Normal-form theory for a laser model with periodic signal and noise injection

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A model for a laser with modulated signal and noise in the semiclassical approach allows the characterization of a Hopf bifurcation point and correspondingly of its spectrum. By means of the normal-form theory we have been able to give explicit normal-form equations that describe the system's dynamic in the vicinity of the bifurcation point. At first order we determine the system's behavior for the case when the signal modulation is coincident with the Hopf frequency and when it does not. In both cases, a characterization of the unfolding parameter is possible. We also consider an injected laser the Hopf frequency of which is a integer multiple of order n of the frequency modulated signal. In this case a perturbative theory of order n is necessary. We present also an example of noise for which this system does not present cooperative amplification. Numerical analysis is consistent with this predictions.

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I. INTRODUCTION

A widely accepted description of a single-frequency, homogeneously broadened laser is the semiclassical model of the Maxwell-Bloch equations for the macroscopic dynamical variables. This approach allows good modeling in several phenomena such as frequency shifts, phase locking of lasers, population-difference pulsations, active and passive mode locking, damped oscillations, ultrashort laser pulses, optical bistability, and routes to chaos of lasers, among other effects [1–9]. In class A lasers (He-Ne, Ar-ion, Kr-ion, dye lasers) it is possible to adiabatically eliminate the polarization and the population difference; in such cases, a differential equation for the electric field in the rotating-wave approximation describes the entire system. In our previous work [10], we presented a model for a laser with a coherent, constant external injection in this approximation, and it was proved that it presents a Hopf-type bifurcation characterized by the gain and loss parameters of the laser, and the detuning, i.e., the normalized difference between the free-running frequency emission of the laser and that of the source of the coherent signal. Two different types of behavior were established, i.e., limit cycles and stationary, fixed-point solutions (phase-locking region). A Hopf's normal-form equation describes the evolution of the field, its unfolding parameter being the one that characterizes the dynamical state. This is also a function of the intensity of the injected field into the laser, the detuning, and the gain-to-loss ratio.

We hereby present a model for a single-frequency laser with external injection of a time-periodic coherent signal, including fluctuations of both the intensity and the frequency detuning between the external signal and the laser when the system is in the vicinity of the bifurcation point. In this case, the system presents a Hopf bifurcation (HB) also. We shall study the effects of this kind of small-amplitude fluctuation when the working point is near the bifurcation point (BP). First we will characterize the HB to zeroth order in the perturbative expansion parameter, which we call ξ, which is a measure of how far the system is from the BP. We are interested in obtaining a normal-form equation (NFE) to describe the system's dynamics in the critical variables. Near the BP, a first-order perturbative theory in ξ is necessary. This approach will explicitly determine the unfolding parameter and the corrections to the Hopf frequency. At first order it will also be possible to determine the system's behavior for the case of resonant and nonresonant signal modulation. We understand by resonant, the case in which the modulation frequency equals the Hopf frequency, and by nonresonant, when it does not. The resonant mode can model a case of an injection from an identical laser working at its own Hopf bifurcation point. We shall also consider an injected laser whose Hopf frequency is a multiple of order n of the periodically modulated signal. In this case, a perturbative theory of order n in ξ is necessary. The critical-variables dynamics near the BP is essentially determined by a NFE [10] with a constant driving term, and in this way the analysis is equivalent to that given in Ref. [11] in the vicinity of the HB point, in the sense that they both use normal-form techniques, the latter applied to an equivalent combination of the critical modes used here. To give a more realistic description, we shall also consider fluctuations in both the amplitude of the injection and in the detuning if the system is working

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near the HB point. In particular, we are interested in low-intensity noise. We shall present a statistical study in order to evaluate, in some cases, if there are cooperative phenomena between the noise terms and the injected modulation, so it is necessary to evaluate stationary correlation functions and intensity temporal mean values of the electric field. From these theoretical studies, it is possible to make an explicit evaluation of the principal spectral features that result from a numerical analysis. The conclusions of this theory are consistent with the numerical evaluation.

II. DETERMINISTIC MODEL

A periodically injected, class A laser system can be modeled in the simplest way by the following differential equation for the complex electric field, which results from a straightforward generalization of the one given in [10],

$$\dot{E} = [i + g(E)]\, E + F_b + \xi\, (1 + a\cos\omega t) , \tag{2.1}$$

where the overdot denotes d/dt, and where

$$g(E) = -\kappa + \frac{\gamma}{1+|E|^2} \tag{2.2}$$

is the net saturated gain, expressed as a function of the gain γ and loss κ adimensional parameters. Time is adimensional, expressed in units of $\Delta\nu$, i.e., the frequency offset between the laser and the external signal. When the periodic modulation is off ($a = 0$) the system is forced with the intensity given by

$$F = F_b + \xi ,$$

F_b being the constant injected intensity that is capable of driving the system to the BP. Here ξ represents a constant, small-amplitude perturbation around the BP.

In a simplified model [12], a sinusoidal perturbation of a single frequency ω is the most suitable way to describe the modulation effects on the system. We shall consider small-amplitude oscillations $a\xi$ compared to the constant perturbation ξ. As previous works show [10, 11], at fixed gain and loss the system will reach a HB for $\xi = 0$. Then, the intensity which drives the system to this point is

$$F_b^2 = [\kappa^2(\sqrt{\sigma}-1)^2 + 1]\, (\sqrt{\sigma}-1) , \tag{2.3}$$

where $\sigma = \gamma/\kappa$ is the gain-to-loss parameter. In this condition it can be shown that the laser electric field has amplitude and phase given by the expressions

$$|E_b|^2 = \sqrt{\sigma} - 1 , \tag{2.4a}$$

$$\varphi = -\cot^{-1}\, g_b , \tag{2.4b}$$

where it is possible to calculate g_b, which we have called net gain at the BP, and it is given by

$$g_b = \kappa\, (\sqrt{\sigma} - 1) . \tag{2.5}$$

The system's oscillation frequency around the BP can be written in terms of g_b as

$$\Omega^2 = 1 - g_b^2 , \tag{2.6}$$

with $0 \le \Omega < 1$. If the system is free from oscillations in the driving field, it has a limit-cycle oscillation behavior for $\xi < 0$, and a locking regime one if $\xi > 0$. In the present work we shall exclude the case $\Omega = 0$, which describes homoclinic orbits solutions to the laser electric field.

We shall make use of a perturbative method near the BP, which puts into evidence the principal characteristics of the system. In the first place, we set a differential equation valid near the BP. The aim is thus to determine a differential equation in normal form, describing the system as a whole, and a nonlinear change of variables with all the system's spectral characteristics.

Performing the perturbative method in the differential equation (2.1) near E_b (i.e., with $E = E_b + v$), it is possible to obtain a differential equation for the perturbative variable $v \in \mathbb{C}$. The new equation and its complex conjugate can be expressed in the canonical base $\{\boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2\}$ of $\mathbb{C}^2$ defining the vector $\mathbf{u} = v\ \boldsymbol{\epsilon}^1 + \bar{v}\ \boldsymbol{\epsilon}^2$. This election of $\mathbb{C}^2$ will allow us to diagonalize the linear part of the following differential equation:

$$\dot{\mathbf{u}} = L_b\mathbf{u} + \mathbf{N}_b(\mathbf{u}) + \xi\, \mathbf{D}(t) , \tag{2.7}$$

where the subindex b is placed to indicate that the coefficients are to be evaluated at the BP. Explicitly, the linear part is

$$L_b = \begin{pmatrix} i & -\kappa E_b^2 \\ -\kappa \overline{E_b}^2 & -i \end{pmatrix} . \tag{2.8}$$

The vector that contains the periodic driving term is

$$\mathbf{D}(t) = (1 + a\cos\omega t)(\boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^2) , \tag{2.9}$$

and the nonlinear term can be written in a condensed form, where repeated indexes indicate the sum, as

$$\mathbf{N}_b = \sum_{r>1} A_j^{j_1\cdots j_r}\, u_{j_1}\cdots u_{j_r}\, \boldsymbol{\epsilon}^j , \tag{2.10}$$

with $j_l = 1, 2$ (for all $1 \le l \le r$) and $u_1 = \overline{u_2} = v$. Note that this expression for the nonlinear part is a multilinear form in v and $\bar{v}$ of degree ≥ 2. The nonlinear part, non-null, second-order coefficients are

$$\begin{aligned} A_1^{11} = \overline{A_2^{22}} &= -\frac{\kappa}{\sqrt{\sigma}}\, \overline{E_b} , \\ A_1^{12} = \overline{A_2^{12}} &= -2\frac{\kappa}{\sqrt{\sigma}}\, E_b , \\ A_1^{22} = \overline{A_2^{11}} &= \frac{\kappa}{\sqrt{\sigma}}\, E_b^3 . \end{aligned} \tag{2.11}$$

The non-null, third-order coefficients are

$$\begin{aligned} A_1^{111} = \overline{A_2^{222}} &= \frac{\kappa}{\sigma}\, \overline{E_b}^2 , \\ A_1^{112} = \overline{A_1^{122}} &= -2\frac{\kappa}{\sigma}\, (1 - \tfrac{\sqrt{\sigma}}{2}) , \\ A_1^{122} = \overline{A_2^{112}} &= 2\frac{\kappa}{\sigma}\, E_b^2 , \\ A_1^{222} = \overline{A_2^{111}} &= -\frac{\kappa}{\sigma}\, E_b^4 , \end{aligned} \tag{2.12}$$

which are the only ones we give here for the purpose of the analysis that follows. As will soon become evident, these are the coefficients which give an explicit expression for the nonlinear change of variables and the NFE for the critical variables. We are now going to determine the critical modes of the system, and to this end we diagonalize the linear operator (2.8) to get the eigenvalues and the critical eigenvectors. Let $\{\chi^1, \chi^2\}$ be the eigenvector base in $\mathbb{C}^2$ which diagonalize L_b, i.e.,

$$L_b \chi^j = \Lambda_j \, \chi^j \; . \tag{2.13}$$

The corresponding eigenvalues are

$$\Lambda_{1,2} = \pm i\Omega \; . \tag{2.14}$$

If $\Omega \neq 0, 1$, in terms of the canonical base ϵ^1, ϵ^2, they are expressed as

$$\chi^j = \chi^j_k \, \epsilon^k \; , \tag{2.15a}$$

where χ^j_k are the coefficients relating to the change of base, given by

$$\begin{aligned} \chi^1_1 = \overline{\chi^2_2} &= \kappa \, E_b^2 \; , \\ \chi^1_2 = \overline{\chi^2_1} &= i(1-\Omega) \; . \end{aligned} \tag{2.15b}$$

For the inverse transformation,

$$\epsilon^l = \epsilon^l_m \, \chi^m \; , \tag{2.16a}$$

the coefficients are

$$\begin{aligned} \epsilon^1_1 = \overline{\epsilon^2_2} &= \Delta^{-1} \, \chi^2_2 \; , \\ \epsilon^1_2 = \overline{\epsilon^2_1} &= \Delta^{-1} \, \chi^2_1 \; , \end{aligned} \tag{2.16b}$$

with $\Delta = 2\Omega(\Omega - 1)$. Note that $\Omega \neq 0$ eliminates the homoclinic-case solution, while $\Omega \neq 1$ does it for the trivial solution $E_b = 0$.

III. ZEROTH-ORDER PERTURBATIVE THEORY IN THE PERIODIC TERMS

The electric-field power spectrum will display peaks whose intensities are related to the coefficients of the nonlinear variables' change. In this section we shall determine those coefficients, making $\xi = 0$ in Eq. (2.1). We shall also determine a NFE for the critical variables at the BP. We thus proceed with the following change of variables (from here on, crossed indexes imply summation):

$$\begin{aligned} \mathbf{u}(\mathbf{w}) &= \sum_{r\geq 1} \mathbf{u}^{[r]}(\mathbf{w}) \; , \\ \mathbf{u}^{[r]}(\mathbf{w}) &= U_j^{j_1 \cdots j_r} \, w_{j_1} \, \cdots \, w_{j_r} \, \chi^j \; , \end{aligned} \tag{3.1}$$

where the supraindex r between square brackets means that the terms in brackets below are of rth order in $\{w_j\}$ $(j = 1,2)$ and $\mathbf{w} = w_j \, \chi^j$ is the new critical vector. To zeroth order in ξ, the $U_j^{j_1 \cdots j_r}$ coefficients are time independent.

We define a NFE for the critical modes in the form

$$\dot{w}_j = \sum_{r\geq 1} F_j^{[r]}(\mathbf{w}) \; . \tag{3.2}$$

The evaluation of the vector $\mathbf{u}^{[r]}$ and the function $F_j^{[r]}$, in powers of $\{w_j\}$, up to the third order, proceeds as follows: For all r, Eq. (2.7) with $\xi = 0$ is

$$\dot{\mathbf{u}}^{[r]} = L \, \mathbf{u}^{[r]} + \mathbf{N}^{[r]}(\mathbf{u}) \; ; \tag{3.3}$$

please note that we have omitted the subindex b. To first order $(r = 1)$, for example,

$$\mathbf{u}^{[1]} = w_j \, \chi^j \tag{3.4}$$

and

$$F_j^{[1]} = \Lambda_j \, w_j \; , \tag{3.5}$$

are obtained. To orders $r > 1$, we can write

$$\begin{aligned} \left(\Lambda_j \, w_j \, \frac{\partial}{\partial w_j} - L\right) \, \mathbf{u}^{[r]} &= \mathbf{I}^{[r]} - F_j^{[r]} \, \chi^j \; , \\ \mathbf{I}^{[r]} &= \mathbf{N}^{[r]} - \sum_{s=2}^{r} (\dot{w}_j)^{[s]} \, \frac{\partial \mathbf{u}}{\partial w_j}^{[r-s+1]} \; , \end{aligned} \tag{3.6}$$

where $\mathbf{u}^{[r]}$ and $F_j^{[r]}$ are functions to be determined. For $r = 2$ we have $\mathbf{I}^{[2]} = \mathbf{N}^{[2]}(\mathbf{u}^{[1]})$ and it can be written as

$$\mathbf{N}^{[2]}(\mathbf{w}) = A_j^{j_1 j_2} \, u_{j_1}^{[1]} \, u_{j_2}^{[1]} \, \epsilon^j \qquad (j_1 \leq j_2) \; . \tag{3.7}$$

This equation, cast into the base $\{\chi^k\}$, is

$$\mathbf{N}^{[2]}(\mathbf{w}) = B_k^{k_1 k_2} \, w_{k_1} \, w_{k_2} \, \chi^k \qquad (k_1 \leq k_2) \; , \tag{3.8}$$

where, by straightforward algebra, it can be shown that

$$B_k^{k_1 k_2} = \chi_{j_1}^{k_1} \, \chi_{j_2}^{k_2} \, \epsilon_k^j \, A_j^{j_1 j_2} + \mathcal{P}(k_1, k_2) \; . \tag{3.9}$$

Here, $\mathcal{P}(k_1, k_2)$ indicates permutation of k_1 with k_2 in the preceding term. With the condition

$$F_j^{[2]} \equiv 0 \tag{3.10}$$

in the NFE (3.2), second-order resonances are eliminated [13]. The coefficients for $\mathbf{u}^{[2]}$ in terms of the $\{w_j\}$ [see Eqs. (3.1) and (3.7)] are

$$U_k^{k_1 k_2} = \left(\sum_i r_i \, \Lambda_i - \Lambda_k\right)^{-1} \, B_k^{k_1 k_2} \; , \tag{3.11}$$

where r_i is the order of $u_k^{[r]}$ $(r_i \leq r)$ in the w_i variable. Explicitly, the non-null coefficients are

$$\begin{aligned} U_1^{11} = \overline{U_2^{22}} = & \frac{(1-\Omega)\, e^{i\varphi}}{2\,\Omega^2 \, \sqrt{\sigma(\sqrt{\sigma}-1)}} \\ & \times\{\sqrt{1-\Omega^2}[(\sqrt{\sigma}-2)(1+\Omega) - 2\Omega] \\ & + i\,(1+\Omega)\,[-(\sqrt{\sigma}-4)\,\Omega + \sqrt{\sigma} - 2]\}, \end{aligned} \tag{3.12a}$$

$$\begin{aligned} U_1^{12} = \overline{U_2^{12}} = & \frac{(1-\Omega)\, e^{-i\varphi}}{\Omega^2 \, \sqrt{\sigma(\sqrt{\sigma}-1)}} \\ & \times\{\sqrt{1-\Omega^2}[-\sqrt{\sigma}(1+\Omega) + 2] \\ & + i\,(1+\Omega)\,[-\sqrt{\sigma}\,\Omega + \sqrt{\sigma} - 2]\}, \end{aligned} \tag{3.12b}$$

$$U_1^{22} = \overline{U_2^{11}} = \frac{(1-\Omega)\,e^{-3i\varphi}}{6\,\Omega^2\,\sqrt{\sigma(\sqrt{\sigma}-1)}} \times \{\sqrt{1-\Omega^2}[\sqrt{\sigma}(1-\Omega)-2] +i\,(1+\Omega)\,[\sqrt{\sigma}\,\Omega+\sqrt{\sigma}-2]\}. \tag{3.12c}$$

These are the coefficients which will enable a calculation of the spectral contributions at 2Ω in the cross-correlation function for the electric field, at order zero in ξ. Up to third order, $\mathbf{I}^{[3]} = \mathbf{N}^{[3]}(\mathbf{u})$ (with $\mathbf{u} = \mathbf{u}^{[1]} + \mathbf{u}^{[2]}$), and the nonlinear terms (2.10) are

$$\mathbf{N}^{[3]}(\mathbf{w}) = \{A_j^{j_1 j_2 j_3}\, u_{j_1}^{[1]}\, u_{j_2}^{[1]}\, u_{j_3}^{[1]} + A_j^{j_1 j_2}\,[u_{j_1}^{[1]}\, u_{j_2}^{[2]} + \mathcal{P}(j_1,j_2)]\}\,\boldsymbol{\epsilon}^j \quad (j_1 \le j_2 \le j_3)\ , \tag{3.13}$$

or, in the alternative variables,

$$\mathbf{N}^{[3]}(\mathbf{w}) = B_k^{k_1 k_2 k_3}\, w_{k_1}\, w_{k_2}\, w_{k_3}\, \boldsymbol{\chi}^k \quad (k_1 \le k_2 \le k_3)\ , \tag{3.14}$$

with

$$B_k^{k_1 k_2 k_3} = \{\chi_{j_1}^{k_1}\, \chi_{j_2}^{k_2}\, \chi_{j_3}^{k_3}\, A_j^{j_1 j_2 j_3} + \chi_{j_1}^{k_1}\, \chi_{j_2}^{l}\, U_l^{k_2 k_3}[A_j^{j_1 j_2} + \mathcal{P}(j_1,j_2)] + \mathcal{P}(k_1,k_2,k_3)\}\,\epsilon_k^j\ , \tag{3.15}$$

where $\mathcal{P}(k_1,k_2,k_3)$ indicates permutations without repetitions of k_1, k_2, and k_3 in the preceding term in the summation. To solve Eq. (3.7) to third order, we choose the term of normal form in a minimal form which does not contain any resonant contributions. We fix it as

$$F_j^{[3]} = B_j^{1j2}\, w_1\, w_j\, w_2\ . \tag{3.16}$$

Note that here there is no sum on repeated indexes. In this way, the non-null coefficients for $\mathbf{u}^{[3]}$, in terms of the set $\{w_j\}$, are

$$U_k^{k_1 k_2 k_3} = \left(\sum_i r_i\,\Lambda_i - \lambda_k\right)^{-1} B_k^{k_1 k_2 k_3}\ . \tag{3.17}$$

These non-null coefficients are

$$U_1^{111} = \overline{U_2^{222}} = -\frac{(1-\Omega)\,e^{2i\varphi}}{6\,\Omega^4\,\sigma(\sqrt{\sigma}-1)}\{(1-\Omega^2)\,\Omega[(4\sigma-24\sqrt{\sigma}+45)\,\Omega^2-\sigma+11\sqrt{\sigma}-26] +i\,\sqrt{1-\Omega^2}\,[(-2\sigma+24\sqrt{\sigma}-45)\,\Omega^4+(3\sigma-24\sqrt{\sigma}+47)\,\Omega^2-\sigma+6\sqrt{\sigma}-8]\}, \tag{3.18a}$$

$$U_1^{122} = \overline{U_2^{112}} = -\frac{(1-\Omega)\,e^{-2i\varphi}}{6\,\Omega^4\,\sigma(\sqrt{\sigma}-1)}\{(1-\Omega^2)\,\Omega[(-4\sigma+24\sqrt{\sigma}-3)\,\Omega^2-5\sigma+\sqrt{\sigma}+26] +i\,\sqrt{1-\Omega^2}\,[(-2\sigma+3)\,\Omega^4+(-3\sigma+18\sqrt{\sigma}-31)\,\Omega^2+5(\sigma-6\sqrt{\sigma}+8)]\}, \tag{3.18b}$$

$$U_1^{222} = \overline{U_2^{111}} = -\frac{(1-\Omega)\,e^{-4i\varphi}}{12\,\Omega^4\,\sigma(\sqrt{\sigma}-1)}\{(1-\Omega^2)[(\sigma-19\sqrt{\sigma}+21)\,\Omega^2-\sigma+6\sqrt{\sigma}-8] +i\,\sqrt{1-\Omega^2}\,\Omega\,[(8\sigma-21\sqrt{\sigma}+24)\,\Omega^2-2\sigma+9\sqrt{\sigma}-18]\}. \tag{3.18c}$$

Such coefficients will enable explicit evaluations at frequency 3Ω, in the cross-correlation function for the electric field, at zero order in ξ. Finally, the nonlinear change of variables up to third order is

$$u_j = (w_i + w_{i_1}\, w_{i_2}\, U_i^{i_1 i_2} + w_{i_1}\, w_{i_2}\, w_{i_3}\, U_i^{i_1 i_2 i_3})\,\chi_j^i\ . \tag{3.19}$$

This procedure can be extended to higher orders. For odd orders, according to Refs. [13] and [14], a minimal form for the NFE coefficients must be chosen in (3.2). For even orders the coefficients ought to be zero for the resonances to be eliminated. As will be shown, a complete description of our model in all the parameter space requires at least a fifth order in perturbation theory. Once the NFE coefficients are determined from (3.10) and (3.16), it is possible for (3.2) to be written at third order, explicitly, as

$$\dot{w}_j = w_j[i\Omega + B_j^{1j2}\,|w_j|^2 + \mathcal{O}(|w_j|^4)]\ , \tag{3.20}$$

where $j = 1,2$. Note that $w_1 = \overline{w}_2$, thus only one equation is enough to completely describe the system's dynamics. The cubic coefficients are

$$B_1^{112} = \overline{B_2^{122}} = \alpha + i\beta\ , \tag{3.21}$$

where α and β are real functions of Ω and σ, given by

$$\alpha = -\frac{(1-\Omega)\sqrt{1-\Omega^2}}{3\,\Omega^2\,\sigma(\sqrt{\sigma}-1)}[(5\sqrt{\sigma}-2)\Omega^2-11\sqrt{\sigma}+14]\ , \tag{3.22a}$$

$$\beta = \frac{(1-\Omega)(1-\Omega^2)}{3\,\Omega^3\,\sigma(\sqrt{\sigma}-1)}[(3\sigma-17\sqrt{\sigma}+5)\Omega^2 -5(\sigma-6\sqrt{\sigma}+8)]\ . \tag{3.22b}$$

Near the BP, the system must be described introducing an unfolding parameter μ. Therefore, near the BP the NFE is

$$\dot{w}_1 = (\mu + i\Omega)\,w_1 + (\alpha + i\beta)\,|w_1|^2\,w_1 + \cdots\ . \tag{3.23}$$

In the case of first-order perturbative theory for the periodic terms the unfolding parameter has an explicit expression that will be given in Sec. IV. For $\mu \simeq 0$, the system reaches a stable fixed-point solution (if $\mu < 0$) or a limit-cycle oscillatory solution (if $\mu > 0$ and $\alpha < 0$). From Eq. (3.22a), the latter is reached in the (Ω, σ) parameter space if

$$\sigma < \left(\frac{2(\Omega^2 - 7)}{5\Omega^2 - 11}\right)^2 \qquad (0 < \Omega < 1) . \tag{3.24}$$

If $\mu > 0$ and $\alpha \geq 0$, it is necessary to evaluate the terms up to the fifth order for the NFE. In such a case, a new limit-cycle solution will be reached whenever the real part of the fifth-order coefficient is negative. If it is not, the calculation should be extended two more orders above. Figure 1 shows three regions in the space (κ, γ), according to the sign of α. Alternatively, α can be expressed as a function of γ and κ through Eqs. (2.4a), (2.5), and (2.6). The region designated II corresponds to $\alpha > 0$, and at least the fifth order is necessary to ensure limit-cycle solutions. Region I corresponds to the case $\alpha < 0$. In this region, a third-order theory has been shown to be enough for the cases examined so far. Region I has, however, been subdivided into subregions I-A and I-B; the reason for this is a restriction for the validity of the third-order NFE, as will become clear when the explicit form of the unfolding parameter is the first-order perturbation theory in the periodic driving term, which will be considered in Sec. IV. Note that the point (1,1) should be excluded since it contains the trivial solution. Region III implies losses greater than gains, and will not be treated here.

The analysis so far has common grounds with the one of Ref. [11], because they both use normal-form techniques. It is possible to demonstrate that their slow

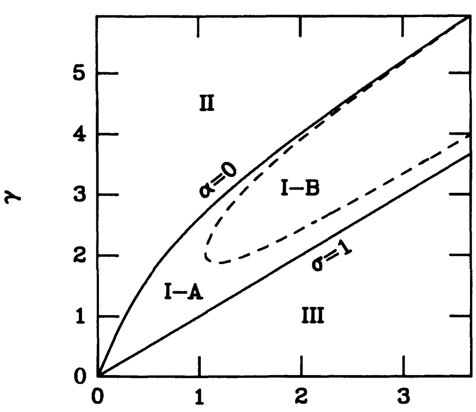

FIG. 1. Parameter space (γ, κ) according to the sign of α for $\mu > 0$. In region II ($\alpha > 0$) at least the fifth order is necessary to ensure limit-cycle solution. Region I (I-A and I-B) corresponds to the case $\alpha < 0$. Here, third-order theory is enough. The dashed curve corresponds to $R = 1$ and is parametrized with ξ. In this case, $\xi = -0.05$. In subregion I-B there is formal convergence of the perturbative series, if $R < 1$. In subregion I-A the normal-form theory is not able to describe the system's dynamical behavior. Region III corresponds to case losses greater than gains. This case is not considered in this work.

variation of the complex amplitudes of the periodic solution and our critical variables are proportional through a combination of the coefficients of the change of base Eq. (2.15a) explicitly defined in Eqs. (2.15b). In these conditions, a redefinition of the coefficients of our NFE (3.23) is necessary in order to compare with the corresponding coefficients of their NFE. The results agree and are consistent in both treatments of the HB, the variation of the coefficients of both NFE with the laser parameter being very much the same up to the third order.

IV. FIRST-ORDER PERTURBATIVE THEORY IN THE PERIODIC TERMS

In order to characterize the dynamics of the critical variables in the vicinity of the BP, it is necessary, as was asserted before, to introduce an unfolding parameter. To get an explicit expression for it, a perturbative theory of at least first order in the parameter ξ is to be developed. Therefore, we include the corrections introduced by the periodic terms to the nonlinear change of variables and the terms added to the NFE. The nonlinear change is now

$$\mathbf{u}(\mathbf{w}, t) = \sum_{r,k=0}^{+\infty} \mathbf{u}^{[k,r]}(\mathbf{w}, t) , \tag{4.1}$$

which replaces Eq. (3.1). Here the supraindexes k, r inside square brackets mean that the terms in brackets below are of order k in ξ and of order r in $\{w_j\}$. The vectors $\mathbf{u}^{[0,r]} \equiv \mathbf{u}^{[r]}$ with ($\mathbf{u}^{[0]} \equiv 0$) are known and time independent, and $\mathbf{u}^{[k,r]}$ ($k > 1$) are time-dependent vectors to be determined. Let us consider Eq. (2.7) written to first order in ξ and the rth order in $\{w_j\}$'s

$$\dot{\mathbf{u}}^{[k,r]} = L\mathbf{u}^{[k,r]} + \mathbf{N}^{[k,r]}(\mathbf{u}) + \delta_{0r}\, \delta_{1k}\, \xi\, \mathbf{D}(t) . \tag{4.2}$$

The NFE is

$$\dot{w}_j = \sum_{r,k=0}^{+\infty} F_j^{[k,r]}(\mathbf{w}, t) , \tag{4.3}$$

where the time-independent functions $F_j^{[0,r]} \equiv F_j^{[r]}$ are known and, besides, $F_j^{[0]} \equiv 0$. The time-dependent functions $F_j^{[k,r]}$ ($k > 1$) are to be determined. When the first-order term in ξ and of order r in $\{w_j\}$ on the right hand side of Eq. (4.1) is considered, we get

$$\dot{\mathbf{u}}^{[1,r]} = \frac{\partial \mathbf{u}}{\partial t}^{[1,r]} + F_j^{[1,r]}\, \boldsymbol{\chi}^j + \Lambda_j\, w_j\, \frac{\partial \mathbf{u}}{\partial w_j}^{[1,r]} + \mathbf{K}^{[1,r]} , \tag{4.4}$$

where

$$\mathbf{K}^{[1,r]} = \sum_{k=0}^{r-1} (\dot{w}_j)^{[1,k]}\, \frac{\partial \mathbf{u}}{\partial w_j}^{[0,r-k+1]} + \sum_{k=1}^{r-1} (\dot{w}_j)^{[0,k+1]}\, \frac{\partial \mathbf{u}}{\partial w_j}^{[1,r-k]} \tag{4.5}$$

Thus, the new equation for determining the periodic functions $F_j^{[1,r]}(\mathbf{w},t)$ and $\mathbf{u}^{[1,r]}(\mathbf{w},t)$ is

$$\left(\Lambda_j\, w_j\, \frac{\partial}{\partial w_j} + \frac{\partial}{\partial t} - L\right)\, \mathbf{u}^{[1,r]} = \mathbf{I}^{[1,r]} - F_j^{[1,r]}\, \boldsymbol{\chi}^j\ , \tag{4.6}$$

where

$$\mathbf{I}^{[1,r]} = \begin{cases} \xi\, \mathbf{D} & \text{if } r = 0 \\ \mathbf{N}^{[1,r]} - \mathbf{K}^{[1,r]} & \text{otherwise.} \end{cases} \tag{4.7}$$

We begin our analysis with this equation at order $r = 0$, rewriting explicitly the change of variable vector as

$$\mathbf{u}^{[1,0]}(\mathbf{w},t) = \xi\, U_j^{(1)}(t)\, \boldsymbol{\chi}^j\ , \tag{4.8}$$

the NFE term as

$$F_j^{[1,0]}(\mathbf{w},t) = \xi\, F_j^{(1)}(t)\ , \tag{4.9}$$

and the driving term as

$$\mathbf{D}(t) = D_j(t)\, \boldsymbol{\chi}^j\ . \tag{4.10}$$

For typographical reasons, we have introduced the supraindex (1) for perturbative order 1 in ξ and order zero in $\{w_j\}$. The equation that determines these two functions is

$$\left(\frac{\partial}{\partial t} - \Lambda_j\right)\, U_j^{(1)} = D_j - F_j^{(1)}\ . \tag{4.11}$$

In the following we present our analysis of two cases.

(a) Nonresonant signal injection: The modulation frequency of the external signal is nonresonant with the Hopf frequency, i.e., $\omega \neq \pm\Omega$. To avoid periodic terms to first order in the NFE, we set

$$F_j^{(1)} \equiv 0\ . \tag{4.12a}$$

Equation (4.11) admits solutions for all j and $k = 0, \pm 1$ given by

$$U_j^{(1)} = \sum_k \frac{D_{j,k}}{ik\omega - \Lambda_j}\, e^{ik\omega t}\ , \tag{4.12b}$$

where

$$D_{j,k} = \frac{1}{T}\int_0^T dt\, D_j(t)\, e^{-ik\omega t}$$

(with $T = 2\pi/\omega$ and $k \in Z$), the Fourier expansion coefficients being

$$D_j(t) = \sum_k D_{j,k}\, e^{ik\omega t}\ .$$

Explicitly, the coefficients $D_{j,k}$ are

$$\begin{aligned} D_{j,0} &= \epsilon_j^1 + \epsilon_j^2\ , \\ D_{j,\pm 1} &= \frac{a}{2}\, D_{j,0}\ , \end{aligned} \tag{4.13}$$

$D_{1,0} = \overline{D_{2,0}}$, $U_1^{(1)} = \overline{U_2^{(1)}}$ and a being proportional to the modulation amplitude [see Eq. (2.1)]. Thus, we can conclude that the spectral contributions in this case are of frequency $0, \pm\omega$.

(b) Resonant injected signal: 1/1 resonance. The modulation frequency is now resonant to the Hopf frequency, i.e., $\omega = \pm\Omega$. In this case it is not possible to eliminate the first-order periodic terms in the NFE in a way consistent with the T-periodic solutions in the nonlinear change of variables. Explicitly, these solutions are

$$F_1^{(1)}(t) = \overline{F_2^{(1)}}(t) = D_{1,1}\, e^{i\Omega t}\ , \tag{4.14a}$$

$$U_1^{(1)}(t) = \overline{U_2^{(1)}}(t) = \tfrac{i}{\Omega}\, D_{1,0} + \tfrac{i}{2\Omega}\, D_{1,-1}\, e^{-i\Omega t}\ , \tag{4.14b}$$

which are the only possible solutions allowed for a T-periodic NFE. Other nonperiodic solutions without a normal-form term would drive the system to a nonstationary regime. From Eq. (4.14b) it is concluded that no new contributions appear in the electric-field spectrum in regions where the system is in a limit-cycle regime. Instead, there will be just an enhancement of the peaks corresponding to the already existing frequencies $0, \pm\Omega$. This analysis is valid up to the perturbative order just considered.

We proceed now with Eq. (4.6) but letting $r = 1$. The interest now lies in adding a multiplicative nonresonant term in the NFE at the BP, with a constant unfolding parameter, in such a way that it allows for limit-cycle or stationary fixed-point solutions. It is therefore necessary that the NFE, with the periodic driving term near the BP considered, should be phase invariant, expressing the periodic term in the change of variables as

$$\mathbf{u}^{[1,1]}(\mathbf{w},t) = \xi\, U_j^{(1)h}(t)\, w_h\, \boldsymbol{\chi}^j\ , \tag{4.15}$$

and the NFE term as

$$F_j^{[1,1]}(\mathbf{w},t) = \xi\, F_j^{(1)h}(t)\, w_h\ . \tag{4.16}$$

Since $\mathbf{I}^{[1,1]} = \mathbf{N}^{[1,1]}(\mathbf{u})$ (where now $\mathbf{u} = \mathbf{u}^{[0,1]} + \mathbf{u}^{[1,0]}$), we put

$$\mathbf{N}^{[1,1]}(\mathbf{w},t) = \xi\, N_j^{(1)h}(t)\, w_h\, \boldsymbol{\chi}^j\ , \tag{4.17}$$

with

$$N_j^{(1)h} = U_i^{(1)}\, [B_j^{hi} + \mathcal{P}(h,i)]\ . \tag{4.18}$$

The resulting equation for $U_j^{(1)h}$ and $F_j^{(1)h}$ to be determined is

$$\left[\frac{\partial}{\partial t} + (\Lambda_h - \Lambda_j)\right]\, U_j^{(1)h} = N_j^{(1)h} - F_j^{(1)h}\ , \tag{4.19}$$

where $N_j^{(1)h}(t)$ admits a Fourier series expansion with coefficient $N_{j,k}^{(1)h}$. As before, two cases need to be considered. The solution of Eq. (4.19) for both cases (and all h, j) is

$$F_j^{(1)h} = N_{j,0}^{(1)h}\, \delta_{hj}\ , \tag{4.20a}$$

$$U_j^{(1)h} = \sum_{k\pm1} \frac{N_{j,k}^{(1)h}}{ik\omega + \Lambda_h - \Lambda_j}\, e^{ik\omega} + \begin{cases} 0 & \text{if } h = j \\ (\Lambda_h - \Lambda_j)^{-1}\, N_{j,0}^{(1)h} & \text{otherwise.} \end{cases} \tag{4.20b}$$

The 1/1 resonance case (b) admits this solution by letting $\omega = \Omega$. The expansion $\{w_j\}$, up to first order, contains spectral contributions at the frequencies $0, \pm\omega$. Note that in the phase-locking or in the limit-cycle regimes, the spectral contributions of the previous order are enhanced. However, this perturbative order is the one that allows for an unfolding parameter to be set to the lowest order, as well as the first correction term in the Hopf frequency. Starting from the NFE equation (3.20), taking into account the expressions, Eqs. (4.14a) and (4.18), of the perturbative analysis to first order in ξ, we obtain the NFE near the BP. It is clear that these contributions are T periodic; hence a periodic normal-form equation (PNFE) is obtained.

The PNFE that describes the system near the BP is

$$\dot{w}_1 = [\mu + i(\Omega + \varpi)]\, w_1 + (\alpha + i\beta)\, |w_1|^2\, w_1 + \xi\, D_{1,1}\, e^{i\Omega t}\, \delta_{\omega\Omega}\,. \tag{4.21}$$

The unfolding parameter μ and the correction to the frequency ϖ to first order in ξ are related to the new contribution, Eq. (4.18), by

$$\mu + i\varpi = \xi\, N_{1,0}^{(1)1}\,, \tag{4.22}$$

with

$$N_{1,0}^{(1)1} = \overline{N_{2,0}^{(1)2}} = U_{1,0}\, B_1^{11} + U_{2,0}\, B_1^{12}\,.$$

As functions of (Ω, σ) they are expressed as

$$\mu = \frac{-3\xi\, \sqrt{1-\Omega^2}}{\Omega\, \sqrt{\sigma(\sqrt{\sigma}-1)}}\, \sin\varphi\,, \tag{4.23a}$$

$$\varpi = \frac{3\xi\, (1-\Omega^2)(\sqrt{\sigma}-2)}{2\Omega^2\, \sqrt{\sigma(\sqrt{\sigma}-1)}}\, \sin\varphi\,. \tag{4.23b}$$

From Eqs. (2.4b) and (2.6), the field phase at the BP is given by $\varphi = -\cot^{-1}\sqrt{1-\Omega^2}$.

The case $\sigma = 4$ ($\varpi = 0$) divides two well defined regions according to the frequency sign [10]. Here we will only consider the case $\sigma < 4$. For $\xi \lesssim 0$ ($\mu > 0$, $\varpi < 0$) and $\alpha < 0$, the stationary solution of the PNFE is a limit cycle of radius $R = \sqrt{-\mu/\alpha}$, and renormalized Hopf frequency $\Omega_r = \Omega + \varpi + \beta R^2$. Since $w_j \sim R$, the expansion of the electric field in the normal modes will be convergent if $R < 1$ (i.e., $\mu < -\alpha$). Note that μ/ξ and α are both functions of Ω and σ only in the first-order theory. In the parameter space (Ω, σ) it is possible to determine a parametrized curve with ξ for $R = 1$. This curve separates two subregions (I-A and I-B in Figs. 1). In subregion I-B $R < 1$, then, there is formal convergence of the perturbative series. In subregion I-A, the normal form theory is not capable of describing the system's dynamical behavior because $R > 1$ and the convergence of the perturbative series is lost. In Fig. 1, both subregions are displayed in the (κ, γ) parameter space. The separatrix (dotted in the figure) collapses to the straight line $\sigma = 1$ to the left and to the curve $\alpha = 0$ to the right when $\xi \to 0^-$. For $\xi \gtrsim 0$ ($\mu < 0$, $\varpi > 0$) and for all α, the system is in the locking region, with electric-field relaxation times approximately equal to μ^{-1}.

V. NTH-ORDER PERTURBATIVE THEORY IN THE RESONANT PERIODIC TERMS

In this section, we shall develop the perturbative theory that gives the effects of resonant periodic excitations with frequencies $\omega = \pm\Omega/n$ ($n \in N - \{0,1\}$). The relevant contributions to the PNFE appear at nth order in ξ and zero order in $\{w_j\}$. For orders less than n, the spectral contributions are of frequencies $\pm k\Omega/n$ ($0 < k \le n-1$), and the PNFE near the BP is essentially the one given by Eq. (4.21). When terms of order n in ξ and order zero in $\{w_j\}$ are considered, it is possible to set the corresponding PNFE. The equation that determines the periodic functions $F_j^{[n,0]}$ and $\mathbf{u}^{[n,0]}$ is

$$\left(\frac{\partial}{\partial t} - L\right)\, \mathbf{u}^{[n,0]} = \mathbf{N}^{[n,0]} - F_j^{[n,0]}\, \boldsymbol{\chi}^j\,, \tag{5.1}$$

where $\mathbf{N}^{[n,0]} = \mathbf{N}(\mathbf{u}^{[1,0]})$ is known if $n \neq 1$. The change in variables can be written as

$$\mathbf{u}^{[n,0]}(\mathbf{w}, t) = \xi^n\, U_j^{(n)}(t)\, \boldsymbol{\chi}^j\,, \tag{5.2}$$

the PNFE term as

$$F_j^{[n,0]}(\mathbf{w}, t) = \xi^n\, F_j^{(n)}(t)\,, \tag{5.3}$$

and

$$\mathbf{N}^{[n,0]}(\mathbf{w}, t) = \xi^n\, N_j^{(n)}(t)\, \boldsymbol{\chi}^j\,, \tag{5.4}$$

which admits a Fourier expansion with coefficients $N_{j,k}^{(n)}$. Here the terms with supraindex (n) mean terms of order n in ξ and zero order in $\{w_j\}$. Equation (5.1) in components is

$$\left(\frac{\partial}{\partial t} - \Lambda_j\right)\, U_j^{(n)} = N_j^{(n)} - F_j^{(n)}\,, \tag{5.5}$$

and your solutions, for $\omega = \pm\Omega/n$ ($n \neq 0, 1$), are

$$F_1^{(n)} = \overline{F_2^{(n)}} = N_{1,n}^{(n)}\, e^{i\Omega t}\,, \tag{5.6a}$$

$$U_1^{(n)} = \overline{U_2^{(n)}} = \sum_{k=-n}^{n-1} \frac{N_{1,k}^{(n)}}{i(\frac{k}{n} - 1)\Omega}\, e^{i\frac{k}{n}\Omega t}\,. \tag{5.6b}$$

The solutions (5.6a) are the only ones that allow a T-period NFE. Other nonperiodic solutions, without normal-form terms, will drive the system to a nonstationary regime. From Eq. (5.6b) it is concluded that there is an enhancement of the peaks at frequencies $\pm k\Omega/n$ ($0 < k < n-1$). In particular, for the limit-cycle region, the Ω peak is enhanced, while in the phase-locking regime, this is a contribution to the spectrum.

The general PNFE for a $1/n$ resonance (for $n \neq 0$) is

$$\dot{w}_1 = [\mu + i(\Omega + \varpi)]\, w_1 + (\alpha + i\beta)\, |w_1|^2\, w_1 + \xi^n\, M^{(n)}\, e^{i\Omega t}\, \delta_{\omega, \frac{\Omega}{n}} \,, \tag{5.7}$$

where

$$M^{(n)} = \begin{cases} N_{1,n}^{(n)} & \text{if } n \neq 1 \\ D_{1,1} & \text{if } n = 1. \end{cases} \tag{5.8}$$

This PNFE allows for a description of the electric-field dynamics if the system has either a resonant or a nonresonant injection. Note that in the case of resonant injection ($\omega = \Omega/n$) its dynamics are yet to be described by the periodic driving terms, though the constant driving terms are not apparent in the PNFE as they are in the original differential equation (2.1).

Note that the PNFE (5.7) in the resonant ($\omega = \frac{\Omega}{n}$) case is not invariant for a global-phase change. In fact, a symmetry breaking occurs, which can be overcome by performing the known perturbative process. In this way, a nonlinear change of variables is obtained, and therefore a NFE. In the rotating-wave approximation, it is possible to make in Eq. (5.7) the following change of variables:

$$w_1 = z\, e^{i(\Omega t + \psi_n)} \,, \qquad \psi_n = \arg(\xi^n\, M^{(n)}) \,. \tag{5.9}$$

The equation for the critical variables (up to third order) is

$$\dot{z} = (\mu + i\varpi)\, z + (\alpha + i\beta)\, |z|^2\, z + G_n \,, \tag{5.10}$$

where $G_n = |\xi^n\, M^{(n)}|$. This is a NFE with a constant driving term, and thus its analysis is similar to the one of previous works [10] for the case of a laser with constant injection. Through the usual procedure, i.e., by linearizing around stationary solutions z_0 and then diagonalizing the linear part, it is possible to find the critical eigenvalues. Hence, the critical modes will be determined. Calling $\{w'_j\}$ the critical variables, the corresponding NFE near the BP is then

$$\dot{w}'_1 = (\mu' + i\Omega')\, w'_1 + (\alpha' + i\beta')\, |w'_1|^2\, w'_1 + \mathcal{O}({w'}_1^5) \,, \tag{5.11}$$

which can be obtained repeating the above described procedure.

VI. STOCHASTIC MODEL

In this section, stochastic temporal fluctuations in the relevant physical parameters will be included. Hence, we shall consider fluctuations both in the amplitude of the injected signal and in the detuning between the laser and the external source for the injected signal. The temporal mean values of these parameters are, however, well defined. The stochastic differential equation (SDE) is

$$\dot{E} = [i + g(E)]\, E + F_b + \xi\, (1 + a\cos\omega t) + \zeta_A + i\zeta_M E \,, \tag{6.1}$$

where the ζ_A and ζ_M are time-dependent random functions, statistically independent, both with zero mean values and nonzero correlation functions. The additive noise ζ_A accounts for fluctuations in the intensity of the injected signal and the multiplicative noise ζ_M in the detuning.

In the white-noise limit, the correlation functions are

$$\langle \zeta_J(t)\zeta_J(t')\rangle = \mathcal{D}_J\, \delta(\tau - \tau') \tag{6.2}$$

for $J = A, M$; $\mathcal{D}_A = \Delta_c^2 T\epsilon_A/\Delta\nu^2$ and $\mathcal{D}_M = \epsilon_M/\Delta\nu^2$, ϵ_A and ϵ_M being the fluctuations in the intensity and in both frequency lasers, respectively. The parameter Δ_c is the free spectral range of the injected laser cavity and T the transmission coefficient of the coupling mirror.

Near the BP ($E = E_b + v$), the SDE (6.1) is

$$\dot{\mathbf{u}} = L_b\mathbf{u} + \mathbf{N}_b(\mathbf{u}) + \xi\, \mathbf{D}(t) + \mathbf{Z}(\mathbf{u}, t) \,. \tag{6.3}$$

The noise term is

$$\mathbf{Z}(\mathbf{u}, t) = (P_j(t) + P_j^k(t)\, u_k)\, \boldsymbol{\epsilon}^j \,, \tag{6.4}$$

where

$$P_1 = \overline{P_2} = \zeta_A + i\, E_b\, \zeta_M \,, \tag{6.5}$$

being the non-null elements of $P_j^k(t)$

$$P_1^1 = \overline{P_2^2} = i\, \zeta_M \,. \tag{6.6}$$

Equation (6.5) can be rewritten as

$$P_1 = \overline{P_2} = p_R + i\, p_I \,, \tag{6.7}$$

where the correlation functions of the real functions p_R and p_I are given by

$$\langle p_R(t)\, p_R(0)\rangle = (\mathcal{D}_A + \mathcal{D}_M\, |E_b|^2\, \sin^2\varphi)\, \delta(t) \,, \tag{6.8a}$$

$$\langle p_I(t)\, p_I(0)\rangle = \mathcal{D}_M\, |E_b|^2\, \cos^2\varphi\, \delta(t) \,, \tag{6.8b}$$

$$\langle p_R(t)\, p_I(0)\rangle = \langle p_R(0)\, p_I(t)\rangle = -\frac{1}{2}\, |E_b|^2\, \mathcal{D}_M\, \sin(2\varphi)\, \delta(t) \,. \tag{6.8c}$$

Noise terms, expressed in the critical base $\{\boldsymbol{\chi}^i\}$ to the first order in the $\{w_j\}$'s, are

$$\mathbf{Z}(\mathbf{w}, t) = [Q_i(t) + Q_i^h(t)\, w_h]\, \boldsymbol{\chi}^i \,, \tag{6.9}$$

with

$$Q_i = P_j\, \epsilon_i^j \tag{6.10}$$

and

$$Q_i^h = \chi_i^j\, P_j^k\, \epsilon_k^h \,. \tag{6.11}$$

Now the general SNFE is

$$\dot{w}_1 = [\mu + i(\Omega + \varpi)]\, w_1 + (\alpha + i\beta)\, |w_1|^2\, w_1 + G_n\, e^{i\psi}\delta_{\omega, \frac{\Omega}{n}} + Q_1(t) + Q_1^1(t)\, w_1 + Q_1^2(t)\, w_2 \,. \tag{6.12}$$

It is noticeable that this equation is non-T periodic due to the noise terms; in fact, Q_1 and $Q_1^2\, w_2$ break the global-phase invariance. In Eq. (6.12) we shall consider the additive noise terms only, because the multiplicative noise terms are of higher order in the $\{w_j\}$. The term considered contains information on the amplitude as well as on the detuning fluctuations. Furthermore, without loss of generality, we can consider the nonresonant case. Should we analyze the other case, a renormalized SNFE is to be worked out, according to Sec. V.

A simple expression for the noise term Q_1 is to be obtained in a particular case. The phase φ at the BP (2.4b) is defined in the interval $(\pi/2, 5\pi/4)$ since $0 < g_b < 1$. Let $g_b \gtrsim 0$ ($\Omega \lesssim 1$); in this case $\varphi = \pi/2 + g_b + \mathcal{O}(g_b^3)$. In the good cavity limit ($\kappa \to 0$, $\gamma \to 0$ but keeping $\sigma = $ const), $g_b \to 0$, then the only non-null correlation function is $\langle p_R(t)\, p_R(0)\rangle$, thus $P_1 \simeq p_R$ and its correlation function will be

$$\langle p_R(t)\, p_R(0)\rangle = (\mathcal{D}_A + \mathcal{D}_M\, |E_b|^2)\, \delta(t)\ . \tag{6.13}$$

From Eq. (6.10)

$$Q_1 = \overline{Q_2} = \nu\, p_R\ , \tag{6.14}$$

it is obtained as $\nu = \epsilon_1^1 + \epsilon_1^2 = \Delta^{-1}\, [\kappa \overline{E_b}^2 + i(1-\Omega)]$. Making the change of variables $w_1 \to w_1\, e^{i \arg \nu}$, Eq. (6.12) turns out to be

$$\dot{w}_1 = [\mu + i(\Omega + \varpi)]\, w_1 + (\alpha + i\beta)\, |w_1|^2\, w_1 + p(t)\ , \tag{6.15}$$

where $p(t) = |\nu|\, p_R(t)$, and its correlation function is

$$\langle p(t)\, p(0)\rangle = \mathcal{P}\, \delta(t)\ , \tag{6.16}$$

with the noise intensity

$$\mathcal{P} = |\nu|^2\, (\mathcal{D}_A + \mathcal{D}_M\, |E_b|^2)\ . \tag{6.17}$$

A well established result [15, 16] states that the only non-null contributions to the stationary correlation function of the w_j^n ($j = 1, 2$) process are the cross correlations, i.e.,

$$C_j^{(n)}(t - t') = \langle w_j^n(t)\, \overline{w}_j^{\,n}(t')\rangle\ , \tag{6.18}$$

with $C_1^{(n)}(\tau) = \overline{C_2^{(n)}}(\tau)$. So, the electric-field cross-correlation function, to order zero in ξ, is

$$\langle E(t)\, \overline{E}(t')\rangle = |E_b|^2 + \sum_{n=1}^{+\infty} c_n^j\, C_j^{(n)}(t - t')\ , \tag{6.19}$$

with

$$c_n^j = \begin{cases} |\chi_1^j|^2 & \text{if } n = 1 \\ |U_k^{j\cdots j}\, \chi_1^k|^2 & \text{if } n > 1, \end{cases} \tag{6.20}$$

where the symbol $j \cdots j$ indicates n-times repetition of the index j. In Figs. 2 and 3 the coefficients c_2^j and c_3^j ($j = 1, 2$) are displayed, respectively. Besides, $c_1^1 = 1 - \Omega^2$ and $c_1^2 = (1 - \Omega)^2$. The first term in Eq. (6.19), i.e., the zero-frequency one, is the electric-field intensity at the BP, and those under the sum sign are the most relevant contributions of frequencies $\pm n\Omega$ to order zero in ξ, it being formally convergent power series in R^2. Thus, up to second order in R, the temporal mean of the electric-field intensity is

$$\langle I(t)\rangle = |E_b|^2 + c_1^j\, C_j^{(1)}(0) + \mathcal{O}(R^4)\ . \tag{6.21}$$

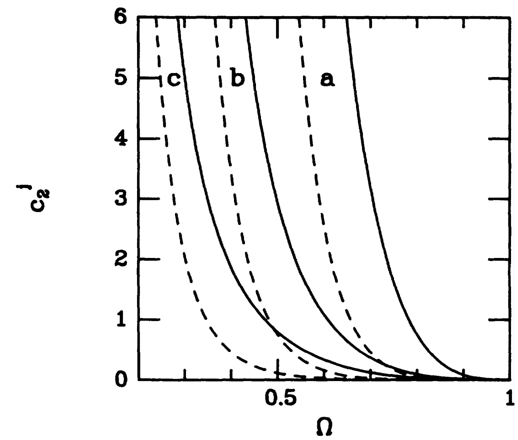

FIG. 2. Coefficients of the cross-correlation function for the electric field at the 2Ω spectral contribution [Eq. (6.19)], as a function of Ω. The solid curves correspond to $j = 1$ and the dashed curves correspond to $j = 2$. The value of σ is (a) 1.2, (b) 2, (c) 3.

In the limit-cycle regime, the electric-field correlation function [Eq. (6.19)] is completely determined by the w_j^n process correlation function [Eq. (6.18)]; then

$$\begin{aligned} C_1^{(n)}(\tau) &= \overline{C_2^{(n)}}(\tau) \\ &= R^2\, e^{in\Omega_r\tau}\, \exp\Big\{ n^2 \Big[-\frac{\mathcal{P}}{4}|\tau| \\ &\quad + \frac{\mathcal{P}}{16R^2}\, (1 + e^{-2R^2|\tau|}) \Big] \Big\}, \end{aligned} \tag{6.22}$$

in particular, the temporal mean of the intensity becomes

$$\langle I(t)\rangle \simeq |E_b|^2 + 2\, R^2\, (1 - \Omega)\, \exp\left(\frac{\mathcal{P}}{8R^2}\right)\ . \tag{6.23}$$

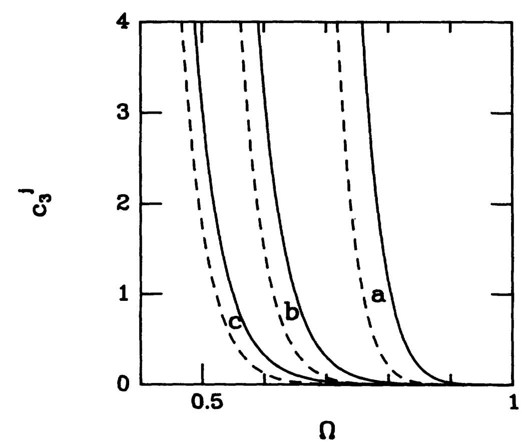

FIG. 3. Coefficients of the cross-correlation function for the electric field at the 3Ω spectral contribution [Eq. (6.19)], as a function of Ω. The solid curves correspond to $j = 1$ and the dashed curves correspond to $j = 2$. The value of σ is (a) 1.2, (b) 2, (c) 3.

It is thus possible to evaluate the signal-to-noise ratio (SNR) from Eq. (6.22) at $\tau = 0$. In this case, for the nth-order contribution, the ratio is

$$(S/N)_n = \frac{C_1^{(n)}}{\mathcal{P}} = \frac{R^2}{\mathcal{P}} \exp\left(\frac{n^2\,\mathcal{P}}{8\,R^2}\right) .$$

It is evident that any maximum occurs for any $\mathcal{P}$ value. A direct calculation shows that the amplification factor, defined in Ref. [17] as the ratio between the output power to the power injected at that frequency, increases monotonically with the noise intensity $\mathcal{P}$.

VII. NUMERICAL SIMULATION

The numerical integration of Eq. (6.1) was done by standard methods [18]. The Stratonovich prescription was used due to the presence of multiplicative noise terms. The transients were carefully avoided in each of the 200 simultaneous realizations that simulate the stochastic process, and those realizations were used to evaluate the Fourier coefficients for each integration step. The power spectral density was obtained through the use of the Wiener-Khintchine theorem.

In Figs. 4(a) and (b) some results for the resonant $\omega = \Omega$ case are shown. In Fig. 4(a), the system is in the limit-cycle regime at peaks of Ω frequency, which come from two sources, the Hopf frequency of the system and the external signal ones. The former have a characteristic width that depends on the limit-cycle radius and on the noise intensity; they become apparent in the zeroth-order perturbative analysis above. The latter contributions enhance the already existing ones. Comparing this situation with the constant-drive one, the spectral peaks in the former are magnified in intensity and better defined in frequency because a narrowing has taken place. In Fig. 4(b), the results of the locking regime are shown. In this case, only the central peak is enhanced, though peaks appear at $n\Omega$ frequencies, as shown in Sec. IV. This phenomenon is what can be called periodic phase locking.

In the nonresonant case, a characteristic spectral feature is the generation of tones, i.e., combination of the exciting frequency and the Hopf frequency. This can be accounted for by the theory; in fact, a perturbative analysis at higher orders in $\{w_j\}$ and ξ allows the general description of the tones. Making an nth-order perturbation in the $\{w_j\}$ and an mth one in ξ (operation indicated as before by the symbol $[n, m]$), we can see that the cross-correlation functions of the w_j^n $(j = 1, 2)$ processes give contributions at frequencies $(-1)^{j+1}(n\Omega + k\omega)$, where $-m \le k \le m$. Note that the $[0, m]$ terms are pure oscillations. In all of these cases, the theory gives a satisfactory answer for the contribution at Ω, because the calculations were made up to first order in ξ.

In Fig. 5, a nonresonant case is displayed, and in its spectrum the system is in the limit-cycle regime. Now the peaks that are related to the Hopf frequency (they are multiples of Ω), and those coming from the external signal ($k\omega$), are clearly displayed. It is also apparent that peaks at combination tones are present, according to the previous analysis. In the locking regime, the situation is similar to the case shown in Fig. 4(b), except that now the peaks are at multiples of the driving frequency.

The values for the oscillation frequency of the system in the limit-cycle regime in the regions where the theory is valid, obtained through the numerical simulation, are coincident with the analytically evaluated Ω_r (see Sec. IV), the relative error being less than 1%. The simulation is robust, since it allows for results even at regions well

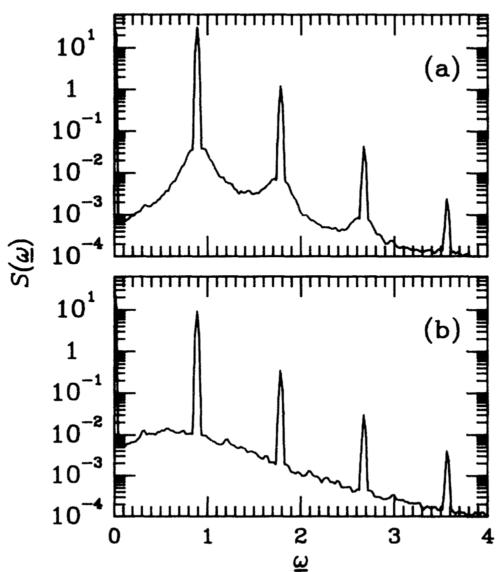

FIG. 4. Power spectral density vs adimensional $\underline{\omega}$ for the resonant case, i.e., $\Omega = \omega$. The parameters are $\sigma = 1.5$, $\mathcal{D}_A = \mathcal{D}_M = 10^{-3}$ in both cases. In part (a) the system is in the limit-cycle regime, $\xi < 0$. In part (b) the system is in the locking regime, $\xi > 0$. Note the peaks at the frequency $n\Omega$. This can be called periodic phase locking.

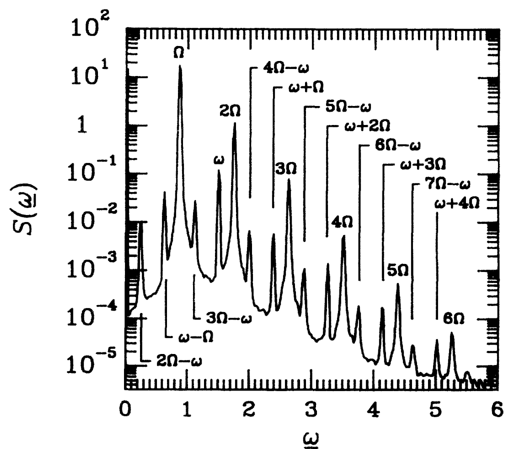

FIG. 5. Power spectral density vs adimensional $\underline{\omega}$ for the nonresonant case, i.e., $\Omega \neq \omega$. The parameters are $\sigma = 1.5$, $\mathcal{D}_A = \mathcal{D}_M = 10^{-3}$. The system is in the limit-cycle regime, $\xi < 0$. Note that the peaks are at the Hopf frequency, at the modulation frequency of the injection, and at the combination tones $(-1)^{j+1}(n\Omega + k\omega)$.

out of the limits of validity of the theory, its results being in qualitative agreement with its predictions.

VIII. CONCLUSIONS

We have investigated a model for a periodically modulated, injected signal laser in the presence of noise. By means of the normal-form theory, we have been able to completely characterize its dynamical behavior, giving an explicit PNFE for both cases of resonant and nonresonant injection, which allows for elimination of the Hopf resonances up to all perturbations orders, and for finally finding an autonomous NFE. A characterization of the unfolding parameter, which allows the system to get to the BP, was also given as well as the system's Hopf frequency Ω_r.

We have presented an analysis of the system with fluctuations in the intensity of the driving field and the detuning. By means of a nonlinear transformation of variables near the BP, we were able to pick the resonances off the original SDE (6.1). In the very low-intensity noise limit, the resonant contributions are principally due to deterministic causes. If the random functions represent colored-noise processes, in a previous paper [10] it was shown that there is a frequency shift and a broadening of the resonant peaks. From the SNFE, it is possible to evaluate the statistical properties of the system, i.e., correlation functions, moments, transition probabilities, etc. We have given explicit expressions for the electric-field correlation function, power spectral density, and thence intensity and peak intensity. It was also shown that no cooperative phenomena between the driving field and the noise exist in the analyzed case, in the limit-cycle regime, as the SNR does not have any maximum for the noise intensity. This is probably due to the fact that the deterministic dynamics for the critical variables are globally phase invariant and the white noise does not structurally break it. We are currently investigating this behavior in the phase-locking regime.

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