosh(r)\sinh(r)$, where r is the usual squeezing parameter, and $\phi_{LS}=\phi_L-\phi_S$ is the relative phase between the laser field and the injected squeezed vacuum. $\phi_{LS}(0)$ is the initial value of this angle. When this angle is zero, the laser mode sees reduced phase fluctuations, i.e., the error ellipse is aligned such that the long axis is aligned with the laser field. The equivalent Fokker-Planck equation for the Glauber-Sudarshan P distribution is

$$\begin{aligned}\frac{\partial P(\alpha,t)}{\partial t}=\frac{1}{2}\Bigg\{&-\frac{\partial}{\partial\alpha}(A-C-B|\alpha|^2)\\&-\frac{\partial}{\partial\alpha^*}(A-C-B|\alpha|^2)\\&-CMe^{i\phi_{LS}(0)}\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\alpha^{*2}}\right]\\&+2(A+CN)\frac{\partial^2}{\partial\alpha^*\partial\alpha}\Bigg\}P(\alpha,t)\ .\end{aligned}\tag{4}$$

For $\phi_{LS}(0)=0$, and using the Cartesian representation defined by the transformation,

$$\alpha=x'+iy'\ ,\quad \alpha^*=x'-iy'\ ,\quad x'=\tfrac{1}{2}(\alpha+\alpha^*)\ ,$$
$$y'=\frac{1}{2i}(\alpha-\alpha^*)\ ,\tag{5}$$

and letting $a=\sqrt{2}(A-C)/\sqrt{AB}$, $b=C/A$; $x^2=\sqrt{2B/A}\,x'^2$; $y^2=\sqrt{2B/A}\,y'^2$, and $\tau=\sqrt{AB/2}t$, we obtain

$$\frac{\partial P}{\partial \tau}=\frac{1}{2}\left\{-\frac{\partial}{\partial x}[a-(x^2+y^2)]x-\frac{\partial}{\partial y}[a-(x^2+y^2)]y+\left[1+\frac{b}{2}(e^{2r}-1)\right]\frac{\partial^2}{\partial x^2}+\left[1+\frac{b}{2}(e^{-2r}-1)\right]\frac{\partial^2}{\partial y^2}\right\}P . \tag{6}$$

Using these expressions we finally obtain the equivalent set of Ito stochastic differential equations for $\phi_{LS}(0)=0$ initially.

$$dx=\tfrac{1}{2}[a-(x^2+y^2)]x\,dt+\left[1+\frac{b}{2}(e^{2r}-1)\right]^{1/2}dW_x , \tag{7}$$

$$dy=\tfrac{1}{2}[a-(x^2+y^2)]y\,dt+\left[1+\frac{b}{2}(e^{-2r}-1)\right]^{1/2}dW_y . \tag{8}$$

Here the dW_i are independent Wiener processes. We note that the equations for $\phi_{LS}(0)=\pi/2$ are obtained by setting $r\rightarrow -r$ in the above equations, but that more complicated expressions are required for arbitrary $\phi_{LS}(0)$. The squeezing parameter r only appears in the noise terms, and does not affect the deterministic parts. We also note that the noise terms are positive definite for $b\leq 2$, and hence to describe the laser above threshold ($b\leq 1$) the normal P representation is adequate, and we need not use the positive-P representation. As $I=(x^2+y^2)$ we have

$$\frac{dI}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}=(a-I)I . \tag{9}$$

Neglecting noise, we have two steady-state values $I_{ss}=a$, and $I_{ss}=0$, just as in the case where ordinary vacuum fluctuations are incident on the output mirror. These correspond to the laser above and below threshold, respectively. The phase diffusion rate D_0 for the ordinary laser is [1]

$$D_0=\frac{1}{4}\frac{(A+C)}{\bar{n}}=\frac{1}{4}\frac{BC}{A-C}(1+C/A) . \tag{10}$$

Letting $Dt=\tilde{D}\tau$, we have using our scaled time

$$\tilde{D}_0=\frac{b}{2a}(1+b)\approx\frac{1}{a} , \tag{11}$$

when b is approximately 1, i.e., for a laser not too far above threshold. In terms of our scaled time, the relaxation time is

$$\tau_R=\frac{a}{2b}e^{-2r}\approx\frac{a}{2}e^{-2r} . \tag{12}$$

This is a characteristic parameter of the laser with injected squeezed vacuum which will be referred to repeatedly in what follows. A discussion on the physical origin of this time scale appears in Sec. VI E. Note that for no squeezing ($r=0$), $\tau_R\approx 1/\tilde{D}_0$, whereas if $r>0$, $\tau_R<1/\tilde{D}_0$. The origin of this relaxation time will be discussed in Sec. VI.

III. RESULTS OF SIMULATIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

A. Algorithm

The Ito stochastic differential equations were simulated on the computer using a simple Euler integration method. The Wiener processes were simulated as $dW_{x,y}=\xi_{x,y}\sqrt{dt}$, where $\xi_{x,y}$ is a random number that is generated from a Gaussian distribution with unit mean [7]. We have checked that making time increments smaller and the number of trajectories larger does not change the result substantially. Typical numbers used were 10^4 trajectories and 10^5 time steps. To further verify that our program functioned properly, our simulations exhibit the laser intensity building from zero intensity to the steady-state value of a, independent of the value of r. We give results here for different choices of r and for $\phi_{LS}(0)=0$ and $\pi/2$. For $\phi_{LS}(0)=0$, noise in the x direction (amplitude) is enhanced while the noise strength in the y direction (phase) is reduced. (Of course, the concepts of phase and amplitude noise are only meaningful for large photon numbers, as in a laser.)

B. Phase-diffusion rate

Figure 2 shows the dependence of the phase-diffusion rate on time. In the usual theory of the laser linewidth it is assumed that the phase-diffusion rate is independent of time. This is certainly not the case for a laser with injected squeezed vacuum. Here the laser linewidth cannot be calculated via the usual Schawlow-Townes argument. Figure 3 shows a plot of the phase-diffusion rate for times small compared to the relaxation time. In this transient

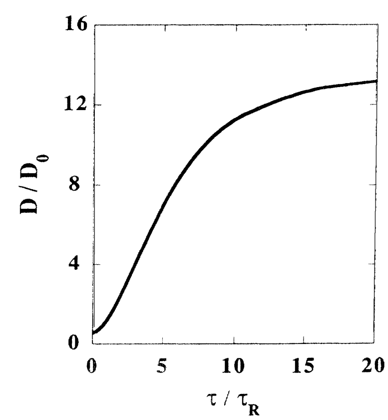

FIG. 2. Phase-diffusion rate as a function of time for $r=2$, $a=100$, $b=0.95$, and $\phi_{LS}=0$ initially. The phase-diffusion rate is equal to one-half the Schawlow-Townes rate D_0 for short times, and grows to an enhanced value, approximately $e^{2r}/4\approx 13.1D_0$.

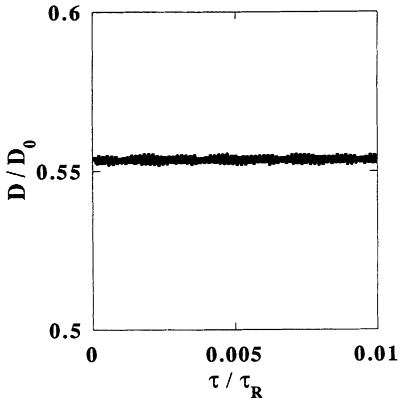

FIG. 3. Phase-diffusion rate as a function of time for $r=3$, $a=100\,000$, and $b=0.95$, for times short compared to the relaxation time, which is approximately 130 sec.

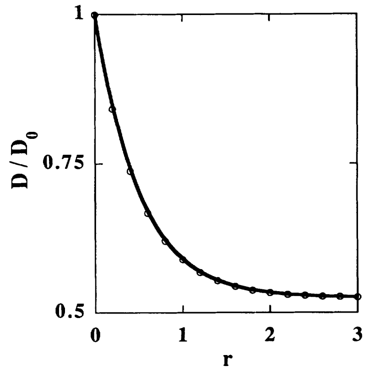

FIG. 4. Plot of the time averaged phase-diffusion rate (averaged over times short compared to the relaxation time) versus the strength of the squeezing r. The circles represent the numerical data, and the solid line the analytical approximation, Eq. 13.

regime, the phase-diffusion rate is indeed approximately one half of $\tilde{D}_0$. In Fig. 4 we present a plot of the value of the transient phase-diffusion rate $\tilde{D}_T$ as a function of the strength of the injected squeezing, r. $\tilde{D}_T$ is defined as a time average of the phase-diffusion rate over times small compared to the relaxation time. Recall that the relaxation time decreases with increased squeezing, so that the time over which quiet phase noise persists is reduced as the amount of squeezing is increased. The amount of phase noise reduction during that period is of course increased as the amount of squeezing is increased. Error bars are smaller than the symbol designating the data point, and typically are in the neighborhood of 1%. Note that as the amount of squeezing is increased, the phase-diffusion rate is indeed reduced to one half the value when there is no injected squeezing. The solid line in Fig. 4 is an approximate solution for the phase-diffusion rate valid for short times [4],

$$\tilde{D}=\frac{1}{a}\left[1+\frac{b}{2}(e^{-2r}-1)\right]\approx\frac{1}{2a}(1+e^{-2r})=\frac{\tilde{D}_0}{2}(1+e^{-2r})\ . \tag{13}$$

There is a striking agreement between this approximate solution and the direct numerical simulations. To see how this solution arises, consider the Fokker-Planck equation written in polar coordinates,

$$\frac{\partial P}{\partial t}=\left\{-\frac{\partial}{\partial I}[(a-I)I+2(1+bN)]-\frac{bM}{I}\frac{\partial}{\partial\phi_{LS}}\sin(2\phi_{LS})+2[1+bN+bM\cos(2\phi_{LS})]\frac{\partial^2}{\partial I^2}I -2bM\frac{\partial^2}{\partial I\partial\phi_{LS}}\sin(2\phi_{LS})+\frac{1}{2I}\frac{\partial^2}{\partial\phi_{LS}^2}[1+bN-bM\cos(2\phi_{LS})]\right\}P\ . \tag{14}$$

Assume that the initial P distribution is narrowly localized around the phase $\phi_{LS}=0$. Equation 14 can be used to calculate the evolution of the phase uncertainty $\langle\Delta\phi_{LS}^2\rangle=\langle\phi_{LS}^2\rangle$, multiplying both sides by ϕ_{LS}^2, and integrating over I and ϕ_{LS}. The result, after integrating by parts and setting all of the surface terms to zero, is

$$\frac{d}{d\tau}\langle\phi_{LS}^2(t)\rangle=\left\langle\frac{1+bN-bM\cos(2\phi_{LS})}{I}\right\rangle+2bM\left\langle\frac{\phi_{LS}\sin(2\phi_{LS})}{I}\right\rangle\ . \tag{15}$$

Next, assume that the average values of products of I and ϕ_{LS} approximately factorize, and that $\langle 1/I\rangle\approx 1/a$ (small intensity fluctuations). Also, if $P(I,\phi_{LS},t)$ remains, for short times, localized near $\phi_{LS}=0$, the trigonometric functions may be expanded, with the result

$$\frac{d}{d\tau}\langle\phi_{LS}^2(t)\rangle\approx\frac{1}{a}\left[1+\frac{b}{2}(e^{-2r}-1)\right]+6\frac{bM}{a}\langle\phi_{LS}^2(t)\rangle\ . \tag{16}$$

If the initial relative phase is free from uncertainty, and equal to precisely zero, Eq. (16) yields the result (13) for the short-time phase-diffusion rate [compare to the definition (1)]. For longer times, the phase uncertainty would appear to grow exponentially. The factor $6bM/a \approx 3be^{-2r}/2a$ (large r approximation) equals $(3/4)(\tau_R)^{-1}$, where τ_R is the relaxation time defined in Eq. (12). A simple physical argument of the origin of this time scale is given in Sec. VI E. For arbitrary ϕ_{LS}, one obtains

$$D=\frac{D_0}{2}(1+\sinh^2 r-\sinh r\cosh r\cos 2\phi_{LS}) \tag{17}$$

as first discussed in Ref. [4].

At later times, Eq. (17) clearly does not describe a simple phase-diffusion process, as evidenced by the amplitude-phase coupling term, which is dependent on M. Eventually, at times long compared to the relaxation time, the average phase-diffusion rate is found, from the numerical simulation, to increase to the value

$$D_{ss} \approx (1/4)e^{2r}D_0 , \tag{18}$$

for large r. Again, D_{ss} is a time average of the phase-diffusion rate, taken after many relaxation times. For $\phi_{LS}=\pi/2$, (Fig. 5) we find that the transient phase-diffusion rate is initially increased but the average phase-diffusion rate decreases with time to D_{ss}, which is the value one obtains by averaging Eq. (17) over ϕ_{LS}. This rate is also the average value of the phase-diffusion rates at $\phi_{LS}=0$ and $\phi_{LS}=\pi/2$. We show in Sec. IV that the laser phase is equally distributed over 2π at long times.

For any initial choice of ϕ_{LS}, the average phase-diffusion rate is D_{ss}. The laser phase is found to be uniformly distributed over 2π, and hence the time average of the phase-diffusion rate D_{ss} is the average of the values of the phase-diffusion rate at all angles, essentially an average of the quiet value ($\phi_{LS}=0$), and the noisiest value (for $\phi_{LS}=\pi/2$). The laser linewidth will be determined essentially by the steady-state phase-diffusion rate, and hence the laser linewidth is approximately broadened by a factor of $\frac{1}{4}e^{2r}$, for large r, with injected squeezing.

C. Intensity fluctuations

We have also considered the intensity fluctuations of this system. Pedrotti and Gea-Banacloche have predicted that the steady-state intensity fluctuations should be enhanced by a factor of $\frac{1}{2}\sinh^2(r) \approx \frac{1}{8}e^{2r}$ above the value in the absence of squeezing ($\Delta I^2/I^2=2/a^2$ in our units). This result was derived using a truncation scheme for the coupled moment equations resulting from the master equation, as well as an assumption that the laser locked at $\phi_{LS}=\pi/2$. In Fig. 6 we present a plot of $\Delta I^2/I^2$ as a function of time for initial values of ϕ_{LS} of 0 and $\pi/2$, obtained from the simulation of the stochastic differential equations. For ϕ_{LS} initially 0, the asymptotic value disagrees with Ref. [6] by a factor of 2; the intensity fluctuations are twice as large as predicted earlier. This value is also reached if the initial value of $\phi_{LS}=\pi/2$, or for any other value as well. This is explained in later sections; here we note that one assumption of Ref. [6], the locking of the laser phase to $\phi_{LS}=\pi/2$ is contradicted by our numerical results, which show that the laser phase uniformly explores all 2π.

IV. NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATION

A. Algorithm

In this section, we discuss the numerical solution and analytical approximation to $P(I,\phi_{LS})$. In Ref. [5] it was shown how the numerical solutions to the Fokker-Planck equation could be obtained in polar coordinates; using the variables I and ϕ_{LS} (field intensity and relative phase, respectively), and expanding $P(I,\phi_{LS})$ in a doubly infinite series, using trigonometric functions for the ϕ_{LS} dependence and Laguerre functions for the I dependence. The resulting eigenvalue equations for the coefficients can then be solved numerically by a continued fraction method. This approach works well for relatively small

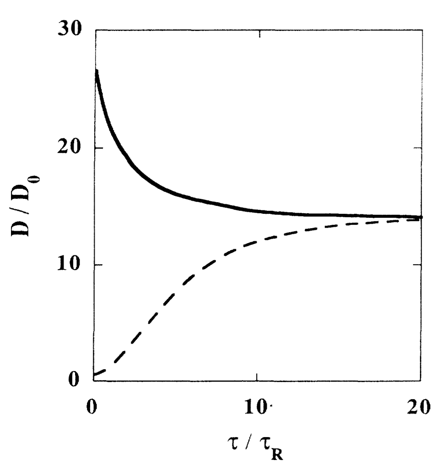

FIG. 5. Phase-diffusion rate as a function of time for $r=2$, $a=100$, and $b=0.95$. The dashed line is for $\phi_{LS}=0$ initially, the solid line for $\phi_{LS}=\pi/2$.

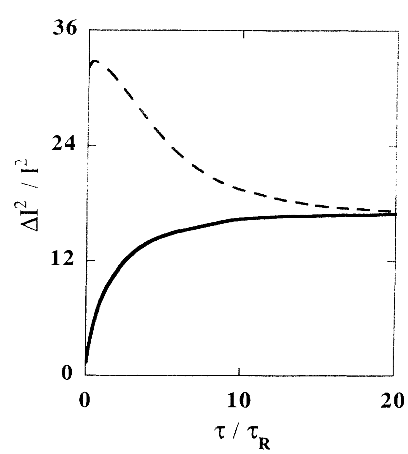

FIG. 6. Intensity fluctuations as a function of time, for times long compared to the relaxation time. The y axis is scaled to the value without squeezing. Results for $a=100$, $b=0.95$, $r=2$, and $\phi_{LS}=0$ (solid line) and $\phi_{LS}=0$ (dashed line).

values of a, but becomes cumbersome for $a \gg 1$; the reason is that the Laguerre functions are not really well suited to the problem, and the expansion converges exceedingly slowly for large a (more than 150 terms are needed for $a=10$). As we wish to consider larger values of a, another approach is necessary.

Here, we retain the Fourier expansion for the variable ϕ_{LS}, which converges fairly rapidly when r, the squeezing parameter, is not too large, and simply discretize the function in the I variable, replacing the derivatives with respect to I by a finite difference scheme. The solution to the time-dependent problem requires then the diagonalization of an $N \times N$ matrix, where $N = n_{\text{pts}} n_{\text{funcs}}$ is the product of n_{pts}, the number of points in the interval $I_{\min}$ to $I_{\max}$ where the function is evaluated, and n_{funcs}, the number of trigonometric functions kept in the Fourier series. This method is relatively straightforward to program, and the lowest eigenvalues of the matrix can be directly identified with the eigenvalues of the Fokker-Planck equation (finding the eigenvalues and eigenvectors is much more complicated in the continued-fraction method). For large a large matrices are again needed: as an extreme example, the calculation for $a=100$ and $r=1$ required $n_{\text{funcs}}=5$ and $n_{\text{pts}}=100$, leading to a 500×500 matrix which can, however, be diagonalized in less than an hour using MATHEMATICA on a small workstation.

B. Approximate analytic approximation for the P distribution

The numerical calculations show that, in fact, for large a the steady-state P distribution is well described by the following very simple form

$$P_0(I,\phi_{LS}) = \frac{1}{2\pi^{3/2}} \frac{e^{-(I-a)^2/2\Delta I(\phi_{LS})^2}}{2\Delta I(\phi_{LS})} , \tag{19}$$

where

$$\Delta I(\phi_{LS})^2 = 2 + 2b \sinh^2(r) + 2b \sinh(r)\cosh(r)\cos(2\pi_{LS}) . \tag{20}$$

This approximation is already fairly good for $a=10$, provided r is not too large ($r \approx 0.5$ or so). The exponential part of Eq. (19) can be derived easily from the Fokker-Planck equation in the variables I, ϕ_{LS}, by changing variables to $x = I - a$ and keeping only the terms proportional to a.

The prefactor $1/\Delta I(\phi_{LS})$ in Eq. (19) (which, we should point out, we have not succeeded at deriving by any simple analytical means) has the effect of making the phase distribution

$$P_0(\phi_{LS}) = \int_0^\infty P_0(I,\phi_{SL}) dI = \frac{1}{2\pi} , \tag{21}$$

that is, the phase distribution is flat, all the phases appear to be equally probable; the greater height of the distribution at $\phi_{LS} = \pm\pi/2$, where it is narrowest, is compensated by its greater width at $\phi_{LS}=0$, π, where it is lowest (see Fig. 7). This is supported by the results of our numerical calculations; direct numerical solution of the Fokker-Planck equation and the simulation of the Ito equations. That is, although the calculated phase distributions do show a very slight modulation, this effect decreases as the precision of the calculation is increased.

C. Phase diffusion

A numerical solution for the steady-state P distribution is presented in Figs. 7(a) and 7(b). At first glance, it appears that the laser phase spends more time in the noisy phase region around $\phi_{LS} = \pi/2$. However, if one integrates over the radial variable to find a probability distribution for the phase, it is $1/(2\pi)$ for all angles, meaning that any relative phase of the laser field is as likely to occur as any other. Alternatively, we may say that the laser phase is uniformly distributed. For r as large as one (injected light 85% squeezed), the numerical calculations

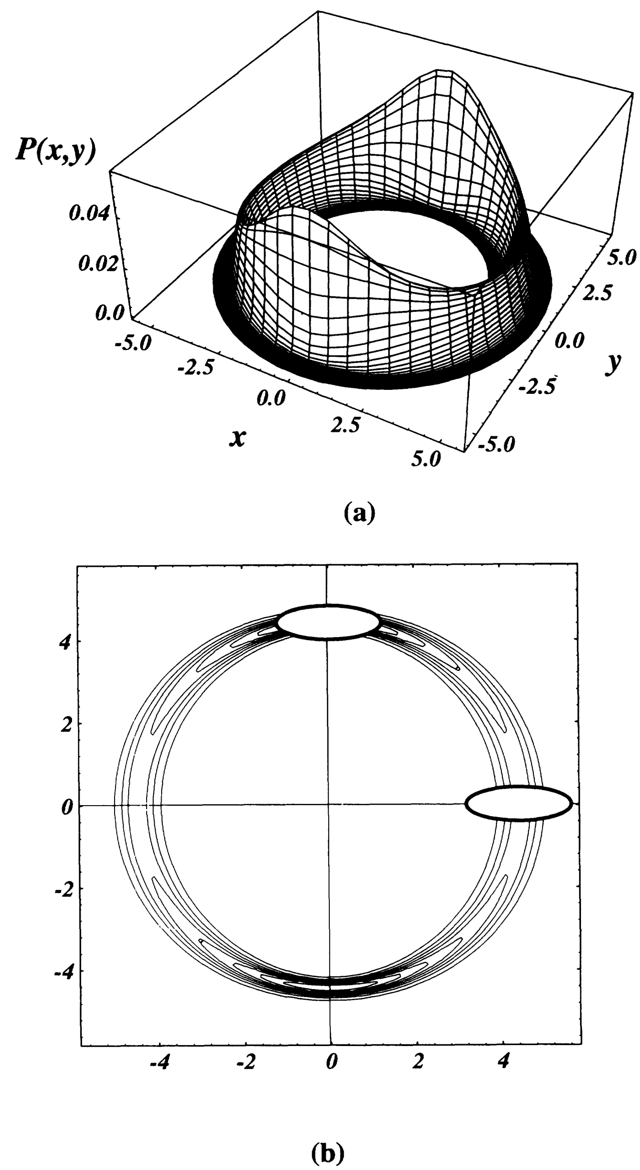

FIG. 7. Plot of the steady-state P distribution, calculated numerically using the method discussed in Sec. IV, (a) three-dimensional view, (b) a contour plot.

place an upper bound on the modulation of the phase distribution at less than 1%. We conclude, therefore, that the steady-state solution of the Fokker-Planck equation does not support the idea that locating at any particular phase takes place at all in this system.

It is obvious from our discussion that there is no steady-state value for the phase, in contrast to the case of a squeezed pump [3]. As every phase is equally likely, we see that the average phase-diffusion rate is just the average of all possible values, $D_{ss} \approx (e^{2r}/4)D_0$.

D. Intensity fluctuations

Equation (19) indicates that, for a given phase ϕ_{LS}, the spread in intensity of the function P_0 is given by $\Delta I(\phi_{LS})$. The conditional variance of the Gaussian distribution $[\Delta I^2(\phi_{LS})]$ oscillates between a maximum at $\phi_{LS}=0$, $\Delta I^2(0)=2+b(e^{2r}-1)$ and a minimum at $\phi_{LS}=\pi/2$, $\Delta I^2(\pi/2)=2-b(1-e^{-2r})$. Note that for large r the minimum value of $\Delta I^2(\phi_{LS})$ approaches $2-b$, which is positive as long as the laser is above threshold ($b<1$) and, in fact, as long as the condition $b<2$, necessary for a positive definite diffusion matrix, holds

The overall intensity variance predicted by (20) is the average of the conditional variance

$$\Delta I(\phi_{LS})^2/I^2 = (2/a^2)[1+b\sinh^2(r)+b\sinh(r)\cosh(r)\cos(2\phi_{LS})] \tag{22}$$

over the uniform phase distribution, i.e.,

$$\Delta I^2/I^2=(2/a^2)[1+b\sinh^2(r)] \ . \tag{23}$$

This also coincides with the average of the conditional variances at $\phi_{LS}=0$ (laser field in phase with squeezed vacuum) and $\phi_{LS}=\pi/2$ (laser field in quadrature with squeezed vacuum). When r is equal to zero, one obtains the standard result for an ordinary laser, i.e., a variance of two in the normalized units; converting again to ordinary units one can write, for the size of the photon number fluctuations in the cavity

$$\Delta n^2=(\Delta n)_0^2\left[1+\frac{C}{A}\sinh^2(r)\right] = \frac{C}{B}\left[1+\frac{C}{A}\sinh^2(r)\right] , \tag{24}$$

where the subscript 0 refers to the situation with no injected squeezed light. In the limit of large squeezing, and for a laser not too far above threshold ($A \approx C$), the injection of squeezed light, therefore, increases the photon number variance by a factor of about $e^{2r}/4$. This is in agreement with our earlier results obtained from numerical integration of the corresponding Ito stochastic differential equations.

V. INTERPRETATION OF RESULTS

We present here a simple graphical picture (Fig. 8) of the behavior of this system along the lines of Schawlow-Townes arguments. Initially, $\phi_{LS}=0$, so the cavity field is coupled to fluctuations that have small phase noise. It is possible to think of laser phase noise as being in part due to amplified vacuum fluctuations and part due to spontaneous emission into the lasing mode. That part of the laser phase noise due to amplified vacuum fluctuations is "quenched" by the reduced phase fluctuations in the squeezed vacuum entering the cavity, and as a result, as long as the phases are right, the effect on the laser phase of a spontaneous emission "kick" is reduced. As a result, the laser phase slowly diffuses away from the optimum point $\phi_{LS}=0$, and as it does so, the probability of events that significantly alter the phase of the laser field is increased, as the projection of the error ellipse along the direction perpendicular to the field is increased. Essentially, the large amplitude noise becomes phase noise in the laser field. Thus the phase-diffusion rate begins to increase, and eventually $\phi_{LS}=\pi/2$. This is not a stable steady state as the phase-diffusion rate is very large compared to the nominal rate, and the laser phase is very quickly kicked away from this point. The laser phase does not "lock" to the injected squeezed field as it would to an injected signal of a well-defined phase. As it is kicked away, the phase-diffusion rate decreases again until eventually ϕ_{LS} is kicked significantly away from $\phi_{LS}=\pi/2$, perhaps even back to 0 or π. In the limit of larger r, the width of the noise ellipse goes to zero, but the probability for a spontaneous emission event perpendicular to the field can only be reduced to one half the unsqueezed value. This is due to the fact that we are only reducing the vacuum fluctuations that are injected into the cavity, but we are not able in this scheme to modify the other part of the spontaneous emission [10].

Essentially, as the laser phase explores the full 2π available to it, the intensity fluctuations are sometimes enhanced (by e^{2r}) and sometimes decreased (by e^{-2r}) relative to the laser with no squeezing, and likewise for the instantaneous phase-diffusion rate. Hence the time averaged expressions for the phase-diffusion rate and variance of the intensity are essentially enhanced by $\frac{1}{2}(1+\cosh 2r)$ over the value when no squeezed light is injected. For large squeezing, this results in an enhancement of $\approx(1/4)e^{2r}$.

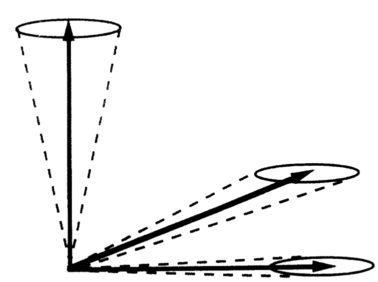

FIG. 8. Schematic of the geometrical vector kick model.

VI. VECTOR KICK MODEL

A. Physical description

The results of this paper can be obtained in a simple semiclassical pictorial analysis. This results from constructing a simple mathematical model that is based on the physics discussed in the preceding section. We represent the complex electric field in the cavity as $E=E_0e^{i\phi_{LS}}$ as a vector as shown in Fig. 9. We further use units such that the square of the field yields a photon number, i.e., the field is scaled by the electric field per photon. The length of the vector is the magnitude of the field E and the phase (ϕ_{LS}) of the field is the angle that the vector makes with the real axis. It is along this axis that the major axis of the error ellipse of the squeezed vacuum lies. The vector sits in a frame rotating at the nominal laser frequency so that ϕ_{LS} would not vary for a monochromatic field. The effect of the two noise sources, spontaneous emission into the lasing mode, and vacuum fluctuations leaking into the cavity, can be represented by vector addition as shown in the phasor diagram (Fig. 9). The field after one spontaneous emission kick E_S and one kick from the vacuum E_V is

$$\begin{aligned} E'e^{i\phi'_{LS}} &= E+E_S+E_V \\ &= E_0e^{i\phi_{LS}}+E_{S0}e^{i\theta_S} \\ &\quad +E_{V0}(\cosh r e^{i\theta_V}+\sinh r e^{-i\theta_V}) . \end{aligned} \tag{25}$$

These kicks will be taken to occur at the cavity photon loss rate (which is approximately equal to the linear gain rate close to threshold). Here the last term represents the injection of the squeezed field which is taken to have minimal noise along the direction $\phi_{LS}=\pi/2$. Our choice is based on the operator Bogoliobuv transformation that maps ordinary vacuum into squeezed vacuum. Further, the phases of the spontaneous field and the squeezed vacuum, θ_S and θ_V, respectively, are assumed random and independent of each other. The amplitudes of the noise contributions, or kicks, are taken to be

$$\frac{E_{S0}}{E_0}=\frac{E_{V0}}{E_0}=\frac{1}{\sqrt{2\bar{n}}}=\frac{1}{\sqrt{2I_0}} , \tag{26}$$

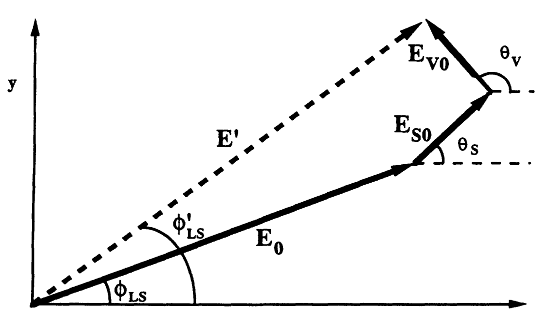

FIG. 9. Schematic of vector kick model of phase diffusion for the laser with injected squeezed vacuum.

with $\bar{n}=I_0$ being the average number of laser photons in the cavity in steady state. This choice is consistent with the interpretation that, with no squeezing, vacuum fluctuations and spontaneous emission each add one half of a photon per cavity lifetime. We will find that this choice is necessary for the results of the vector kick model to reproduce the results from the previous sections. This is also consistent with considering the strength of the noise due to the gain reservoir (spontaneous emission into the cavity mode) being equal to the noise due to the loss reservoir (amplified vacuum fluctuations) near threshold, where the gain is approximately equal to the loss. This is also consistent with the results of a calculation using a symmetrical ordering of operators, as opposed to normal or antinormal ordering [9]. For a good treatment of the effects of operator ordering on the interpretation, or cause of a quantum mechanical process, we point the reader to several references [10,11].

B. Phase diffusion

Multiplying Eq. (25) by $E_0e^{-i\phi_{LS}}$ and taking the imaginary part of the resulting expression leads to

$$\begin{aligned} \phi'_{LS}-\phi_{LS} \approx \frac{1}{\sqrt{2I_0}}[&\sin(\theta_S-\phi_{LS})+\cosh r\sin(\theta_V-\phi_{LS}) \\ &-\sinh r\sin(\theta_V+\phi_{LS})] , \end{aligned} \tag{27}$$

where the fact that each phase kick is small has been used to set $\sin(\phi'_{LS}-\phi_{LS})\approx\phi'_{LS}-\phi_{LS}$. Since the phases of the added fields are random, the average value of the phase kick is zero. Consider now the fluctuations, which result in a nonzero average value of the square of the phase kick. Squaring Eq. (27) and performing an average gives,

$$\begin{aligned} \langle(\phi'_{LS}-\phi_{LS})^2\rangle = \frac{1}{4I_0}[&1+\cosh^2 r+\sinh^2 r \\ &-2\cosh r\sinh r\cos(2\phi_{LS})] . \end{aligned} \tag{28}$$

Here, we have set $I'_0=I_0=E_0^2\equiv\bar{n}$ since the steady-state intensity is maintained at, approximately, a constant value by gain saturation. The spontaneous kicks occur at the gain rate $A\approx C$ (near threshold) and the vacuum kicks occur at the loss rate C, the rate at which the vacuum fluctuations leak in. Thus, we have

$$\frac{d}{dt}\langle(\phi'_{LS}-\phi_{LS})^2\rangle = D , \tag{29}$$

with

$$D=\frac{C}{4I_0}[1+\cosh^2 r+\sinh^2 r-2\cosh r\sinh r\cos(2\phi_{LS})] \tag{30}$$

Note that for the laser phase originally along the direction of minimum phase fluctuations, (i.e., $\phi_{LS}=0$),

$$\begin{aligned} D &= \frac{C}{4I_0}+\frac{C}{4I_0}(\cosh r-\sinh r)^2 \\ &= \frac{D_0}{2}(1+e^{-2r}) . \end{aligned} \tag{31}$$

This agrees with Eq. (13) obtained by more rigorous means. An average over a time long enough so that the laser field wanders through all phases (as indicated by our numerical results) leads to an enhanced phase-diffusion rate of

$$D = D_0 \cosh^2 r \approx \frac{C}{8I_0} e^{2r} \tag{32}$$

for large r.

C. Intensity fluctuations

A similar procedure can be used to obtain the uncertainty in intensity, starting with Eq. (25) and squaring both sides to obtain

$$\begin{aligned} I' - I_0 = [&1 + \sinh^2 r + \cosh r \cos(\theta_S - \theta_V) \\ &+ \sinh R \cos(\theta_S + \theta_v) + \cosh r \sinh r \cos(2\theta_V)] \\ &+ 2\sqrt{I_0/2}[\cos(\phi_{LS} - \theta_S) + \cosh r \cos(\phi_{LS} - \theta_V) \\ &+ \sinh r \cosh(\phi_{LS} + \theta_V)] \ . \end{aligned} \tag{33}$$

The phases of the noise fields are random so the terms involving these phases, (i.e., the interference terms) average to zero. The fields add incoherently to the semiclassical cavity field. That is, the pair of noise events adds the amount,

$$\delta I \equiv \langle I' - I_0 \rangle = 1 + \sinh^2 r \ . \tag{34}$$

Due to gain saturation and the assumption that the number of photons in the squeezed field ($\sinh^2 r$) is much less than I_0, this noise does not appreciably change the average intensity of the laser field. However, in agreement with the results of the main body of this paper, the introduction of the squeezed vacuum does drastically effect the variance in intensity. The fluctuations in intensity can be calculated using Eq. (33) as,

$$\begin{aligned} \delta(I^2) &\equiv \langle (I' - I_0)^2 \rangle \\ &= 4I_0 + 4I_0 \sinh^2 r + 2I_0 \cosh r \sinh r \cos(2\phi_{LS}) \ , \end{aligned} \tag{35}$$

where we have assumed that $I_0 \gg \cosh^2 r$.

We now show that adding this noise term to the usual classical equations of motion for the photon number in a laser field leads to the results presented earlier in this paper. The equation of motion for the expectation value of the photon number is taken to be

$$\begin{aligned} \langle \dot{I} \rangle &= \langle (A - C - BI)I \rangle + C\delta I \\ &= (A - C)I_0 - BI_0^2 - B\Delta(I^2) + C\delta I \ . \end{aligned} \tag{36}$$

Here we have written $I = I_0 + \Delta I$. The last term represents the noise added due to the two noise sources. The equation of motion for $d(I^2)/dt$ is then

$$\begin{aligned} \left\langle \frac{d(I^2)}{dt} \right\rangle &= 2\langle (A - C - BI)I^2 \rangle + C\delta(I^2) \\ &= 2(A - C)(I_0^2 + \Delta I^2) - 2B\langle I^3 \rangle + C\delta(I^2) \ . \end{aligned} \tag{37}$$

Here again, the first terms come from the deterministic change in I due to gain saturation, and the last term represents the change in I due to the two noise sources. To proceed, we use the approximation that $\langle I^3 \rangle \approx I^3 + 3BI\Delta I^2$. Using this, and solving Eqs. (36) and (37) in steady state gives

$$I \approx \frac{A - C}{B} \tag{38}$$

and

$$\begin{aligned} \Delta I^2 &= \frac{C}{2B}[\delta(I^2) - 2I_0 \delta I] \\ &= \frac{C}{B}[1 + \sinh^2 r + \cosh r \sinh r \cos(2\phi_{LS})] \ . \end{aligned} \tag{39}$$

Note that this shows the expected behavior of enhanced number fluctuations at $\phi_{LS} = 0$ and π, and reduced fluctuations at $\phi_{LS} = \pi/2$ and $3\pi/2$. Note that this is not a true steady-state value but rather is an average taken over many cavity lifetimes but before the phase has wandered significantly from ϕ_{LS}. The steady-state value is the average of Eq. (39) over equally weighted phases. Again, our rationale for this is based on numerical results that indicate that the instantaneous laser phase is uniformly distributed over 2π. This gives

$$\begin{aligned} \Delta I^2 &= \frac{C}{B}\left[1 + \sinh^2 r\right] \\ &= (\Delta I^2)_0 \left[1 + \sinh^2 r\right] , \end{aligned} \tag{40}$$

in agreement with Eq. (23). Here we note that if we took the laser phase to be locked at $\phi_{LS} = \pi/2$ or $3\pi/2$, we would obtain the results of Ref. [6], where exactly that assumption was made. In the equations of Ref. [6], if one averages over all possible values of the laser phase, one obtains the result presented in Eq. (40).

D. Dynamical evolution of the laser phase

We now discuss the locking of the laser, or more appropriately the lack of locking in this system. As noted earlier, the phase does not lock to $\phi_{LS} = \pi/2$ or $3\pi/2$ as discussed in earlier treatments of this problem [3,6]. Rather there is a tendency for the phase to drift towards these values but in fact the phase wanders through all values. In steady state, the field is equally likely to be found at any phase. This is not uniform phase diffusion as in the usual laser, as we have seen from our previous results, but occurs in a more complicated fashion. The field does tend to be driven towards $\phi_{LS} = \pi/2$ or $3\pi/2$ more often than towards $\phi_{LS} = 0$ and π, but near $\phi_{LS} = 0$ and π the phase tends to stick because of the smaller phase diffusion in that vicinity. That is, the phase does tend to oscillate about $\phi_{LS} = \pi/2$ or $3\pi/2$ with occasional long-lived forays into the regions surrounding $\phi_{LS} = 0$ and π. In Fig. 10, we present the time evolution of the laser phase for representative individual trajectories from the numerical simulation of the stochastic differential equations. All trajectories viewed show similar behavior. The tendency to be driven towards the "poles" can be

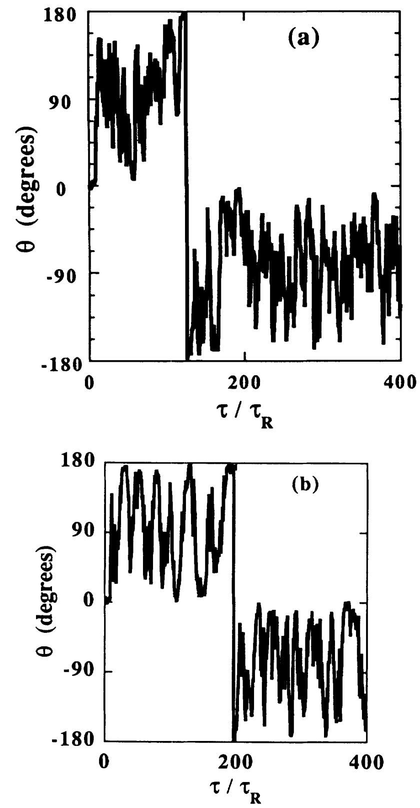

FIG. 10. Output of simulation of stochastic differential equations, showing the temporal evolution of two representative trajectories. Here $a = 100$, $b = 0.95$, and $r = 3$.

seen from the simple geometrical picture. The laser field, at a given time, is equally likely to receive a kick which tends to increase or decrease its phase. However, the size of the resulting phase change due to a single squeezed noise event is not symmetric as can be seen from the figure below. For simplicity, consider the field represented in Fig. 11, where $\phi_{LS} = \pi/4$. Noise events along the long sides of the squeezed noise ellipse contribute to a larger phase change for the event which drives the field towards $\phi_{LS} = \pi/2$ than the one which drives the field towards $\phi_{LS} = 0$. The mathematical reflection of this effect can be seen in Eq. (27), since E' is less, and hence $\phi'_{LS} - \phi_{LS}$ more, for the case in which the field is driven towards $= \pi/2$.

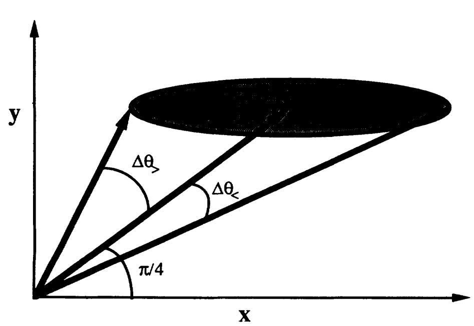

FIG. 11. Vector kick model for $\phi_{LS} = \pi/4$. Here $\Delta\theta_{>}$ is the phase kick associated with spontaneous emission events that would increase the phase noise, and $\Delta\theta_{<}$ is the phase kick associated with spontaneous emission events that would decrease the phase noise.

E. Origin of time scales

In this section, we wish to discuss the origin of the two time scales in the problem. These two times are (i) the time it takes for the laser phase to diffuse out of the quiet region (the relaxation time τ_R we have referred to) and (ii) the time it takes for the laser phase to explore a significant portion of 2π, denoted by τ_{ss}. For times short compared to (i), the laser phase-diffusion rate is approximately constant, and at times long compared to (ii), the laser phase-diffusion rate becomes (for the ensemble average) the steady-state value $\tilde{D}_{ss}$, which is the average over all ϕ_{LS}. Here we present order of magnitude calculations for these two times.

To calculate (i), we begin with the instantaneous phase-diffusion rate at some angle ϕ_{LS},

$$D = \frac{D_0}{2}(1 + \sinh^2 r - \sinh r \cosh r \cos 2\phi_{LS}) \; . \tag{41}$$

We now calculate the angle at which this rate equals the unsqueezed rate of D_0. The result is

$$\begin{aligned} \cos\phi_{LS} &= (\sinh^2 r - 1)/\sinh r + \cosh r) \\ &\approx 1 - \frac{e^{-2r}}{4} \; , \end{aligned} \tag{42}$$

where we have taken the large squeezing limit. Assuming this is a small angle, we can expand $\cos\phi_{LS}$ as $1 - \phi_{LS}^2/2$, and find that

$$\phi_{LS}^2 = \frac{e^{-2r}}{2} \; . \tag{43}$$

Now we calculate the time it takes for the laser phase to diffuse to this value, and use the quiet diffusion rate $(D_0/2)$ to obtain an order of magnitude estimate,

$$\phi_{LS}^2 = 2\left[\frac{D_0}{2}\right]\tau_R = \frac{e^{-2r}}{2} \tag{44}$$

and hence find that the time it takes for the laser phase to wander into the noisy regime, our relaxation time, is approximately

$$\tau_R = \frac{e^{-2r}}{2D_0} \; . \tag{45}$$

This is of course just an order of magnitude estimate, as the laser phase diffuses at a slightly larger rate during the evolution to this point. However, using the nominal phase-diffusion rate of D_0 during this evolution results in a time one half of the value obtained above, and this sets the upper and lower limits on this time.

We now ask about the time that it takes for the laser phase to explore a significant portion of the 2π available to it. If one takes the "significant portion" to be $\pi/2$, then the approximate time to reach steady state is on the order of

$$\tau_{ss}=\frac{\pi^2}{2D_0}e^{-2r}=\pi^2\tau_R \ . \tag{46}$$

Here we see that the time it takes the laser phase to have explored both the noisy and quiet regions of parameter space is about 10 times longer than the time it takes for the laser phase to leave the quiet region, the relaxation time. This order of magnitude estimate is borne out by the numerical results presented in Figs. 5 and 6.

In this model, the noise in the laser is ascribed in equal amounts to effects of vacuum fluctuations leaking into the cavity, and to spontaneous emission into the cavity mode. This decomposition of the effects of the two noise sources corresponds to the result obtained using a symmetrical operator ordering [9]. Dalibard, Dupont-Roc, and Cohen-Tannoudji [11] have argued that to ascribe cause in quantum mechanics, one must use a symmetrical ordering of operators in calculations. Of course, the total field operator and atomic operators commute at equal times, and the choice of the ordering does not matter in the final answer. But when the field is considered as the sum of more than one piece, for example a sum of source and free fields, the individual components of the field do not commute with atomic operators, and hence one can proceed using different ordering schemes. The argument of Dalibard, Dupont-Roc, and Cohen-Tannoudji [11], as we understand it, is that using symmetrical ordering results automatically in a Hermitian operator for each of the terms in the interaction Hamiltonian, and one could in principle make measurements as to the result of those terms. Hence, one may ascribe a definite weight to the various terms, as in the case of spontaneous emission, where the result is that equal weight is given to vacuum fluctuations and radiation reaction. In our case, such a calculation would indicate that the laser phase noise would be ascribed in equal measure to vacuum fluctuations and spontaneous emission into the cavity mode. Hence our vector kick model is consistent with the interpretation that the laser linewidth stems in equal measure from vacuum fluctuations leaking into the cavity and spontaneous emission into the lasing mode. These are independent noise sources, although we note that spontaneous emission into the lasing mode is due in part to vacuum fluctuations in nonlasing modes that couple to the atom.

VII. CONCLUSIONS

We have investigated the behavior of a laser with squeezed vacuum injected into the output mirror. We find that the phase-diffusion rate of the laser is not constant, and hence the laser linewidth is not given as simply as in the usual Schawlow-Townes argument. For a choice of relative laser-squeezing phase $\phi_{LS}=0$, the phase-diffusion rate is reduced to one half of the nominal value given by the Schawlow-Townes formula for a time short compared to the locking time, $\tau_R=(a/2b)e^{-2r}\approx(a/2)e^{-2r}$. At longer times, the phase-diffusion rate equals the steady-state value $D_{ss}\approx(1/4)e^{2r}D_0$, independent of the initial phase of the laser, a value much larger than the Schawlow-Townes value. As the linewidth depends essentially on the integral of the phase-diffusion rate for all times, it seems obvious that the linewidth of the laser will indeed be well above the Schawlow-Townes value, in agreement with earlier predictions. However, the phase noise of a laser can be reduced transiently by a factor of two, as indicated above.

Is this a useful effect? The relaxation time decreases as the amount of squeezing is increased, and so the operational question becomes how long do you need quiet phase light. As an example of how this effect may be useful, consider the following. A He-Ne laser of length 0.5 m, 10-mW output power, and with an output reflector of 95% has a linewidth of about 0.01 Hz. If squeezed vacuum with $r=1$ is injected into this laser, the relaxation time of this device is 13.5 sec. This time is large enough so that this transient effect could be used for a host of spectroscopic and interferometric applications. Of course, we must realize that here we discuss the ultimate quantum limit, which is generally swamped by effects such as Doppler and collisional broadening, or thermal fluctuations in cavity length to name just a few. But we have shown that the quantum limited phase noise of a laser can be reduced significantly for useful periods of time.

We have also examined intensity fluctuations in this system, and found that when the phase of the injected squeezing is chosen so that the laser field initially sees reduced phase noise, the intensity noise that the laser is coupled to is necessarily increased. At longer times when the laser sees increased phase noise, one might intuitively expect that the intensity fluctuations decrease, but they do not as the laser field is not in a minimum-uncertainty state. This qualitative behavior was predicted earlier by Gea-Banacloche and Pedrotti [6], and the results of this work differ by a factor of 2 from the results they obtained using a truncation of moment equations and an assumption about the laser locking at $\phi_{LS}=\pi/2$. Upon relaxation of this latter assumption, their results agree with ours. As with the phase-diffusion rate, the value of the intensity fluctuations in the steady state is independent of the initial choice of the laser phase.

The dynamics of the laser phase evolution has been discussed, and we find that the laser phase does not lock in the usual sense, but that the phase is alternatively pulled towards $\phi_{LS}=0$ and $\pi/2$, and then kicked away. The net result is that the laser phase is uniformly distributed over 2π, but that this results in a more complicated manner than the usual phase diffusion of a laser with no squeezing.

A simple geometrical, vector kick model has been constructed that agrees with the results of the fully quantum mechanical calculations. In this model, one ascribes equal weight to two independent noise sources, spontaneous emission into the lasing mode, and vacuum fluctuations coupled to the cavity mode. This is the result one obtains in a calculation using a symmetrical ordering of operators, after the field has been decomposed into vacuum and source contributions, and our result is consistent with the arguments of Dalibard, Dupont-Roc, and Cohen-Tannoudji [11] on the relation between operator ordering and interpretation in quantum mechanics.

ACKNOWLEDGMENTS

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[1] M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), pp. 281–289.

[2] C. M. Caves, Opt. Lett. **12**, 971 (1987); M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1987).

[3] M. A. Marte, H. Ritsch, and D. F. Walls, Phys. Rev. A **38**, 3577 (1988).

[4] J. Gea-Banacloche, Phys. Rev. Lett. **59**, 543 (1987).

[5] Ch. Ginzel, R. Schack, and A. Schenzle, J. Opt. Soc. Am B **8**, 1704 (1991); Ch. Ginzel, J. Gea-Banacloche, and A. Schenzle, Acta Phys. Pol. A **78** (1990); Ch. Ginzel, J. Gea-Banacloche, and A. Schenzle, Phys. Rev. A **42**, 4164 (1990).

[6] J. Gea-Banacloche and L. M. Pedrotti (unpublished).

[7] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).

[8] C. W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).

[9] M. O. Scully and S. Stenholm, Phys. Scr. **T21**, 119 (1988).

[10] On the subject of interpretations of the source of spontaneous emission, see the review by P. W. Milonni, Phys. Rep. **25**, 1 (1976).

[11] We find especially persuasive the arguments of J. Dalibard, J. Dupont-Roc, and C. Cohen-Tannoudji, J. Phys. (Paris) **43**, 1617 (1982), which suggest that one-half of the spontaneous emission rate comes from vacuum fluctuations and one-half from radiation reaction.

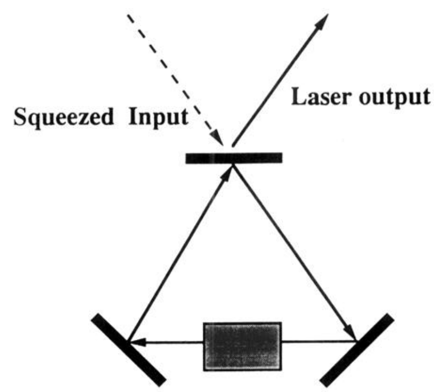


FIG. 1. Schematic of the physical system under consideration, a single-model ring laser with squeezed vacuum injected into the output port.

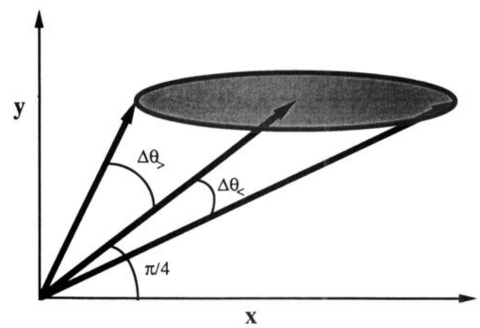
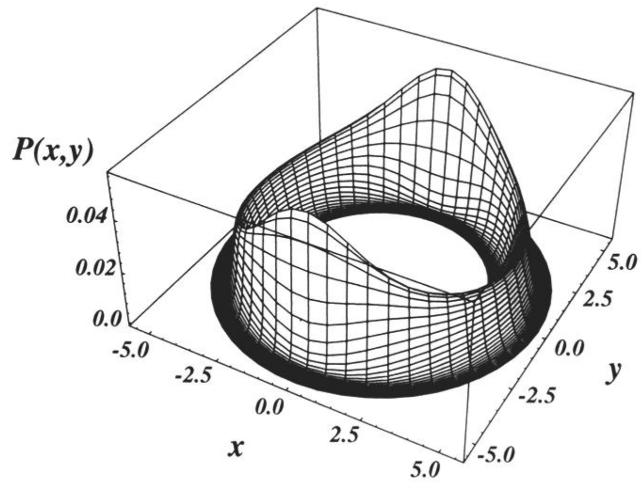
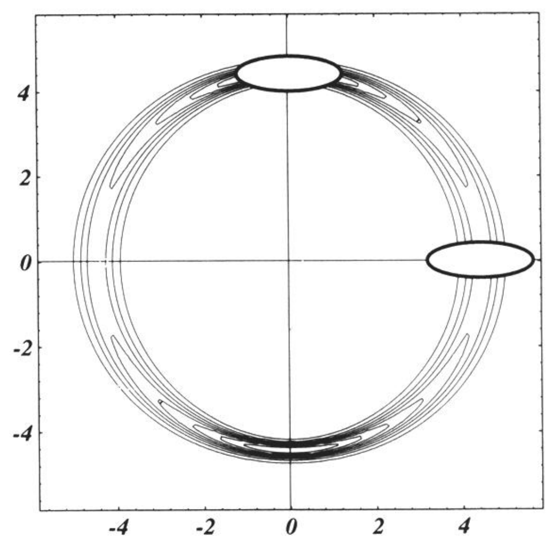


FIG. 11. Vector kick model for $\phi_{LS}=\pi/4$. Here $\Delta\theta_>$ is the phase kick associated with spontaneous emission events that would increase the phase noise, and $\Delta\theta_<$ is the phase kick associated with spontaneous emission events that would decrease the phase noise.



(a)



(b)

FIG. 7. Plot of the steady-state P distribution, calculated numerically using the method discussed in Sec. IV, (a) three-dimensional view, (b) a contour plot.