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Nonperturbative studies of a quantum higher-order nonlinear Schrödinger model using the Bethe ansatz

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We consider the integrability problem for a quantum version of the perturbed nonlinear Schrödinger (NS) equation, including a higher spatial dispersion and nonlinear dispersion of the group velocity (the corresponding classical equations are well known in the nonlinear fiber optics and in other applications). Employing the Bethe ansatz (BA) technique, which is known to yield a complete spectrum including the so-called quantum solitons (multiparticle bound states) of the unperturbed NS system, we find that the particular cases of the model which correspond to integrable classical equations, viz., the derivative NS and Hirota equations, are also fully integrable at the quantum level. In the generic (nonintegrable) case, the model remains integrable in the two-particle sector. In the three-particle sector, the BA produces unphysical states with complex energy. It turns out that the Hamiltonian in this case becomes non-Hermitian. We propose a procedure for finding the physical eigenstates of the system. We build an example of such a state and we show that it describes inelastic scattering.

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I. INTRODUCTION

The nonlinear Schrödinger (NS) equation is a general description of the propagation of small-amplitude envelope waves in weakly dispersive, nonlinear media. It plays an important role in a number of physical applications [1]. One very interesting example (which will be referred to here) is the propagation of solitons in nonlinear optical fibers [2] (see the Appendix). Recently a significant interest in the quantum version of the NS equation was invoked by the experiments, in which quantum properties of the optical solitons in fibers were observed [3]. The number of photons bound in the fiber soliton is typically $10^6 - 10^8$ and such an object seems to be a classical one; however, the quantum features manifest themselves as the quantum fluctuations. In addition one can observe interesting effects, as squeezing, due to the nonlinearity of the medium [3]. In the opposite limit, few-photon bound states (solitons) were recently discussed in [4] and new experiments were proposed in order to observe these states using the nonlinear Fabry-Pérot resonator. The main part of this resonator is a quasi-one-dimensional nonlinear optical cavity. As it was shown, the electro-magnetic field propagation in this cavity is also governed by the quantum NS equation [4]. The photons produced by laser source penetrate the cavity and form there two-particle bound states—diphotons—which are then detected.

A fundamental property of the NS equation is its exact integrability both in the quantum [5] and in the classical versions [6]. There are various approaches to studying quantum integrable systems. One of the simplest and most physically clear ones is based on the Bethe ansatz (BA) and we will follow this technique. The BA method was employed in the early works [5,7] to solve the quantum NS equation. The BA yields wave functions of the bound multiparticle states (quantum solitons) which are constituents of the spectrum of the exactly integrable quantum NS system. Note that the physical solitons, for example, in fibers, are usually defined as wave packets (coherent states) composed of the Bethe eigenstates [8].

The NS equation that appears in physical applications usually contains additional terms that destroy the exact integrability. Such terms appear as generic high-order corrections, when the unperturbed NS equation is the lowest nontrivial order of expansion in powers of small amplitude and small inverse wavelength. Regarding the unperturbed equation as the zeroth-order approximation, one can use a perturbation theory to handle the additional terms. The perturbation theory has been systematically applied to the classical NS equation (see, e.g., the

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review paper [9]). In the quantum case, the perturbation theory may be applied in the semi-classical (WKB) approximation for solitons containing a large number of quanta. For the so-called breathers, i.e., solitons governed by the sine-Gordon equation, a systematic WKB perturbation theory has been employed in Ref. [10]. The NS solitons may be regarded as a small-amplitude limit of the sine-Gordon breathers, so the results obtained in Ref. [10] may be directly applied to them.

The objective of this work is to develop the analysis of the perturbed quantum NS system by means of the BA technique. This technique is not itself perturbative; therefore the theoretical analysis of the quantum fiber solitons developed thus far [8] in terms of the BA technique has been restricted to the pure NS equation. We will show that, nevertheless, this technique is applicable to the perturbed version of the NS model and, even in the nonintegrable cases, it produces results which can be useful for further analysis and which provide a deeper insight into the general problem of what is integrability vs nonintegrability in quantum systems. In this work we try to proceed as far as possible without recourse to the perturbation theory.

One of the physical motivations of our studies is answering the question whether the quantum few-particle solitons survive in a system with higher dispersions. Let us consider, for example, the experiment proposed in [4]. It is natural to expect that the nonlinear cavity of the Fabry-Pérot resonator in this experiment has appreciable coefficients of higher dispersions. Does it mean that the detection of the two-particle bound states is impossible? We conclude that two-particle solitons still exist and thus one may hope to observe them. Nevertheless, when the third photon is added to the cavity, it can destroy the soliton. This means that the photon source should be tuned so that no more than two photons are in the cavity at the same time. As for the nonlinear fibers, our few-photon results are not directly applicable in this case (because of the great number of photons in the fiber solitons), but they explain qualitatively the phenomena of nontrivial scattering of the solitons in fibers.

As a fairly fundamental perturbed model, we will take the higher-order NS equation well known in the theory of optical fibers [11]:

$$i\Psi_t = -\Psi_{xx} + 2i\epsilon_1\Psi_{xxx} + 2c\Psi^*\Psi^2 + 4i\epsilon_2\Psi^*\Psi\Psi_x. \quad (1.1)$$

In Eq. (1.1) Ψ is a complex envelope of the dispersive waves, t and x stand for temporal-like and spatial-like variables, the coefficient c measures nonlinearity of the medium, and the real perturbation parameters ϵ_1 and ϵ_2 account for the higher linear dispersion and for the nonlinear dispersion of the group velocity, respectively.

This paper is organized as follows. In Sec. II we summarize some properties of the classical perturbed NS equation (1.1) which are important for comparison with the results obtained in the quantum case. In Sec. III we apply the BA technique to searching for exact multiparticle eigenstates (quantum solitons) of the Hamiltonian corresponding to the quantum model (1.1). We demonstrate that in the known particular cases, in which the classical perturbed NS equation (1.1) remains exactly integrable, the quantum version is integrable as well. Then, in Sec. IV , which plays a central role in the work, we consider models which are nonintegrable in the classical limit [this is the generic case since the integrability survives only for few particular values of the ratio ϵ_2/ϵ_1 in Eq. (1.1)]. We concentrate on the simplest and most physically meaningful example of the nonintegrable model, namely, Eq. (1.1) with $\epsilon_2 = 0$:

$$i\Psi_t = -\Psi_{xx} + 2i\epsilon_1\Psi_{xxx} + 2c\Psi^*\Psi^2. \quad (1.2)$$

It seems plausible that the general inferences obtained for the model (1.2) remain true for other nonintegrable cases.

It is known that Eq. (1.2) does not have classical soliton solutions. If one takes an initial configuration in a form close to the unperturbed NS soliton, it will decay into radiation exponentially slowly in time [12]. The nonintegrability of Eq. (1.2) at the classical level demonstrates itself also when one analyzes a collision between the quasisoliton pulses: unlike what is well known for the integrable systems, in this case the collision is inelastic, giving rise to emission of radiation [13]. In Sec. IV we demonstrate that the BA equations for the quantum model (1.2) admit an explicit solution for the simplest two-particle bound states. So, at this level there is no difference from the integrable case. However, when trying to solve the BA equations for the three-particle solitons, we find that the BA produces unphysical eigenstates corresponding to complex energies. We show that in this case, while the Hamiltonian becomes non-Hermitian over the entire domain of its definition, it remains Hermitian on a certain subdomain. The BA states do not belong to this subdomain; hence they are unphysical. We propose a procedure by which physical states can be constructed using combinations of the BA states. These physical states describe nontrivial inelastic scattering processes. In turn, the inelastic scattering is the most generic property of the nonintegrable classical systems [9]. We expect this to be a generic picture of the onset of the perturbation-induced nonintegrability in quantum solitonic systems.

In Sec. V we discuss, in some detail, the approach to the classical limit in the quantized model (1.1). We demonstrate that in the particular cases in which the model remains exactly integrable, it goes over into a classical integrable higher NS equation. However, in the nonintegrable case we encounter the same problem which has been mentioned before, that is, the appearance of the formal eigenstates with complex energies.

II. SOME PROPERTIES OF THE CLASSICAL HIGHER NS EQUATIONS

It is well known that the higher NS equation (1.1) is exactly integrable in two particular cases: (i) the derivative NS (DNS) equation with $c = \epsilon_1 = 0$ and (ii) the Hirota equation [14] with $\epsilon_2 = -3\epsilon_1 c$. In case (i) the DNS equation has the exact one-soliton solution in the form [15–17]

$$\Psi_{sol}(x,t) = \frac{4\Delta \sin(\epsilon_2 N) \exp(2\Theta - 2i\sigma - 2i\epsilon_2 N)}{\exp(4\Theta) + \exp(i\epsilon_2 N)}, \quad (2.1)$$

where

$$\Delta^2 = \frac{|\epsilon_2| P}{2\sin(\epsilon_2 N)} \quad (2.2)$$

and

$$\begin{aligned} \Theta &= \mathrm{sgn}(\epsilon_2)\Delta^2 \sin(\epsilon_2 N)(x - x_0), \\ \sigma &= \mathrm{sgn}(\epsilon_2)\Delta^2 \cos(\epsilon_2 N)x + \sigma_0, \\ \frac{dx_0}{dt} &= 4\mathrm{sgn}(\epsilon_2)\Delta^2 \cos(\epsilon_2 N), \\ \frac{d\sigma_0}{dt} &= 2\Delta^4 \cos(2\epsilon_2 N). \end{aligned} \quad (2.3)$$

Here N and P are the soliton's "number of quanta" (wave action) and momentum, respectively, which are given by

$$N = \int_{-\infty}^{+\infty} dx \Psi^* \Psi, \quad (2.4)$$

$$P = -i \int_{-\infty}^{+\infty} dx \Psi^* \Psi_x. \quad (2.5)$$

Being nonrelativistic, the soliton's energy E is related to P and N as follows:

$$E \equiv \int_{-\infty}^{+\infty} dx[\Psi_x^* \Psi_x + (i\epsilon_2 \Psi^{*2} \Psi \Psi_x + \text{c.c.})] = \frac{P^2}{2m}, \quad (2.6)$$

$$m \equiv \frac{\tan(\epsilon_2 N)}{2\epsilon_2}. \quad (2.7)$$

Note that the effective mass m depends on $\epsilon_2 N$ and can change its sign. As N is positive, it follows from Eq. (2.2) that, for $\mathrm{sgn}(\epsilon_2) = \pm 1$, the soliton's momentum is, respectively, either always positive or always negative. At the same time, the soliton's velocity

$$v \equiv \frac{\partial E}{\partial P} = \frac{P}{m} = \frac{dx_0}{dt} \quad (2.8)$$

changes sign together with the mass m at $\epsilon_2 N = \frac{\pi}{2}$. As it follows from Eq. (2.1), the continuous family of the soliton solutions exists in the interval

$$|\epsilon_2 N| < \pi. \quad (2.9)$$

In the following section we will demonstrate that the DNS equation remains exactly solvable at the quantum level and the classical solution described by Eqs. (2.1)–(2.6) may be regarded as the classical (to be specified below) limit of the exact quantum solitons.

III. THE QUANTUM MODEL AND THE BETHE ANSATZ

To quantize the model based on Eq. (1.1), we introduce its Hamiltonian, following the Wick quantization procedure:

$$\hat{H} = \int_{-\infty}^{+\infty} dx[\hat{\Psi}_x^\dagger \hat{\Psi}_x + (i\epsilon_1 \hat{\Psi}_{xx}^\dagger \hat{\Psi}_x + \text{H.c.}) + c\hat{\Psi}^{\dagger 2}\hat{\Psi}^2 + (i\epsilon_2 \hat{\Psi}^{\dagger 2}\hat{\Psi}\hat{\Psi}_x + \text{H.c.})]. \quad (3.1)$$

Here $\hat{\Psi}(x,t)$ is considered as the field operator obeying the following equal-time commutation relations:

$$[\hat{\Psi}(x), \hat{\Psi}(y)] = [\hat{\Psi}^\dagger(x), \hat{\Psi}^\dagger(y)] = 0,$$
$$[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] = \hbar\delta(x-y).$$

The physical meaning of quantization of the envelope field is discussed in the Appendix.

It is well known [5,7,18] that, introducing the annihilation and creation operators

$$\hat{b}(x) \equiv \hbar^{-1/2}\hat{\Psi}(x), \quad \hat{b}^\dagger(x) \equiv \hbar^{-1/2}\hat{\Psi}^\dagger(x), \quad (3.2)$$

one may regard the Hamiltonian (3.1) as describing a gas of nonrelativistic bosons with the purely pairwise local interaction, a classical analog of which is the one-dimensional gas of hard particles interacting only when they collide.

The perturbed classical NS equation (1.1) conserves the "number of particles" and momentum given by Eqs. (2.4) and (2.5), as well as the classical Hamiltonian. In accordance with this, the quantum Hamiltonian (3.1) commutes with the operator of the number of particles

$$\hat{N} = \int_{-\infty}^{+\infty} dx \hat{\Psi}^\dagger \hat{\Psi}. \quad (3.3)$$

Thus, following the general idea underlying the BA technique, one may separate the full Fock space $|F\rangle$ of the quantum system into disjoint N-particle sectors $|F_N\rangle$. Then we introduce, as is usually done [18,19], the N-particle wave functions (matrix elements) according to the definition

$$f_N(x_1,...,x_N) \equiv \frac{1}{\sqrt{n!}}\langle 0|\hat{b}(x_1),...,\hat{b}(x_N)|F_N\rangle, \quad (3.4)$$

where $\hat{b}(x_n)$ are the annihilation operators defined by Eq. (3.2) and x_n are particles' coordinates. The corresponding element of the Fock space $|F_N\rangle$ may be expressed as

$$|F_N\rangle = \frac{1}{\sqrt{n!}} \int dx^N f_N(x_1,...,x_N)\hat{b}(x_1)^\dagger,...,\hat{b}(x_N)^\dagger|0\rangle \quad (3.5)$$

and the usual (linear) Schrödinger equation for the evolution of the N-particle states is given by

$$i\hbar\frac{d}{dt}|F_N\rangle = \hat{H}|F_N\rangle. \quad (3.6)$$

The projection of the second-quantization Hamiltonian $\hat{H}$ on the N-particle sector is given by

$$i\hbar\frac{\partial f_N}{\partial t} = H_N f_N, \quad (3.7)$$

$$H_N \equiv -\hbar\Delta_N + 2\hbar i\epsilon_1 \sum_{l=1}^{N} \frac{\partial^3}{\partial x_l^3} + 2\hbar^2 \sum_{l<m} \delta(x_l - x_m) \left[c + i\epsilon_2 \left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m} \right) \right]. \tag{3.8}$$

Equation (3.7) with the effective Hamiltonian (3.8) is the quantum Schrödinger equation for N particles interacting through the modified pairwise δ-like potential. When the perturbing term proportional to ϵ_2 is included, the effective potential acquires an additional term proportional to a derivative of the δ function. In terms of the gas of classical particles, this means that a collision gives rise to additional shifts of the colliding particles depending upon the particles' momenta. The perturbing term proportional to ϵ_1 changes the kinetic energy of the particles.

The Schrödinger equations of this type were investigated in a general form by Gutkin [20] using the BA method. The essential point of the BA technique is that in the regions $x_l \neq x_m$ the interaction potential is zero and the wave function is simply the sum of the free-particle wave functions. The problem is to match the solutions in the different regions (sectors). If such a matching is possible in all the configuration space, the system is solvable by the BA technique. It has been demonstrated by Gutkin that in the general case only the symmetric (antisymmetric) form of the BA is valid for equations such as (3.7). The solutions with another kind of symmetry cannot be found by means of the BA. As we work with bosons, these circumstances do not lead to additional difficulties. It turns out that all the necessary coefficients for the BA solution may be obtained from the two-particle problem. Hence, following [20], we will, first of all, consider the two-particle states ($N = 2$). At $x_1 < x_2$, Eq. (3.7) has constant coefficients and in this region we seek a solution of the form

$$f_2(x_1, x_2) = \exp[i(\lambda x_1 + \mu x_2)]. \tag{3.9}$$

In the region $x_1 > x_2$, the same solution should have the representation

$$f_2(x_1, x_2) = A(\lambda, \mu) \exp[i(\lambda x_1 + \mu x_2)] + B(\lambda, \mu) \exp[i(\mu x_1 + \lambda x_2)]. \tag{3.10}$$

The *matching coefficients* $A(\lambda, \mu)$ and $B(\lambda, \mu)$ play a key role in the BA technique. Substituting Eqs. (3.9) and (3.10) into Eq. (3.7), demanding continuity of the wave function $f(x_1, x_2)$ at $x_1 = x_2$, and integrating the equation over an infinitesimal vicinity of this point, one can readily find

$$A(\lambda, \mu) = \frac{(\lambda - \mu) + 3\epsilon_1(\lambda - \mu)(\lambda + \mu) - i\hbar[c - \epsilon_2(\lambda + \mu)]}{(\lambda - \mu) + 3\epsilon_1(\lambda - \mu)(\lambda + \mu)}, \tag{3.11}$$

$$B = 1 - A. \tag{3.12}$$

Having found the two-particle matching coefficients (3.11) and (3.12), one can construct the N-particle symmetric BA as follows:

$$f_\Lambda|(x_1 \le x_2 \le \cdots \le x_N) = \sum_\omega \left(\prod_{l<m} \frac{A(k_{\omega(m)}, k_{\omega(l)})}{A(k_m, k_l)} \right) \times \exp(i[k_{\omega(1)} x_1 + \cdots + k_{\omega(N)} x_N]) = \sum_\omega C_\omega \exp(\cdots). \tag{3.13}$$

In Eq. (3.13) Λ stands for the set of the wave numbers k_n (it is assumed that $k_l \neq k_m$ for $l \neq m$) and $\sum_\omega$ implies summing over all permutations of the numbers $(1, ..., N)$.

Note that the wave function (3.13) can be viewed as an exact scattering state of N free particles, which is a delocalized wave function. At the same time, by their nature, quantum solitons should be described by localized wave functions. The crucial step that makes it possible to use the BA for the quantum solitons is the fact that we can analytically continue the wave function of free particles to complex values of the wave numbers, so that the wave function be localized [7]. To do so we denote $\mathrm{Im}(k_n) \equiv q_n$ for the complex wave numbers. The conditions for boundedness of the analytically continued wave function (3.13) in the domain $x_1 < x_2 < \cdots < x_N$ are

$$q_{\omega(1)} \le 0, q_{\omega(1)} + q_{\omega(2)} \le 0, ..., \textstyle\sum_{j=1}^{l<N} q_{\omega(j)} \le 0, \quad \sum_{j=1}^{N} q_{\omega(j)} = 0. \tag{3.14}$$

It is implied that the inequalities (3.14) must hold *together* for all the permutations ω. Generally speaking, these inequalities are incompatible, and if they do not hold for some permutations, the corresponding coefficients C_ω must vanish [see Eq. (3.13)]. This idea has been successfully employed to build the multisoliton states of the quantum NS model [7,21]. Applying this approach to the perturbed model, it can be shown that the conditions necessary for vanishing of the "surplus" coefficients C_ω may be expressed as follows. The full set Λ of the complex wave numbers k_n should be separated into subsets $\Lambda_l = \{k_{l_n}\}$, $1 \le n \le N_l$, so that $\sum_l N(l) = N$, and inside each subset the following equations hold:

$$A(k_{l_1}, k_{l_2}) = 0, \quad A(k_{l_2}, k_{l_3}) = 0, \quad \vdots \quad A(k_{l_{N(l)-1}}, k_{l_{N(l)}}) = 0, \tag{3.15}$$

$$\sum_{j=1}^{N(l)} q_{l_j} = 0. \tag{3.16}$$

Note that some of the subsets Λ_l may contain only one wave number ($N_l = 1$), which must then be real due to Eq. (3.16). Each subset Λ_l with $N_l > 1$ corresponds to a quantum soliton, so that an eigenstate parametrized by the subsets $\{\Lambda_l\}$ describes a scattering state of these solitons [21], the subsets with $N_l = 1$ corresponding to an "admixture" of free particles to this multisoliton state. The scattering of the solitons and free particles is purely elastic, as it does not change the wave numbers involved [21,7].

The equations (3.15) for the complex wave numbers can be obtained in another way. Considering the Bethe equations for a system of quantum particles with the δ-like interactions in a box with periodic boundary conditions [18], one can verify that Eqs. (3.15) are a limit of those equations for the infinite spatial period.

We proceed to solve the system (3.15) for the set $\{k_j\}$, in order to obtain the wave function corresponding to a one-soliton state containing N_l particles. One can see that the system (3.15) and (3.16) is not complete, as Eq. (3.16) is actually not a complex equation but just an imaginary part of some complex equation. To complete the system, we introduce the soliton's total momentum $\hbar K_l$ which is real,

$$K_l \equiv \sum_{n=1}^{N(l)} \mathrm{Re}(k_{l_n}), \tag{3.17}$$

which parametrizes the solution. Now, the last equation of the system will be the following:

$$\sum_{n=1}^{N(l)} k_{l_n} = K_l. \tag{3.18}$$

For an arbitrary integer N_l and real K_l, the solution cannot be found in an explicit form. However, it can be readily found for the particular cases in which the corresponding classical perturbed NS equation is exactly solvable. First we consider the Hirota equation with $\epsilon_2 = -3c\epsilon_1$. In this case, using Eq. (3.11), we obtain that Eqs. (3.15) take exactly the same form as for the unperturbed NS model:

$$\begin{aligned} k_{l_1} - k_{l_2} &= i\hbar c, \\ k_{l_2} - k_{l_3} &= i\hbar c, \\ &\vdots \\ k_{l_{N(l)-1}} - k_{l_{N(l)}} &= i\hbar c. \end{aligned} \tag{3.19}$$

The solution, which exists for all real K_l [see Eq. (3.17)] and for any integer N_l, is [18]

$$k_{n_l} = \frac{K}{N_l} + \frac{1}{2}[N_l - (2n-1)]i\hbar c \tag{3.20}$$

(the finite set of the wave numbers k_n on the complex plane is referred to as a "*string*" [18]). The boundedness conditions (3.14) are satisfied if $c < 0$. Thus the Hirota equation remains integrable in its quantized form and its spectrum is actually the same as that of the unperturbed model.

Similarly, the system of equations (3.15) and (3.18) can also be solved for the DNS model, i.e., that with $\epsilon_1 = 0$. We obtain the following solution:

$$k_n = \left(K - \frac{Nc}{2\epsilon_2}\right) \frac{\sin(\phi)}{\sin(N\phi)} e^{-i(N+1-2n)\phi} + \frac{c}{2\epsilon_2}, \tag{3.21}$$

where $\phi \equiv \tan^{-1}(\hbar\epsilon_2)$. Note that the solution (3.21) goes over into Eq. (3.20) in the limit $\epsilon_2 \to 0$. The energy E of the DNS soliton corresponding to the string (3.21) can be calculated as follows:

$$\begin{aligned} E = \hbar \sum_{n=1}^{N} k_n^2 &= [1 - N\tan(\phi)\cot(N\phi)] \left(\frac{c\hbar}{\epsilon_2}K - \frac{\hbar N c^2}{4\epsilon_2^2}\right) \\ &\quad + \hbar\tan(\phi)\cot(N\phi)K^2, \end{aligned} \tag{3.22}$$

where $P = \hbar K$ is the soliton's momentum and, again, $\phi \equiv \tan^{-1}(\hbar\epsilon_2)$.

We can show that the string (3.21) satisfies the conditions (3.14) (i.e., the soliton exists) only for

$$K \geq \frac{Nc}{2\epsilon_2}, \tag{3.23}$$

$$N\phi \leq \pi \tag{3.24}$$

($c < 0$ and $\epsilon_2 > 0$). When $N\phi > \pi$, the string forms a closed circle and the conditions (3.14) break down.

Finally we would like to mention that the existence of complete set of the Bethe eigenstates for arbitrary N is equivalent to the full integrability of the quantum system since one can easily show that the values

$$I_n = \sum_{j=1}^{N} k_j^n, \quad n = \{0, 1, ..., N\} \tag{3.25}$$

remain integrals of motion. When $N \to \infty$ one obtains the infinite number of conservation laws. The problem of how to represent I_n in the secondary-quantized form was considered in [22].

IV. THE NONINTEGRABLE CASE

Proceeding to the generic nonintegrable case, we will concentrate on the simplest nonintegrable model corresponding to the classical equation (1.2). For this model, Eqs. (3.15) take the form

$$\begin{aligned} k_1 - k_2 + 3\epsilon_1(k_1^2 - k_2^2) &= ic\hbar, \\ k_2 - k_3 + 3\epsilon_1(k_2^2 - k_3^2) &= ic\hbar, \\ &\vdots \\ k_{N-1} - k_N + 3\epsilon_1(k_{N-1}^2 - k_N^2) &= ic\hbar. \end{aligned} \tag{4.1}$$

Unlike the linear equations (3.19), the nonlinear system of Eqs. (4.1) and (3.18) does not have a simple analytic solution in a general form. Therefore we start the analysis from the simplest case $N = 2$. The solution is then fairly simple:

$$k_{1,2} = \frac{K}{2} \pm \frac{ic\hbar}{2(1+3\epsilon_1 K)}. \quad (4.2)$$

If we choose the signs $c < 0$ and $\epsilon_1 > 0$, the boundedness conditions (3.14) for the two-particle quantum soliton are satisfied on the semiaxis $K > -\frac{1}{3\epsilon_1} \equiv K_{cr}$ (recall the real quantity $K \equiv k_1 + k_2$ determines the soliton's momentum $P = \hbar K$). The total energy of this soliton E_2 can be readily calculated to be

$$E_2 = \frac{\hbar}{2}(K^2 + \epsilon_1 K^3) - \frac{1}{2}c^2\hbar^3(1+3\epsilon_1 K)^{-1}. \quad (4.3)$$

Note that the first term in Eq. (4.3) is the kinetic energy of two free particles, while the second one is their binding energy. The latter diverges when the total momentum K approaches the boundary value of the soliton's existence. This divergence is actually a consequence of unboundness from below of the kinetic energy in this case. It may be renormalized by adding to the Hamiltonian density a term proportional to $\Psi^*\Psi_{xxxx}$. At $K < K_{cr}$, there are only the delocalized eigenstates describing scattering of two free particles. Following the approach of Ref. [7], one can show that the eigenstates (4.2) constitute a complete set in the two-particle sector of the quantum model (1.2). Thus the two-particle sector of this model remains integrable, which also follows from the existence of two fundamental conservation laws.

In the sector $N = 3$, we eliminate k_1 and k_2 from Eqs. (3.15)–(3.17) and obtain

$$k_{1,3} = \frac{K-k_2}{2} \pm \frac{ic\hbar}{[1+3\epsilon_1(K-k_2)]}. \quad (4.4)$$

For the sake of convenience we introduce two new variables $Q \equiv K + \frac{1}{2\epsilon_1}$ and $q \equiv k_2 + \frac{1}{6\epsilon_1}$. Then the equation for q is

$$-3q^4 + 4Qq^3 + 2Q^2q^2 - 4Q^3q + Q^4 = \left(\frac{2c\hbar}{3\epsilon_1}\right)^2. \quad (4.5)$$

A straightforward analysis of the algebraic equation (4.5) demonstrates that it has two real and two complex roots for q if

$$|Q| > Q_0 \equiv \left(\frac{3\sqrt{3}}{\sqrt{3}+2}\right)^{1/4} \sqrt{\left|\frac{c\hbar}{3\epsilon_1}\right|} \quad (4.6)$$

and four complex roots in the opposite limit. A direct check of the inequalities (3.14) demonstrates that two solutions out of the four satisfy these inequalities. For $K > Q_0 - \frac{1}{2\epsilon_1} \equiv K_0$, the energy for both localized eigenstates is real, while for $K < K_0$ the energy proves to be *complex*, taking complex conjugate values for the two localized eigenstates. Thus we can formally build the complete set of the eigenstates of the Hamiltonian in the three-particle sector, but it contains the unphysical states corresponding to complex energies.

The appearance of eigenstates with complex energy obviously means that our Hamiltonian (3.8) is non-Hermitian. To understand this let us find the domain of H_3, namely, the set of wave functions, on which the Hamiltonian H_3 is determined. First, we separate the center of mass movement. To do this, we introduce new coordinates

$$R = \frac{x_1 + x_2 + x_3}{3}, \quad (4.7)$$

$$y_1 = x_1 - x_2, \quad (4.8)$$

$$y_2 = x_2 - x_3 \quad (4.9)$$

and try wave functions in the form

$$f(x_1, x_2, x_3) = \exp(iKR) g(y_1, y_2). \quad (4.10)$$

Then we obtain a Hamiltonian which governs a movement of three particles in their center of mass frame:

$$H_{3,K} = H_{free} + 2\hbar c[\delta(y_1) + \delta(y_2) + \delta(y_1 + y_2)], \quad (4.11)$$

$$H_{free} = -2(1+2\epsilon_1 K)\left(\frac{\partial^2}{\partial {y_1}^2} + \frac{\partial^2}{\partial {y_2}^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\right) + 6i\epsilon_1 \frac{\partial^2}{\partial y_1 \partial y_2}\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right). \quad (4.12)$$

The wave function symmetry conditions read now as follows:

$$\begin{aligned} g(y_1, y_2) &= g(-y_2, -y_1), \\ g(y_1, y_2) &= g(y_1 + y_2, -y_2), \\ g(y_1, y_2) &= g(-y_1, y_1 + y_2). \end{aligned} \quad (4.13)$$

This means that it is enough to know the wave function g in the quadrant $Q_3 = (y_1 < 0, y_2 < 0)$.

The Hamiltonian (4.11) contains singular potentials so, acting on an arbitrary continuous function g, it can produce a singular function in general. It can be avoided only if g satisfies the following boundary conditions:

$$-2(1+2\epsilon_1 K)\left(\frac{\partial g}{\partial y_1} - 2\frac{\partial g}{\partial y_2}\right) + 6i\epsilon_1\left(2\frac{\partial^2 g}{\partial y_1 \partial y_2} - \frac{\partial^2 g}{\partial y_1^2}\right) = -2\hbar c g \quad (\text{for } y_1 < 0,\ y_2 = 0), \quad (4.14)$$

$$-2(1+2\epsilon_1 K)\left(\frac{\partial g}{\partial y_2} - 2\frac{\partial g}{\partial y_1}\right) - 6i\epsilon_1\left(2\frac{\partial^2 g}{\partial y_1 \partial y_2} - \frac{\partial^2 g}{\partial y_2^2}\right) = -2\hbar c g \quad (\text{for } y_2 < 0,\ y_1 = 0). \quad (4.15)$$

These conditions determine the domain of the Hamiltonian (4.11), which consist of all smooth, integrable on the quadrant Q_3 functions satisfying (4.14) and (4.15). Let us call this domain U. To check the Hermiticity of the Hamiltonian we evaluate the following expression:

$$\begin{aligned} \Delta &\equiv \langle z|H_{3,K}g\rangle - \langle H_{3,K}z|g\rangle \\ &= 6\int_{-\infty}^{0} dy_1 \int_{-\infty}^{0} dy_2 z^*(H_{free}g) \\ &\quad -6\int_{-\infty}^{0} dy_1 \int_{-\infty}^{0} dy_2 (H_{free}z^*)g. \end{aligned} \quad (4.16)$$

Using the conditions (4.14) and (4.15) we obtain

$$\Delta = 18i\epsilon_1[z^*(g_{y_2} - g_{y_1}) + g(z^*_{y_2} - z^*_{y_1})]|_{y_1=y_2=0}. \quad (4.17)$$

From (4.17) we see that the Hamiltonian (4.11) is Hermitian only on subdomain V, which consists of functions g belonging to U and satisfying the following condition:

$$g_{y_1}(0,0) = g_{y_2}(0,0). \tag{4.18}$$

So we see that in this nonintegrable case the Hamiltonian becomes non-Hermitian. We can make it Hermitian only restricting it on the subdomain V. Moreover, one can show that the Bethe ansatz solutions in this case do not belong to V and consequently they are not physically meaningful. Actually, V is not an invariant space of the physical states since the condition (4.18) does not commute with the Hamiltonian (4.12). The physical domain is determined by the following infinite consequence of equations:

$$\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right) H^n_{free} \,|_{y1=y2=0} \quad (\text{for } n = 0, 1, 2, ..., \infty). \tag{4.19}$$

We cannot now determine the whole physical domain by solving these equations, but, nevertheless, we are able to build some physical eigenstates of the system combining different BA solutions which have the same energy and total momentum. Indeed, in the three-particle sector every scattering BA state (not containing solitons) is parametrized by three real wave numbers k_1, k_2, and k_3. Fixing two integrals of motion E and K leaves one parameter free. Thus we have an infinite degeneracy of scattering states with the same energy and momentum. Let us take two such BA states g and h that do not belong to V. Then it can be directly shown that the state $f = g + \alpha h$, where $\alpha \equiv \frac{g_{y_2}(0,0)-g_{y_1}(0,0)}{h_{y_1}(0,0)-h_{y_2}(0,0)}$, does belong to V and, moreover, satisfies all the conditions (4.19). So f is a physical state. The state f mixes different sets of the wave vectors k_j and describes a nontrivial scattering process, which is the most characteristic feature of nonintegrable systems. For example, let us take for g and h, respectively, the BA state describing a two-particle soliton plus a free particle and another Bethe state corresponding to three free particles. The wave numbers corresponding to the first state are $k_{1,2} = \pm ic\hbar/2$ and arbitrary k_3. Here we take the soliton's momentum equal to zero so that the total momentum K of this state is equal to k_3. Then the second BA state is specified by the real wave numbers q_1, q_2, and $q_3 = k_3 - q_1 - q_2$. Equating the energies, we obtain q_1 as a function of q_2 and k_3. This means that there exists a continuous spectrum of degenerated states h, which can be used for building the physical ones. Let us take, for example, $q_2 = 0$. Then

$$q_1 = \frac{k_3}{2} \pm \frac{\sqrt{k_3^2(4 + 12\epsilon_1 k_3)^2 - 4c^2\hbar^2(4 + 12\epsilon_1 k_3)}}{2(4 + 12\epsilon_1 k_3)}. \tag{4.20}$$

When $k_3^2(4 + 12\epsilon_1 k_3) > c^2\hbar^2$, two real solutions for q_1 exist and hence we can combine the corresponding BA states. As a result, we obtain an infinite number of physical eigenstates which are degenerated by energy and total momentum. All these states describe nontrivial inelastic scattering processes, in which the two-particle soliton can be destroyed.

V. THE CLASSICAL LIMIT

In this section we will discuss the classical limit of the results obtained above. For the integrable cases, two question can be raised. One is to obtain the classical soliton solution as a limit of some quantum objects. The other is to calculate the classical soliton's energy and momentum. The first problem was thus far solved only for the unperturbed NS model [17,23,8]; other cases seem to require extensive calculations. Here we calculate the soliton's parameters only (the second problem) and we will show that the classical energy, momentum, and existence conditions obtained from our approach coincide with the results of the other approaches.

As it has been demonstrated in Ref. [23], a classical soliton $\Psi_{sol}(x,t)$ can be obtained as a limit of a certain matrix element taken between the quantum multiparticle states:

$$\Psi_{sol}(x,t) = \lim\left(\hbar\int_{-\infty}^{+\infty} d\mu \langle \hbar K, N|\hat{\Psi}(x,t)|\hbar(K+\mu), N+1\rangle\right), \tag{5.1}$$

where $\hat{\Psi}(x,t)$ is realized as the operator-valued function obeying the quantum NS equation and the limit process is defined as follows:

$$\hbar \to 0\ ,\ N \to \infty\ ,\ K \to \infty,\ \hbar N \to N_{cl},\ \hbar K \to P_{cl}. \tag{5.2}$$

The finite quantities N_{cl} and P_{cl} are the values of the number of particles and momentum for the classical soliton. Other authors [8] performed this limit process in a somewhat different form, but the general observation fits every version: the main contribution to the Ψ_{sol} gives the integration over a nearest vicinity of the value K, which determines the classical value of the momentum P_{cl}. So we can assume that the classical soliton's parameters can be obtained from studying the quantum states (strings) when approaching the limit (5.2). To do this, we will directly analyze the classical (i.e., continuum) limit of the chain of the string equations (3.15) for the general model (3.1). We substitute (3.11) into the system (3.15) and rewrite each equation with all the terms proportional to $\hbar$ on the right-hand side. Let $\hbar \to 0$. Since the left-hand side of each equation is proportional to $k_i - k_j$, one of its solutions will always be such that $k_i \to k_j$. In this limit we call $k_i - k_j$ as dk and replace $k_i + k_j$ with $2k_i$. Introducing the continuous variable $s \equiv \hbar n$ instead of the discrete number n, so that s ranges in the interval $0 < s < N_{cl} \equiv \hbar N$, we can represent the continuum limit of Eqs. (3.15) as follows:

$$\frac{dk}{ds} = -i(c - 2\epsilon_2 k)(1 + 6\epsilon_1 k)^{-1}. \tag{5.3}$$

Generally, one can take other solutions of Eqs. (3.15), for which $k_i - k_j$ is not small. This will produce gaps in the strings, as will be described below. In terms of the continuous function $k(s)$, the momentum of the classical wave field can be written as

$$\int_0^{N_{cl}} k(s)ds = P \tag{5.4}$$

and the energy

$$E = \int_0^{N_{cl}} (k^2 + 2\epsilon_1 k^3)ds. \tag{5.5}$$

The classical momentum (5.4) must always be real in virtue of Eq. (3.16).

The system of the boundedness conditions (3.14) also has a continuum limit which takes the following form: the inequality

$$\mathrm{Im} \int_0^{\nu} k(s)ds \leq 0 \tag{5.6}$$

must be satisfied for all positive $\nu < N_{cl}$.

A general solution to Eq. (5.3) can be represented in the implicit form

$$s(k) = -\frac{3i\epsilon_1}{\epsilon_2}\left[k + \left(\frac{c}{2\epsilon_2} + \frac{1}{6\epsilon_1}\right)\ln\left|k - \frac{c}{2\epsilon_2}\right|\right] + \text{const.} \tag{5.7}$$

For the integrable particular cases, the relation (5.7) can be reversed to express the complex k in terms of real s. For the DNS model, we obtain

$$k(s) = \left(P - \frac{N_{cl}c}{2\epsilon_2}\right)\epsilon_2 \csc(N_{cl}\epsilon_2) \times \exp[-i\epsilon_2(N_{cl} - 2s)] + \frac{c}{2\epsilon_2}, \tag{5.8}$$

which exactly coincides with the continuum limit of the discrete solution (3.21). For the Hirota model, as well as for the unperturbed NS one,

$$k(s) = \frac{P}{N_{cl}} - ic\left(s - \frac{N_{cl}}{2}\right), \tag{5.9}$$

which can be obtained too as the continuum limit of the string (3.20). The "continuum strings" (5.8) and (5.9) are depicted in Figs. 1 and 2. The "classical strings" (5.8) and (5.9) for the integrable models are always symmetric relative to the real axis on the k plane (see Figs. 1 and 2) and, as it follows from Eqs. (5.4) and (5.5), the energy of the corresponding classical solitons is real.

Applying the limit process (5.2) to the quantum DNS soliton (3.21), we conclude that the result coincides, up to a Galilean transformation, with the classical soliton described by Eqs. (2.1)–(2.3). In the same limit, the energy of the quantum soliton (3.22) with $c = 0$ goes over into the classical expression (2.6); at $c \neq 0$, the classical energy, obtainable from Eq. (2.6) by means of the Galilean transformation, also coincides with the limit of

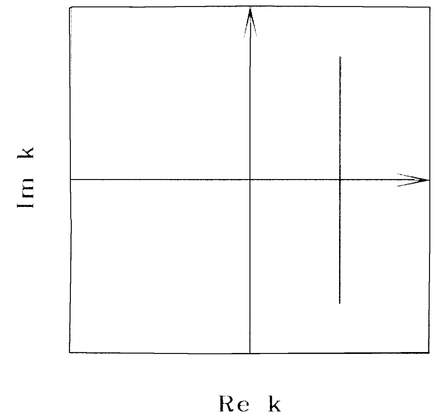

FIG. 1. The "string" for the unperturbed NS and Hirota equations.

Eq. (3.22). Finally, the inequality (3.24), bounding the region of existence of the quantum solitons, in the classical limit goes over into the one (2.9) bounding the existence region of the classical DNS solitons. It is also easy to demonstrate that the soliton of the classical Hirota equation [i.e., Eq. (1.1) with $\epsilon_2 = -3c\epsilon_1$] is exactly the limit of the quantum soliton (3.20). The same results may be obtained when calculating the soliton's energy and momentum (5.5) and (5.4) for the classical "strings" (5.8) and (5.9) obtained above.

To relate the classical strings to the classical solitons, we notice the following: in the classical limit, a soliton with the energy E and momentum P is represented, in terms of the inverse scattering transform, by a complex root λ (which is a corresponding eigenvalue of the scattering problem). We have shown that, for all integrable cases (considered in the paper), $k(s = N_{cl})$, which is the end point of the string is equal to 2λ.

Another problem is to analyze the classical limit of the nonintegrable cases. As for the integrable models, the classical strings may be obtained and they have a more rich structure. For the simplest nonintegrable model (1.2), the expression $k(s)$ can also be found explicitly:

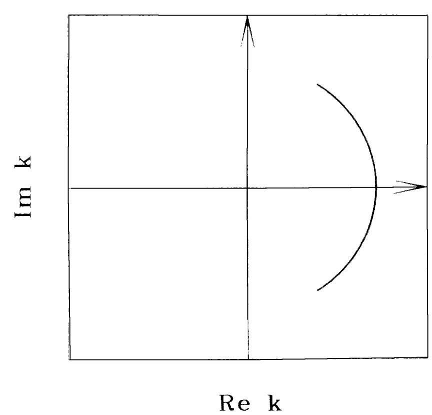

FIG. 2. The string for the DNS equation.

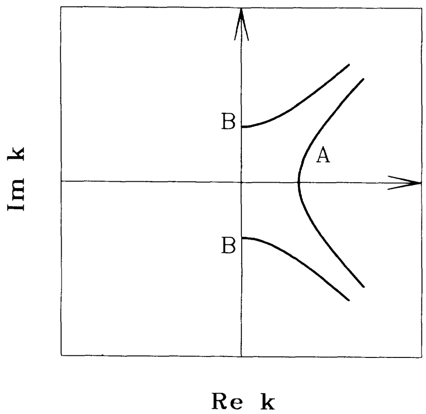

FIG. 3. Two symmetric strings for the simplest nonintegrable equation (1.2).

$$k(s) = \pm\sqrt{-\frac{1}{3}ics\epsilon_1^{-1} + C} - \frac{1}{6\epsilon_1}, \tag{5.10}$$

where C is a complex constant. As it follows from (5.10), such strings may have gaps in the form $k(s-0) = -k(s+0) - \frac{1}{3\epsilon_1}$ and they satisfy the requirement that total momentum be real. The gaps may leave the string symmetric (Fig. 3), but in the general case they destroy the symmetry (Fig. 4). Discontinuous nonsymmetric strings give rise to complex values of the energy. This is the same effect that we saw in the three-particle sector. In the nonintegrable case the above built "strings" correspond to unphysical states and additional investigation is needed in order to build the physical ones.

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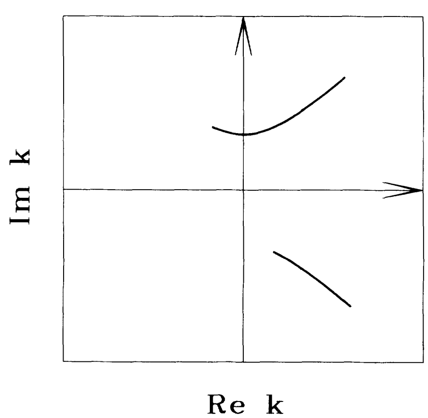

FIG. 4. The nonsymmetric string for the simplest nonintegrable equation (1.2).

APPENDIX

In this section we will show how the quantum nonlinear Schrödinger equation can be obtained directly from quantization of the electromagnetic field in the nonlinear optical fiber. Usually, the NS equation is derived from the classical field equations by employing the multi-scale method. After this the quantization of the obtained envelope field is performed. This quantization is formal and the quantum meaning of the envelope field remains unclear. Our approach is based on the paper of Drummond and Carter [24], where the stochastic NS equation was derived from the foundations of a correctly quantized field theory. Following this approach, we will just show that the fully quantized NS equation can be obtained in the same way.

Consider the model describing propagation of a linearly polarized wave in the lowest order transverse mode [24] in a dielectric nonmagnetic fiber. The dispersionless Hamiltonian with the quartic nonlinearity inside a volume V is

$$\hat{H} = \int_V : \left[\frac{1}{8\pi\epsilon}\hat{\mathbf{D}}^2 + \frac{1}{8\pi}\hat{\mathbf{B}}^2\right] : d^3\mathbf{x} - \frac{\chi}{4\epsilon^4}\int_V : \hat{\mathbf{D}}^4 : d^3\mathbf{x}, \tag{A1}$$

where :: means the normal ordering of quantum fields. The correct quantization of the electrodynamics in nonlinear dielectric media was done in Ref. [25] and the interacting-field commutation relations were found to be

$$[\hat{D}_j(\mathbf{x}) \ , \ \hat{A}_k(\mathbf{x}')] = i\hbar\delta_{jk}\delta(\mathbf{x}-\mathbf{x}'), \tag{A2}$$

where $\mathbf{A}$ is the vector potential in the Coulomb gauge. This means that the generalized momentum in this model is the field $\mathbf{D}$ but not $\mathbf{E}$, as in the vacuum quantum electrodynamics. So the fields $\mathbf{A}$ and $\mathbf{D}$ have to be expanded into the harmonic-oscillator form

$$\hat{\mathbf{D}}(\mathbf{x}) = i\hbar^{\frac{1}{2}}\sum_n \left(\frac{\omega_n}{2}\right)^{\frac{1}{2}} \hat{a}_n\mathbf{u}_n(\mathbf{r})\exp(ik_nz) + \text{H.c.}, \tag{A3}$$

$$\hat{\mathbf{A}}(\mathbf{x}) = \hbar^{\frac{1}{2}}\sum_n \left(\frac{1}{2\omega_n}\right)^{\frac{1}{2}} \hat{a}_n\mathbf{u}_n(\mathbf{r})\exp(ik_nz) + \text{H.c.}, \tag{A4}$$

where $k_n \equiv k_0 + n\Delta k$, $\Delta k \equiv \frac{2\pi}{L}$, and L is the fiber's length. The operator $\hat{a}_n$ is a longitudinal-mode operator for the normalized transverse mode $\mathbf{u}_n(\mathbf{r})$, with a frequency ω_n. The position vector $\mathbf{x}$ is represented as $(\mathbf{r}, z)$. The functions $\mathbf{u}(\mathbf{r})$ are normalized as follows:

$$\int_V \mathbf{u}_n^*\mathbf{u}_n \, d^3\mathbf{x} = 1. \tag{A5}$$

The dispersion is now taken into account by inserting the function $\omega_n(k_n) = ck_n/\sqrt{\epsilon(k_n)}$ into expansions (A3) and (A4).

Substituting Eqs. (A3) and (A4) into Eq. (A1), one obtains the following Hamiltonian:

$$\hat{H} = \hbar \sum_n \omega_n \hat{a}_n^\dagger \hat{a}_n - \frac{\chi}{4\epsilon^4} \int_V : \hat{\mathbf{D}}^4 : d^3\mathbf{x}$$
$$\equiv \hbar \sum_n \omega_n \hat{a}_n^\dagger \hat{a}_n + \hbar^2 \sum_{lmjf} c_{lmjf} \hat{a}_l^\dagger \hat{a}_m^\dagger \hat{a}_j \hat{a}_f$$
$$\times\ \delta(k_l + k_m - k_j - k_f) + (\text{other terms}). \quad \text{(A6)}$$

The "other terms" mean all the quartic terms with unequal numbers of creation and annihilation operators. Their role and the coefficients c_{lmjf} will be discussed later.

Now we do the main step of the derivation. Our main assumption is that almost all the excitations are distributed near some wave number k_0, which is the share of excitations with k lying far from k_0 is small. This assumption actually means that we consider a carrier wave with long-wavelength modulations. Then we can expand the function $\omega_n \equiv \omega(k_0 + n\Delta k)$ in powers of $n\Delta k$ and restrict ourselves to the first terms of the expansion

$$\omega_n = \omega(k_0 + n\Delta k) = \omega_0 + n\Delta k \omega' + \frac{(n\Delta k)^2}{2}\omega'' + \cdots, \quad \text{(A7)}$$

where $\omega' \equiv d\omega/dk|_{k_0}$ and $\omega'' \equiv d^2\omega/dk^2|_{k_0}$.

The next step is to represent our model in the interaction picture, which, as the free Hamiltonian $\hat{H}_0$ is taken, is the following:

$$\hat{H}_0 = \hbar \sum_n \omega_0 \hat{a}_n^\dagger \hat{a}_n. \quad \text{(A8)}$$

Then the interaction-picture Hamiltonian will be

$$\hat{H}_I = \hbar \sum_n (\omega_n - \omega_0) \hat{a}_n^\dagger(t) \hat{a}_n(t)$$
$$+ \hbar^2 \sum_{lmjf} c_{lmjf} \hat{a}_l^\dagger(t) \hat{a}_m^\dagger(t) \hat{a}_j(t) \hat{a}_f(t)$$
$$\times\ \delta(k_l + k_m - k_j - k_f) + (\text{other terms}), \quad \text{(A9)}$$

where

$$\hat{a}_n(t) \equiv \exp\left(\frac{-i\hat{H}_0 t}{\hbar}\right) \hat{a}_n \exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) = \hat{a}_n \exp(-i\omega_0 t). \quad \text{(A10)}$$

Substituting Eq. (A10) into Eq. (A9) one can see that the first two terms have no time dependence, while the "other terms" contain rapidly oscillating coefficients. Averaging the Hamiltonian (A9) over the time T much greater than the carrier wave period $T_0 \equiv \frac{2\pi}{\omega_0}$, one obtains a time-independent Hamiltonian without the "other terms." This is a very important point. The Hamiltonian (A9) does not commute with the number operator $\hat{N} = \sum_n \hat{a}_n^\dagger \hat{a}_n$ and, in turn, the number of photons is not conserved. Nevertheless, the number of photons averaged in time is constant, and we will deal with the averaged picture. Under our main assumption (that the excitations are distributed near the wave vector k_0), we can replace, in the first approximation, the coefficients c_{lmjf} with the constant c, which is actually equal to c_{0000}. Employing the expansion (A7), we obtain the final expression for the Hamiltonian averaged in time:

$$\langle \hat{H}_I \rangle_T = \hbar \sum_n \left(\omega'(k_n - k_0) + \frac{\omega''}{2}(k_n - k_0)^2 \right) \hat{a}_n^\dagger \hat{a}_n$$
$$+ \hbar^2 \sum_{lmjf} c \hat{a}_l^\dagger \hat{a}_m^\dagger \hat{a}_j \hat{a}_f \delta(k_l + k_m - k_j - k_f). \quad \text{(A11)}$$

Performing the inverse Fourier transform

$$\hat{\Psi}(z) \equiv \hbar^{\frac{1}{2}} \sum_n \hat{a}_n \exp[i(k_n - k_0)z], \quad \text{(A12)}$$

one obtains the unperturbed NS Hamiltonian in the laboratory coordinates in terms of the field operator $\hat{\Psi}(z)$.

Expanding the coefficients $c_{lmjf} \equiv c(k_l, k_m, k_j, k_f)$ near the point $k_l = k_m = k_j = k_f = k_0$, one can obtain, as the next approximation, the following expression:

$$c_{lmjf} = c + c^{(1)}(k_l + k_m + k_j + k_f - 4k_0), \quad \text{(A13)}$$

where $c^{(1)} \equiv \frac{\partial c_{lmjf}}{\partial k_l}|_{k_0}$. Substituting Eq. (A13) into the Hamiltonian and making the inverse Fourier transform (A12), we directly obtain the nonlinear perturbing term considered in the present work. In the same way, employing the expansion (A7) up to the third power, we obtain the linear perturbing term with the third derivative.

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