Dense configuration of solitons in resonant four-wave mixing

Alexander A. Zabolotskii
Institute of Automation and Electrometry, Siberian Branch of Russian Academy of Sciences, 630090 Novosibirsk, Russia
(Received 18 May 1992; revised manuscript received 14 March 1994)

A set of integrable four-wave mixing models arising in a medium with a resonant transition is presented. A suitable modification of the inverse scattering transform is constructed and used to find a common one-phase periodic solution of the evolution equations. The Whitham equations describing the dynamics of slowly changing parameters are found. Their solution is used for a description of the evolution of light pulses with sharp leading edges. The possible application of the results for the analysis of the evolution of dense packets of solitons is discussed.

PACS number(s): 42.65.Hw, 42.65.Dr, 42.65.Re

I. INTRODUCTION

The description of long intense pulses in nonlinear media is one of the interesting and important problems of theoretical physics. Such nonlinear optical phenomena as Raman scattering (RS), two-photon propagation (TPP), near resonance four-wave mixing (FWM), and the degenerate case of FWM–phase-wave conjugation have been studied in a large number of publications (see, for example, [1]). The above mentioned nonlinear phenomena, if they take place in a nondissipative, nondispersive medium having its own frequency approximately equal to the sum or difference of carrying frequencies of one or two pairs of interacting wave packets, are described by mathematically close models [2–5] . A very important property of these models arising under a set of assumptions is the integrability by means of the inverse scattering transform (IST) [6]. The IST has been effectively used for the analysis of solitonic and self-similar decay-type regimes of stimulated RS, TPP, and FWM. Note that Raman solitons have been discussed previously by several authors, who found the simplest special solutions, for example [7]. The N-soliton family of the stimulated RS model was found by Meinel [8]. Solitary waves are also known for some special FWM models [9]. The simplest periodic solutions of evolution equations with periodic boundary conditions are found for some special cases of the RS and FWM models [10].

The utility of exact mathematical methods results, to a large extent, from the practical importance of these isolated, localized nonlinear pulses and precise yet simple description of these pulses. However, physical situations arise which involve a high density of solitons, for example, finite length nonlinear oscillators such as Josephson ones or propagations of shocks in a conservative lattice [11,12] and plasma, see [6], Chap. 4. Numerical simulations demonstrate that the sharp leading front of the shock wave is smoothed by a high frequency nonlinear wave train which may be interpreted as a dense configuration of many solitons, see Refs. [11–13]. It is also demonstrated that a long intense pulse having a sharp leading edge splits asymptotically into a train of solitons or breathers. This is a common feature of different integrable models, both dispersive and nondispersive. It is natural to suggest the existence of the same phenomena in the interaction of power wave packets described by means of an integrable version of the nondispersive FWM models. We shall call corresponding intense pulses with a sharp leading front propagating in nondispersive media nondispersive shock waves (NSW). Note that the effective lengths of nonlinear interaction in experimental observation of the nonlinear stage of four-wave interactions are not usually enough to observe the complete decay of NSW into a set of isolated solitons (breathers). Therefore analysis of the intermediate stage, the dynamics of which qualitatively differs from that of isolated pulses, is required. Owing to overlapping of solitons in a dense packet their behavior in a dense packet strictly differs from that in isolation. So, the approaches used in the papers cited above [2–9] and related theories cannot be applied for analysis of NSW dynamics.

Investigation of nonlinear dynamics of densely packed solitons is usually intractable analytically, because they involve a large number degrees of freedom of solitons. However, the problem can be solved approximately, following the ideas of Whitham [14]. The first stage consists in construction of the periodic solutions of the system of evolution equations under periodic boundary conditions. Then, it is assumed that this periodic solution possesses slowly (in comparison with the period of oscillations or solitonic length and duration) changing parameters. Averaging of some integrals over the period of fast oscillations yields a set of equations in partial derivatives for these parameters. Solution of this system allows one to study the slow modulation of the solitonic train. Such so-called Whitham equations have been used for description of shocks in the dispersive media [6,15]. Unfortunately, this approach leads, in common cases, to tremendous calculations for each new model and solution. In the works of Date, Forest, McLaughlin, Novikov, Flaschka, Krichever, and co-workers [6,16–19] the exact methods of solution of evolution equations under periodic boundary conditions have been developed and used for construction of a common quasiperiodic solution of integrable models. These authors studied Korteveg–de Vries [16,17], sine-Gordon [18], and nonlinear Schrödinger equations [19],

1050-2947/94/50(4)/3384(10)/$06.00

and demonstrated that the exact methods allow one to find the Whitham equations directly in a diagonal form.

We shall follow an approach of Date, Forest, McLaughlin and co-workers, Refs. [16–18], which is more convenient for our purposes and transparent than that of the authors of Refs. [6,19] . The approach of these authors will be suitably modified for integrable FWM models. A set of new common periodic solutions will be found here. Whitham equations will be constructed as well. Their solution will be used for analysis of dynamics of dense configuration of solitons near the leading edge of NSW. Note that these results are obtained first not only for FWM phenomena, but for stimulated RS, TPP, phase-conjugation, and close nonlinear models also.

The second section is devoted to description of the physical models under consideration. In the next section the exact method is suitably modified and used for construction of periodic solutions. The Whitham equations are derived in the fourth section. The behavior of solitons forming near the leading edge of steplike pulses is analyzed in Sec. V . Discussion of results and their possible application are presented in the last part of this paper.

II. BASIC EQUATIONS

Let four light packets propagate in the medium along the z axis,

$$E(z,t)=\sum_{j=1}^{2}\{P_j\exp[i(k_jz-\Omega_jt)] \\ +S_j\exp[i(l_jz-\omega_jt)]\}+\text{ c.c.} \tag{2.1}$$

Here P_j, S_j are the envelopes, Ω_j, ω_j are the carrying frequencies, and k_j, l_j are the carrying vectors of packets, respectively. We shall consider the special cases of the FWM models, which lead to integrable systems of evolution equations—the resonant interaction of field (2.1) with the molecular transitions. Note that equivalent systems appear in other physical situations as well.

So, consider the interaction of two pairs of counter-propagating waves with the following resonant conditions:

$$\Omega_j\mp\omega_j=\omega_0+\nu, \tag{2.2}$$

where detuning $\nu\ll\omega_0,\omega_j,\Omega_j,\ j=1,2$, see Fig. 1. Following Giordmine and Kaiser [20], the coupled wave equations, describing the scattering of light from phonons during the FWM, will be derived. Let us consider a lattice of fixed noninteracting ions, sufficiently diluted as to require no local corrections. The interaction of light with the single phonon mode will be described in the harmonic-oscillator approximation with external force. The equation is

$$\partial_t^2q+\Gamma\partial_tq+\omega_0^2=\kappa_{ij}E_iE_jM^{-1}. \tag{2.3}$$

The macroscopic polarization, which is proportional to $\kappa_{ij}E_jq$, includes only a nonlinear component of polarization associated with the variation of q . κ_{ij} is a scattering tensor. Consider only the isotropic case, i.e., κ_{ij} is proportional to δ_{ij}. Γ is a phenomenological relaxation constant and M is the ion mass. The assumption is that the time scale of nonlinear processes and detuning ν are such that one may neglect the relaxation and adiabatically eliminate q from Eq. (2.3). Substituting this expression for q in the Maxwell equations one obtains an equation which in rotating wave and slow envelope approximations is transformed to the system of equations presented below. For a scheme of interaction (a) in Fig. 1 we have the following equations for the field envelopes:

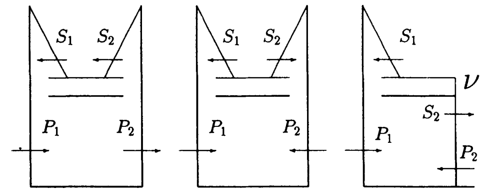

FIG. 1. The schemes of FWM in the two-level medium. The horizontal lines denote the energy levels. The thick line corresponds to the envelope P_i, carrying frequency Ω_i, carrying wave vector k_i, group velocity $\pm V_i$ (upper sign corresponds to propagation of light from left to the right and vice versa, as it is shown by arrows) and thin line corresponds to respective quantities S_i, ω_i, l_i, $\pm U_i$. The index $i=1$ ($i=2$) denotes right (left) pair.

$$\begin{aligned}(\partial_z+V_1^{-1}\partial_t)P_1&=i[\alpha_{11}P_1|S_1|^2+\alpha_{12}P_2S_2^*S_1\exp(i\Delta z)],\\ (\partial_z+V_2^{-1}\partial_t)P_2&=i[\beta_{11}P_2|S_2|^2\\ &\quad+\beta_{12}P_1S_1^*S_2\exp(-i\Delta z)],\\ (\partial_z-U_1^{-1}\partial_t)S_1&=i[\alpha_{22}S_1|P_1|^2\\ &\quad+\alpha_{21}S_2P_2^*P_1\exp(-i\Delta z)],\\ (\partial_z-U_1^{-1}\partial_t)S_2&=i[\beta_{22}S_2|P_2|^2+\beta_{21}S_1P_1^*P_2\exp(i\Delta z)],\end{aligned} \tag{2.4}$$

where $\Delta=k_2-l_2-k_1+l_1$, Δ is a wave detuning, $V_{1,2}$, $U_{1,2}$ are group velocities,

$$\begin{aligned}\alpha_{11}&=\frac{2\pi\Omega_1^2n_0N_0}{k_1\hbar\nu}|\kappa(\Omega_1)|^2,\ \alpha_{22}=\frac{2\pi\omega_1^2n_0N_0}{l_1\hbar\nu}|\kappa(\Omega_2)|^2,\\ \alpha_{12}&=\frac{2\pi\Omega_1^2n_0N_0}{k_1\hbar\nu}\kappa(\Omega_1)^*\kappa(\Omega_2),\\ \alpha_{21}&=\frac{2\pi\omega_1^2n_0N_0}{l_1\hbar\nu}\kappa(\Omega_2)^*\kappa(\Omega_1),\end{aligned} \tag{2.5}$$

where N_0 is the atomic density, and n_0 is the constant difference between energy populations of levels. One gets analogous expressions for constants β_{ij} after replacements in formulas (2.5):

$$\Omega_1\leftrightarrow\Omega_2,\ \omega_1\leftrightarrow\omega_2,\ k_1\leftrightarrow k_2,\ l_1\leftrightarrow l_2.$$

We assume that $\kappa_{ij}=\delta_{ij}\kappa$ and κ is real. Analogous systems of evolution equations can be obtained for schemes (b) and (c) in Fig. 1. Thus for (b) the equations are

$$\begin{aligned}
&(\partial_z + V_1^{-1}\partial_t)P_1 = i[\alpha_{11}P_1|S_1|^2 + \alpha_{12}P_2S_2^*S_1\exp(i\Delta z)],\\
&(\partial_z + U_1^{-1}\partial_t)S_2 = i[\beta_{22}S_2|P_2|^2 + \beta_{21}S_1P_1^*P_2\exp(i\Delta z)],\\
&(\partial_z - U_1^{-1}\partial_t)S_1 = i[\alpha_{22}S_1|P_1|^2 + \alpha_{21}S_2P_2^*P_1\exp(i\Delta z)],\\
&(\partial_z - V_2^{-1}\partial_t)P_2 = i[\beta_{11}P_2|S_2|^2\\
&\qquad +\beta_{12}P_1S_1^*S_2\exp(-i\Delta z)],
\end{aligned} \tag{2.6}$$

where $\Delta = k_2 - l_2 - k_1 + l_1$. For (c) the equations are

$$\begin{aligned}
&(\partial_z + V_1^{-1}\partial_t)P_1 = i[\alpha_{11}P_1|S_1|^2 + \alpha_{12}P_2S_2S_1\exp(i\Delta z)],\\
&(\partial_z + U_2^{-1}\partial_t)S_2 = i[\beta_{22}S_2|P_2|^2\\
&\qquad +\beta_{21}S_1^*P_2^*P_1\exp(-i\Delta z)],\\
&(\partial_z - U_1^{-1}\partial_t)S_1 = i[\alpha_{22}S_1|P_1|^2 + \alpha_{21}S_2^*P_2^*P_1\exp(i\Delta z)],\\
&(\partial_z - V_2^{-1}\partial_t)P_2 = i[\beta_{11}P_2|S_2|^2\\
&\qquad +\beta_{12}P_1S_1^*S_2^*\exp(-i\Delta z)],
\end{aligned} \tag{2.7}$$

where $\Delta = k_2 + l_2 - k_1 + l_1$. It is enough to restrict the consideration to these cases only. This is due to the fact that only they are mathematically nonequivalent. Other integrable models of the FWM differ from those considered above by the choice of resonance conditions (2.2) and signs of group velocities of waves. All of them may be transformed to one of the three models presented above after simple redesignation of the field envelopes and variables. Note that some combinations of resonance conditions and signs of velocities lead to nonintegrable variants of the FWM. See, for more details, Sec. V of this paper.

The exact method application to the system under consideration is based on the existence of the so-called Lax representation of this system. This imposes the restriction on the group velocities, i.e., that the fields propagating in the same direction are equal to each other. This means that refractive index $n(\omega)$ does not depend upon the frequency, more exactly $n(\omega_1) = n(\omega_2) = n(\Omega_1) = n(\Omega_2)$ (the medium is nondispersive). It is easy to demonstrate using resonance conditions (2.2) and schemes presented in Fig. 1 that for a nondispersive medium $\Delta = 0$.

All these integrable systems of equations can be written as the following set of equations characterized by the integer parameters $\delta = \pm 1$, $\epsilon = \pm 1$:

$$\begin{aligned}
&\partial_T R_+ = i[\zeta_1 R_+ F_3 + \delta R_3 F_+ \exp(i\Delta z)],\\
&\partial_T R_3 = -\partial_X F_3\\
&\qquad = i/2[R_+F_-\exp(-i\Delta z) - R_-F_+\exp(i\Delta z)],\\
&\partial_X F_+ = i[\zeta_2 F_+ R_3 + \epsilon F_3 R_+ \exp(-i\Delta z)],
\end{aligned} \tag{2.8}$$

where

$$F_- = F_+^*,\ R_- = R_+^*,\ \partial_\eta = \partial_z + V^{-1}\partial_t,$$

$$\partial_\xi = \partial_z - V^{-1}\partial_t,$$

$$a^2 = \frac{\alpha_{11}\alpha_{22}}{\beta_{11}\beta_{22}},\quad u = \frac{\kappa(\Omega_1)}{\kappa(\Omega_2)}.$$

For scheme (a) in Fig. 1 we have also

$$V = V_{1.2} = U_{1,2},\ \delta = \epsilon = 1,$$

$$s_1 = S_1,\ p_1 = P_1,\ s_2 = \mp S_2(\beta_{11}/\alpha_{11})^{1/2},$$

$$p_2 = \mp P_2(\beta_{22}/\alpha_{22})^{1/2},$$

$$\zeta_1 = -\frac{u^2 + a^2}{2au},\quad \zeta_2 = -\frac{1 + u^2a^2}{2au},$$

$$T = \alpha_{11}\int_{-\infty}^{\eta} I_1(\eta)\,d\eta,\quad X = \alpha_{22}\int_{-\infty}^{\xi} I_2(\xi)\,d\xi,$$

$$I_1(\eta) = u/a\,|s_1|^2 + a/u\,|s_2|^2,$$

$$I_2(\xi) = au\,|p_1|^2 + 1/(au)|p_2|^2,$$

$$F_3 = (a/u\,|s_2|^2 - u/a\,|s_1|^2)/I_1,$$

$$R_3 = (ua\,|p_2|^2 - 1/(ua)\,|p_1|^2)/I_2,$$

$$F_+ = 2s_1s_2^*/I_1\exp\{i[X(u^2 - a^2) + T(1 - u^2a^2)]/(2au)\},$$

$$R_+ = 2p_1p_2^*/I_2\exp\{i[X(u^2 - a^2) + T(1 - u^2a^2)]/(2au)\}.$$

For scheme (b) in Fig. 1, T, $V_{1.2}$, and $U_{1,2}$ are the same as above, $\delta = \epsilon = -1$,

$$s_i = S_i,\ p_i = \pm P_i(\alpha_{jj}/\beta_{jj})^{1/2},\ i \neq j,\ i,j = 1,2;$$

$$\zeta_1 = -\frac{1 + a^2}{2au},\ \zeta_2 = -\frac{(1 + a^2)u}{2a},$$

$$X = \beta_{11}\int_{-\infty}^{\xi} I_2(\xi)\,d\xi,$$

$$I_1(\eta) = 1/a|p_1|^2 - a|s_2|^2,\ I_2(\xi) = a|p_2|^2 - 1/a|p_2|^2,$$

$$F_3 = -u(a|s_2|^2 + 1/a|p_1|^2)/I_1,$$

$$R_3 = -(1/a|s_1|^2 + a|p_2|^2)/(uI_2),$$

$$F_+ = 2up_1s_2(I_1)^{-1}\exp[i(X + T)(1 - a^2)/(2a)],$$

$$R_+ = 2p_2s_1(I_2u)^{-1}\exp[i(X + T)(1 - a^2)/(2a)].$$

For scheme (c) in Fig. 1, T, X, s_i, p_i, V_i, $U_{1,2}$ are the same as above, $\delta = -\epsilon = 1$,

$$\zeta_1 = u\frac{a^2-1}{2a},\ \zeta_2 = \frac{a^2-1}{2au},$$

$$I_1(\eta) = 1/a|s_1|^2 - a|p_2|^2,$$

$$I_2(\xi) = 1/a|p_1|^2 + a|s_2|^2,$$

$$F_3 = -(a|p_2|^2 + 1/a|s_1|^2)/(uI_1),$$

$$R_3 = u(a|s_2|^2 - 1/a|p_1|^2)/I_2,$$

$$F_+ = 2p_2s_1/(uI_1)\exp[-i(X+T)(1+a^2)/(2a)],$$

$$R_+ = 2up_1s_2^*/(I_2)\exp[-i(X+T)(1+a^2)/(2a)].$$

Note that for the special case of initial conditions, such that $I_1F_3(T,0) = \text{const}$ [for example, for constant initial envelopes $P_{1,2}$ in case (a)], nonzero detuning Δ may be avoided by using simple gauge transform

$$R_\pm \to R_\pm \exp\left(\pm i\zeta_0 \int_0^X R_3(Y,T)\,dY\right),$$

$$F_\pm \to F_\pm \exp\left(\pm i\zeta_0 \int_0^X R_3(Y,T)\,dY\right),$$

which results in renormalization of coupling constants characterized by the nonlinear frequency modulation:

$$\zeta_{1,2} \to \zeta_{1,2} \pm \zeta_0,\ \zeta_0 = \Delta[\alpha_{11}I_1F_3(T,0)]^{-1}.$$

The special case of the model described above, $\zeta_{1,2} = \delta = \epsilon = 1$, corresponds to the phase-conjugate or degenerate four-wave mixing model, Ref. [1]. For $u^2 = 1$, $\Delta = 0$, $\delta = 1$, $\epsilon = \pm 1$ system (2.8) describes the stimulated RS (upper sign of ϵ) and TPP (lower sign of ϵ). The important special case of the model above is that of counterpropagation of two waves having two orthogonally polarized components, so that envelopes $P_{1,2}$ denote polarization components of one wave and $S_{1,2}$ correspond to that of another wave. In such a model required integrability conditions imposed on the group velocities and wave detuning ($\Delta = 0$) are fulfilled automatically. Note that for all considered schemes of interaction (pseudo)spins $\vec{F}$, $\vec{R}$ satisfied the following normalization conditions:

$$\delta|R_+|^2 + R_3^2 = 1, \tag{2.9}$$
$$\epsilon|F_+|^2 + F_3^2 = 1. \tag{2.10}$$

III. CONSTRUCTION OF PERIODIC SOLUTION OF THE EVOLUTION EQUATIONS

In this section we shall present a modification of the approach of the authors of Refs. [16–18], which is more convenient for our aims than alternatives in the literature. Although we restrict present consideration to the one-phase solution, see Refs. [6,19], the generalization to more common cases is straightforward.

As was first shown in Ref. [5] system (2.8) can be presented in the form of the compatibility condition of two systems of linear equations (Lax representation):

$$\partial_y\Phi = \begin{pmatrix} -i(\delta\lambda-\zeta_1)F_3 & (\lambda+\varphi)F_+ \\ -\epsilon(\lambda+\bar{\varphi})F_- & i(\delta\lambda-\zeta_1)F_3 \end{pmatrix}\Phi, \tag{3.1}$$

$$\partial_x\Phi = \frac{1}{\lambda}\begin{pmatrix} i(\lambda\zeta_2-\epsilon)R_3 & (\lambda+\varphi)R_+\delta\epsilon \\ -\delta(\lambda+\bar{\varphi})R_- & i(\lambda\zeta_2-\epsilon)R_3 \end{pmatrix}\Phi, \tag{3.2}$$

where λ is a spectral parameter [6]. Φ is a two-component function. $\varphi = \alpha + i(\delta\epsilon-\alpha^2)^{1/2}$, $\bar{\varphi} = \alpha - i(\delta\epsilon-\alpha^2)^{1/2}$, $\alpha = -\delta(\zeta_1+\zeta_2)/2$. For special limiting cases ($\zeta_1 = \zeta_2$, $\delta = 1$) this Lax pair is formally equivalent to that obtained by Kaup [2] for RS ($\epsilon = 1$) and TPP ($\epsilon = -1$).

We introduce, following an approach of Forest and McLaughlin [12], quadratic eigenfunctions

$$f = -i/2(\phi_1\psi_2 + \phi_2\psi_1),\quad g = \phi_1\psi_1,\ h = \phi_2\psi_2, \tag{3.3}$$

where $\phi_{1,2}$ and $\psi_{1,2}$ denote different solutions of linear systems (3.1), (3.2). These functions are easily shown to satisfy the system

$$\begin{aligned}
&\partial_y f = -i[(\lambda+\varphi)F_+h - \epsilon(\lambda+\bar{\varphi})F_-g],\\
&\partial_x f = -i/\lambda[(\lambda+\varphi)R_+h - \epsilon(\lambda+\bar{\varphi})R_-g],\\
&\partial_y g = -2i[(\delta\lambda-\zeta_1)F_3g - (\lambda+\varphi)F_+f],\\
&\partial_x g = 2i/\lambda[(\lambda\zeta_2-\epsilon)R_3g + (\lambda+\varphi)R_+f],\\
&\partial_y h = 2i[(\delta\lambda-\zeta_1)F_3h - \epsilon(\lambda+\bar{\varphi})F_-f],\\
&\partial_x h = -2i/\lambda[(\lambda\zeta_2-\epsilon)R_3h + \epsilon(\lambda+\bar{\varphi})R_-f].
\end{aligned} \tag{3.4}$$

As it immediately follows from (3.4) the function

$$P(\lambda) = f^2 + gh \tag{3.5}$$

is independent of both variables, i.e., $\partial_y P(\lambda) = 0$, $\partial_x P(\lambda) = 0$. The form of periodic solution is determined by dependence of P on λ, which in turn is determined by the symmetry properties of a nonlinear system. The simplest (one-phase) nontrivial periodic solution corresponds to the following form of polynomial P:

$$P(\lambda) = \prod_{k=1}^{4}(\lambda-\lambda_k) = \sum_{j=0}^{4}P_j\lambda^j, \tag{3.6}$$

where λ^j denotes the jth degree of complex parameter. λ_j are some spectral data. It is natural to suppose, that functions f, g, h satisfying system (3.4) may be presented in polynomial form. It may be shown that functions f, g, and h should have the form

$$f = \sum_{k=0}^{2} f_k\lambda^k,\ g = (\lambda+\varphi)(g_0+g_1\lambda),$$
$$h = (\lambda+\bar{\varphi})(h_0+h_1\lambda). \tag{3.7}$$

Substituting these expansions in system (3.4) we found

$$F_+h_1 = \epsilon F_- g_1,\ R_+h_0 = \epsilon R_- g_0,$$
$$F_3 g_1 = \delta F_+ f_2,\ F_3 h_1 = \delta\epsilon F_- f_2, \quad (3.8)$$
$$R_3 g_0 = \delta R_+ f_0,\ R_3 h_0 = \delta\epsilon R_- f_0.$$

From (3.5)–(3.7) we have for the zero and fourth degrees of λ

$$g_1 = F_+,\ g_0 = \delta\epsilon\sqrt{P_0}R_+,$$
$$h_1 = \epsilon F_-,\ h_0 = \delta\sqrt{P_0}R_-, \quad (3.9)$$
$$f_2 = \delta F_3,\ f_0 = \epsilon\sqrt{P_0}R_3,$$

where P_4 is chosen equal to 1 without restriction of results.

Introduce the function

$$\mu(\lambda, x, y) = -\sqrt{P_0}R_+/F_+, \quad (3.10)$$

which is a zero of $g(\lambda)$, as it easily follows from (3.9), i.e.,

$$g = (\lambda + \varphi)(\lambda - \mu)F_+.$$

Substituting this expression of g in system (3.4) and using the fact that $f^2(\lambda = \mu) = P(\mu)$, we obtain

$$\partial_y g(\lambda = \mu) = 2i(\mu + \varphi)F_+ P^{1/2}(\mu)$$

and close equality for $\partial_x g$. We have from these equalities

$$\partial_y \mu = 2i[P(\mu)]^{1/2}, \quad (3.11)$$

$$\partial_x \mu = -2i\delta\epsilon[P(\mu)/P_0]^{1/2}. \quad (3.12)$$

From (3.11), (3.12) it follows that μ depends on the single variable θ,

$$\partial_\theta \mu = i[P(\mu)]^{1/2},\ \theta = 2\left(x - \delta\epsilon\frac{y}{\sqrt{P_0}}\right). \quad (3.13)$$

This solution corresponds to the simplest (one-phase) periodic solution of the model. From (3.8), (3.13) it follows that both F_3 and R_3 depend on a single variable θ and

$$F_3 = \sqrt{P_0}R_3 + A, \quad (3.14)$$

where A is a real constant. The condition of existence of the finite polynomial decomposition (3.7) imposes, in common, some constraints on a set of spectral data λ_k, $k = 1 - 4$. These constraints can be found from the above equalities (3.5)–(3.7). From expansion of polynomial P in degrees of λ follows algebraic relations

$$2f_0 f_1 - \delta\epsilon\varpi + 2\alpha J|\mu|^2 = P_1,$$
$$2f_0 f_2 + f_1^2 + \delta\epsilon J + J|\mu|^2 - 2\alpha\varpi = P_2, \quad (3.15)$$
$$2f_2 f_1 - \varpi + 2\alpha J = P_3,$$

where $J = \epsilon|F_+|^2$, $\varpi = J(\mu + \mu^+)$. Algebraic system (3.15) yields some constraints on relations between functions. An analogous relation follows from Eqs. (3.10) and (3.14) and (2.9), (2.10),

$$2F_3 A = 1 + A^2 - J(1 - \delta\epsilon|\mu|^2) - P_0. \quad (3.16)$$

Comparing (3.15) and (3.16) one can find that to avoid possible contradictions the following equations should be satisfied:

$$A^2 = P_0 - \delta\epsilon P_2 + 2\alpha\delta\epsilon(2\alpha - P_3) + 1, \quad (3.17)$$

$$\delta\epsilon P_1 - P_3 = 2\alpha[\delta\epsilon P_2 + 2\alpha\delta\epsilon(2\alpha - P_3) - 2]. \quad (3.18)$$

We have $f_1 = 2\alpha\delta F_3$, as well. The first constraint (3.17) may be considered as a definition of constant A. Thus, only one constraint (3.18) remains. A periodic solution may be obtained directly from the relations presented above. However, it is still rather complex for further analysis and physical applications. Therefore let us consider the special cases of a polynomial P:

$$P^{(1)}(\mu) = (\mu^2 + 2\alpha\mu + \delta\epsilon)(\mu + \xi_1)(\mu + \xi_2),\ A = 0,$$
$$P^{(2)}(\mu) = [(\mu + \alpha)^2 - \lambda_1^2][(\mu + \alpha)^2 - \lambda_2^2],$$
$$A^2 = \prod_{j=1}^{2}(\alpha^2 - \lambda_j^2 - \delta\epsilon),$$
$$P^{(3)}(\mu) = \prod_{j=1}^{2}(\mu^2 + 2\lambda_j\mu + 1),\ A^2 = -4\lambda_1\lambda_2\delta\epsilon,$$

where $\xi_{1,2}$ are arbitrary complex and $\lambda_{1,2}$ are pure real or pure imaginary, or complex conjugate constants. All roots of polynomial $P^{(k)}$ satisfy the above constraints. It means that, in common, only two roots are independent. To solve Eq. (3.13) using handbooks (for example, [21]) it is convenient to use the Table presented in the Appendix, where the relations between the form of polynomial $P^{(1-3)}$ and that of the polynomial depending on only real parameter μ and real roots is shown.

Functions F_3, R_3, $|R_+|$, $|F_+|$ are easily expressed in the terms of A, P_0, $\mu(\theta)$ by means of (3.14), (2.9), (2.10). For example, for F_3 we have

$$F_3 = \frac{A \pm [P_0 + \delta\epsilon(A^2 - P_0 - 1)|\mu|^2 + |\mu|^4]^{1/2}}{1 - \delta\epsilon|\mu|^2}. \quad (3.19)$$

An inversion of integral

$$\int_{\mu_0}^{\mu}[P(\mu)]^{-1/2}\,d\mu = i(\theta - \theta_0) \quad (3.20)$$

may be expressed in terms of Jacobian functions by means of Table I presented in the Appendix and a handbook. For $A = 0$ (case $P^{(1)}$) (3.19) reduces to

$$F_3 = \pm\left(\frac{P_0 - \delta\epsilon|\mu|^2}{1 - \delta\epsilon|\mu|^2}\right)^{1/2}. \quad (3.21)$$

Consider the special case (a) of $P^{(1),(A)}$, see Appendix. Let $\mu = -\alpha + i\eta$, $\lambda_1 = -\lambda_2 = \gamma$, $\lambda_3 = -\lambda_4 = \beta$, here function η and constants α, β, γ are real. An integral (3.20) for $\mu_0 = -\alpha$ yields

$$\mu = -\alpha + i\beta\,\mathrm{sn}(\gamma\theta, \beta/\gamma),\quad \beta < \gamma.$$

From (3.19) we have

$$F_3 = \frac{A \pm [(2\alpha^2 - \beta^2 \mathrm{cn}^2)(2\alpha^2 - \gamma^2 \mathrm{dn}^2) + 4\alpha^2(\alpha^2 + \beta^2 \mathrm{sn}^2)]^{1/2}}{1 - \delta\epsilon(\alpha^2 + \beta^2 \mathrm{sn}^2)}, \tag{3.22}$$

where $\mathrm{sn} = \mathrm{sn}(\gamma\theta, \beta/\gamma)$, $\mathrm{cn} = \mathrm{cn}(\gamma\theta, \beta/\gamma)$, $\mathrm{dn} = \mathrm{dn}(\gamma\theta, \beta/\gamma)$ are the Jacobian functions with modulus β/γ, Ref. [20]. For the special case of FWM (considered, for example, in Ref. [10]) $\alpha = 0$, expression (3.22) reduces to

$$F_3 = \frac{A \pm \gamma\beta \mathrm{cn}\,\mathrm{dn}}{1 - \delta\epsilon\beta^2 \mathrm{sn}^2}.$$

The form of polynomial P corresponding to case (a) of $P^{(1),(A)}$ in Table I leads to the simple solution

$$F_3 = \pm\beta \frac{\mathrm{cn}}{\mathrm{dn}},\ R_3 = \pm\frac{\mathrm{cn}}{\mathrm{dn}},$$

$$F_+ = \pm\frac{(1-\beta^2)^{1/2}}{\mathrm{dn}} \exp(i\vartheta_1),$$

$$R_+ = \pm\frac{(1-\beta^2)^{1/2}\,\mathrm{sn}}{\mathrm{dn}} \exp(i\vartheta_2),$$

$$\vartheta_{1,2}(\theta) = \vartheta_{1,2}(0) \pm \zeta_{1,2} \ln[(1+\beta\,\mathrm{sn})/\mathrm{dn}],$$

$$\beta^2 = P_0 < 1,\ A = 0,\ \mathrm{sn} = \mathrm{sn}(\theta,\beta),$$

$$\mathrm{cn} = \mathrm{cn}(\theta,\beta),\ \mathrm{dn} = \mathrm{dn}(\theta,\beta).$$

For the more common case of polynomial $P^{(1),(A)}$, such that $\mu = i\chi - \alpha$, $\chi = \chi(\theta)$, $\rho = \varphi - \alpha$, $\bar{\rho} = \alpha - \bar{\varphi}$, $\xi_{1,2} = \alpha + i\eta_{1,2}$, where χ, ρ, $\bar{\rho}, \eta_{1,2}$ are real and $\rho > \chi \geq \eta_1$ we obtain

$$\chi = \frac{\rho(\eta_1 - \bar{\rho}) + \bar{\rho}(\rho - \eta_1)\,\mathrm{sn}^2(\Theta, K)}{\eta_1 - \bar{\rho} + (\rho - \eta_1)\,\mathrm{sn}^2(\Theta, K)}, \tag{3.23}$$

where $\Theta = 2(\theta - \theta_0)[(\rho - \eta_2)(\eta_1 - \bar{\rho})]^{1/2}$,

$$K^2 = \frac{(\rho - \eta_2)(\eta_1 - \bar{\rho})}{(\rho - \eta_1)(\eta_2 - \bar{\rho})}.$$

IV. WHITHAM EQUATIONS

The exact solutions obtained in a preceding section describe nonlinear waves repeating themselves after some period T. Description of smoothed shock waves or any modulated wave train may be performed in quasiclassical approximation. In this approximation it is assumed that length and duration of a train or region of oscillations is much more than that of each soliton or other nonlinear spikes filling the region of oscillations. We suggest that characteristic parameters of periodic solution (in our case they are the roots of polynomial P : λ_i, $i = 1-4$) are slow functions of variables x, y, i.e., their scales change much more than that of single pulsation. These slowly changing parameters obey equations which may be found by averaging of some integrals over the period of fast pulsations T. As a result one is able to reduce the problem of analysis of the complex system with many degrees of freedom to a few evolution equations.

As is mentioned above, an exact method of solution of evolution equations under periodic boundary conditions is very effective for deriving the modulation Whitham equations, because it allows one to obtain them in a diagonal form directly. The first step of such a procedure is analogous to that of generating infinite sequences of the conservation laws. Using the Lax representation (3.1), (3.2) of the model considered we get the following generating formulas:

$$\partial_x\left(\frac{\lambda+\varphi}{g}F_+\right) = \delta\epsilon\,\partial_y\left(\frac{\lambda+\varphi}{\lambda g}R_+\right), \tag{4.1}$$

$$\partial_x\left(\frac{\lambda+\bar{\varphi}}{h}F_+\right) = \delta\epsilon\,\partial_y\left(\frac{\lambda+\bar{\varphi}}{\lambda h}R_+\right). \tag{4.2}$$

Both Eqs. (4.1) and (4.2) yield the same result, therefore we shall consider only (4.1). Following Flaschka, Forest, and McLaughlin [17], we introduce a new normalization for functions f, g, h: $f^2 + hg = 1$. Using (3.11) and (3.12) we have from (4.1)

$$\partial_x\left(\frac{P(\lambda)^{1/2}}{\lambda - \mu}\right) = \partial_y\left[\frac{\delta\epsilon P(\lambda)^{1/2}}{\sqrt{P_0}}\left(\frac{1}{\lambda} - \frac{1}{\lambda-\mu}\right)\right]. \tag{4.3}$$

The period of oscillations T is determined by the following integral:

$$\begin{aligned} T &= \int d\theta \\ &= \int [-P(\mu)]^{-1/2}\,d\mu \\ &= 2K(k)[(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)]^{-1/2}, \end{aligned} \tag{4.4}$$

where $K(k)$ is a complete elliptic integral of the first kind with modulus k : $k^2 = [(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)]/[(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)]$, λ_k are the roots of polynomial P such that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Integration in (4.4) is performed along the curve whose circle cuts between λ_1 and λ_2 or λ_1 and λ_2. Averaging over the period of fast oscillations T is performed by using the following relations:

$$\begin{aligned} \left\langle \frac{1}{\lambda - \mu} \right\rangle &= \frac{1}{T}\int \frac{1}{\lambda - \mu}\,d\theta \\ &= \frac{1}{T}\int \frac{1}{\lambda-\mu}[-P(\mu)]^{-1/2}\,d\mu. \end{aligned} \tag{4.5}$$

Setting successively $\lambda = \lambda_n$, $n = 1-4$ we obtain from (4.4) and (4.5)

$$\lim_{\lambda \to \lambda_n} \left\langle \frac{1}{\lambda - \mu} \right\rangle = -2\partial_{\lambda_n}(\ln T). \tag{4.6}$$

The limits $\lambda \to \lambda_n$ yield the singularities in the differentials

$$\partial_x \sqrt{P(\lambda)},\ \partial_y \sqrt{P(\lambda)}.$$

As is shown in Ref. [17] the conditions of vanishing of corresponding coefficients in (4.6) are fulfilled if the spectral parameters obey the Whitham equations, see, for details, Ref. [17]. These equations for our case are

$$\partial_x \lambda_n + \frac{1}{V_n}\partial_y \lambda_n = 0, \tag{4.7}$$

where

$$\frac{1}{V_n} = \frac{1}{V_0}\left[1 - \left(\lambda_n \left\langle \frac{1}{\lambda_n - \mu}\right\rangle\right)^{-1}\right], \tag{4.8}$$

$$\left\langle \frac{1}{\lambda_1 - \mu}\right\rangle = \frac{(\lambda_2-\lambda_4)E(k) - (\lambda_1-\lambda_4)K(k)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)K(k)},$$

$$\left\langle \frac{1}{\lambda_2 - \mu}\right\rangle = \frac{(\lambda_1-\lambda_3)E(k) - (\lambda_2-\lambda_3)K(k)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)K(k)},$$

$$\left\langle \frac{1}{\lambda_3 - \mu}\right\rangle = \frac{(\lambda_2-\lambda_4)E(k) - (\lambda_2-\lambda_3)K(k)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)K(k)},$$

$$\left\langle \frac{1}{\lambda_4 - \mu}\right\rangle = \frac{(\lambda_1-\lambda_3)E(k) - (\lambda_1-\lambda_4)K(k)}{(\lambda_1-\lambda_4)(\lambda_3-\lambda_4)K(k)},$$

where $V_0 = \delta\epsilon\sqrt{P_0} = \delta\epsilon(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^{1/2}$. $E(k)$ is a complete elliptic integral of the second kind with the same modulus k as above. Note that only three λ_n are independent in a common case and only two of them in simplified cases considered above. Therefore one or two equations should be excluded from the above system, then the concretized cases will be considered.

V. SOLUTION OF WHITHAM EQUATIONS

In this section the Whitham equations are studied for the case of solution (3.23) and steplike initial pulses, i.e., $F_+ \to$ nonzero const for $y \to -\infty$ and $F_+ \to 0$ for $y \to +\infty$. The evolution of such pulses corresponds to forming a set of densely packed oscillations near the leading sharp edge. Numerical simulations performed for various models demonstrate that the form shapes of these nonlinear pulses tend to solitonic ones (see the next section of this paper). The solitonic limit of solution (3.23) corresponds to $\eta_1 \to \eta_2$, $k \to 1$. Let $\eta_{1,2} = \eta \pm \varepsilon/2$, then

$$k^2 \approx 1 - \varepsilon(\rho - \bar\rho)[(\rho-\eta)(\eta-\bar\rho)]^{-1} + O(\varepsilon^2). \tag{5.1}$$

From (4.8) one has

$$V_0/V_{1,2} \approx 1 \pm 2\varepsilon \ln(\varepsilon) + O(\varepsilon). \tag{5.2}$$

The similarity solution of Eq. (4.7), where λ_i depends on one variable x/y, follows from representation of these equations in a form

$$(V_i - x/y)\partial_y \lambda_i = 0. \tag{5.3}$$

From (4.8) and (5.3) we have

$$\begin{aligned}\varrho = V_0 y/x - 1 &= \frac{1}{\eta}\frac{4(\eta - \rho + \varepsilon/2)\varepsilon K(k)}{(\rho-\eta)E(K) - \varepsilon K(k)} \\ &\approx \frac{4\varepsilon}{\eta}\ln\left(\frac{Q}{\varepsilon}\right) + O(\varepsilon),\end{aligned} \tag{5.4}$$

where $Q = 4(\rho-\eta)(\eta-\bar\rho)(\rho-\bar\rho)^{-1}$. In the logarithmic approximations solution of Eq. (5.4) has a form

$$\varepsilon \approx \varrho\eta[4\ln(Q/\varrho)]^{-1}. \tag{5.5}$$

The length L of a soliton formed near the leading edge of a train is

$$\begin{aligned}L \approx V_0 T \approx 4V_0 \ln\left\{\frac{4Q}{\varrho\eta}\left[\ln\left(\frac{Q}{\varrho}\right)\right]\right\} \\ \times[(\rho-\eta)(\eta-\bar\rho)]^{-1/2}.\end{aligned} \tag{5.6}$$

Expressions (5.2)–(5.6) show that characteristic parameters of the solitons and distances between them logarithmically increase with y. This affirmation is valid for all types of periodic solutions discussed in Sec. III. A back edge of the train is determined by the form of the pulse injected into the nonlinear medium. For a steplike pulse the back edge consists of quasiharmonic oscillations, which are associated with limit $k \to 0$.

In another limit of the same solution (3.23) $\eta_1 \to \bar\rho$, $k \to 0$ the leading front of the pulse transforms into quasiharmonic oscillations and a solitonic packet is formed near the back edge of the pulse. This limit may be related with propagation of a "turned around" shock wave.

VI. DISCUSSION

Exactly solvable models occupy a special place in theoretical physics. Quite universal in application, they constitute the foundation for progress in theoretical physics. The exactly solvable models are applied to nonlinear optics, plasma physics, hydrodynamics, etc. Moreover, the inverse transform is a single modern analytical tool by means of which the complex evolution equations can be investigated in detail. The integrability of the system of evolution equations plays a crucial role in the present study. Using an exact method one may effectively construct common periodic solutions and Whitham equations. The integrability of the model is based on a set of idealizations. The model is a one-dimensional, nondispersive, and nondissipative version of FWM. However, these properties are not enough, in common, for application of the inverse scattering transform. For example, the scheme of the FWM depicted in Fig. 2 leads to a system of equations which does not admit a Lax representation. Close models of FWM are used for analysis of propagation of solitary waves in optical fibers, plasma, and so on, see [1]. A common feature of these models of FWM, which prevents application of inverse scattering trans-

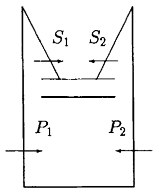

FIG. 2. The scheme of FWM leading to the nonintegrable evolution equations. The designations are the same as in Fig. 1.

form, is the existence of self-interaction terms. In terms of Sec. II of the present paper they are $F_3F_\pm$, $R_3R\pm$. But the periodic solutions of the integrable FWM model presented above can be used for nonintegrable schemes of FWM, if the solutions depend on a single automodel variable θ (one-phase solution). Consider, for example, the scheme of interaction depicted in Fig. 2. The evolution equations in this case are

$$\begin{aligned}(\partial_z + V^{-1}\partial_t)P_1 &= i\alpha_{11}(P_1|S_1|^2 + \kappa P_2 S_2^* S_1),\\ (\partial_z + V^{-1}\partial_t)S_1 &= i\alpha_{22}(S_1|P_1|^2 + 1/\kappa S_2 P_2^* P_1),\\ (\partial_z - V^{-1}\partial_t)P_2 &= i\beta_{11}(P_2|S_2|^2 + 1/\kappa P_1 S_1^* S_2),\\ (\partial_z - V^{-1}\partial_t)S_2 &= i\beta_{22}(S_2|P_2|^2 + \kappa S_1 P_1^* P_2).\end{aligned} \tag{6.1}$$

There $V_1 = U_1 = -V_2 = -U_2 = V$. Constants and functions have the same meaning as above. We transform system (6.2) into the following form:

$$\begin{aligned}\partial_T R_+ &= iR_3(\zeta_2 R_+ + F_+),\\ \partial_T R_3 &= -\partial_X F_3 = i/2(R_- F_+ - R_+ F_-),\\ \partial_X F_+ &= iF_3(\zeta_1 F_+ + R_+),\end{aligned} \tag{6.2}$$

where

$$F_- = F_+^*,\ R_- = R_+^*,\ \zeta_1 = \frac{aI_1}{2I_2}\left(\kappa + \frac{1}{\kappa}\right),$$

$$\zeta_2 = \frac{I_2}{2aI_1}\left(\kappa + \frac{1}{\kappa}\right),$$

$$T = 4\beta_{11}I_1(z - Vt),\ X = 4\alpha_{11}I_2(z + Vt),$$

$$I_1 = \kappa|S_1|^2 + \alpha_{22}|P_1|^2/(\kappa\alpha_{11}),$$

$$I_2 = 1/\kappa|S_2|^2 + \kappa\beta_{22}|P_2|^2/\beta_{11},$$

$$F_3 = 1/(4a)(\alpha_{22}|P_1|^2/(\kappa\alpha_{11}) - \kappa|S_1|^2),$$

$$R_3 = a/4(\kappa\beta_{22}|P_2|^2/\beta_{11} - 1/\kappa|S_2|^2),$$

$$\begin{aligned}F_+ &= \sqrt{\alpha_{22}}P_1^* S_1/(2a\sqrt{\alpha_{11}}I_1)\\ &\quad\times \exp[i(\kappa - 1/\kappa)(I_1X/I_2 - I_2T/I_1)],\end{aligned}$$

$$\begin{aligned}R_+ &= a\sqrt{\beta_{22}}P_2^* S_2/(2\sqrt{\beta_{11}}I_2)\\ &\quad\times \exp[i(\kappa - 1/\kappa)(I_1X/I_2 - I_2T/I_1)].\end{aligned}$$

Here $I_{1,2}$ are constants, $a^2 = (\beta_{11}\alpha_{22})/(\beta_{22}\alpha_{11})$. $\kappa, P_{1,2}, S_{1,2},\ldots$ are the same as in the above schemes. For a function depending on one variable θ we have

$$F_3 = V_0 R_3 + A. \tag{6.3}$$

Using (6.3) one can transform system (6.2) in system of (2.8) with new constants $\zeta_1 \to \zeta_1 V_0^{-1}$, $\zeta_2 \to \zeta_2 V_0$ and new functions :

$$(R_\pm, F_\pm) \to (R_\pm, F_\pm)\exp[\pm i(T/V_0 - X)A].$$

Thus one-phase solutions obtained above after simple redesignations may be used for the nonintegrable case of the FWM scheme.

The resonance conditions (2.2) allow one not only to neglect the self-interaction terms, but lead to enhancement of nonlinear conversion of fields [21]. These self-interaction terms naturally arise in many models of four-wave interaction. Implementation of the resonant atoms in the sample may lead to a situation where these terms can be neglected. The nonintegrable scheme of FWM considered in this section may be realized, for example, in fibers, see, for example, Ref. [10]. The periodic solutions obtained above can be used for description of propagations of dense packets of solitons in fibers for weak dispersion as well.

Direct experimental investigation of NSW dynamics in FWM is unknown, although required parameters are not beyond the achievements of modern technique. The parameters may be the following. For $4S \leftrightarrow 5S$ transition of KI vapor and carrying frequencies $\Omega_1 = 3.5 \times 10^{15}\,\mathrm{s}^{-1}$, $\Omega_2 = 1.8\times 10^{15}\,\mathrm{s}^{-1}$, $\omega_1 = 1.8\times 10^{14}\,\mathrm{s}^{-1}$ and resonance conditions $\Omega_1 + \omega_1 \approx \omega_0$, $\Omega_2 - \omega_2 \approx \omega_0$ one has $\kappa(\Omega_1) \approx 2.2 \times 10^{-23}\,\mathrm{cm}^2$, $\kappa(\Omega_2) \approx 1.2 \times 10^{-22}\,\mathrm{cm}^2$ [22]. If vapor pressure is 10 Torr, then $\alpha_{11} \approx 2.0 \times 10^{-5}$ cgs electrostatic units (CGSE). For intensities of the fields about 10^9 W/cm^2 the length of solitons is about 1–1.5 cm. Note that for this scheme $u \approx 5.45$. In the case of hydrogen, let us consider the scheme of FWM, where $\Omega_i + \omega_i \approx \omega_0$, $i = 1,2$. Let $\Omega_1 = 3.0 \times 10^{15}\,\mathrm{s}^{-1}$, $\Omega_2 = 4.0 \times 10^{15}\,\mathrm{s}^{-1}$, $\omega_1 = 2.0 \times 10^{15}\,\mathrm{s}^{-1}$, $\nu N_0 \approx 4.5 \times 10^{15}$ CGSE. Hence $\kappa(\Omega_1)^2 \approx \kappa(\Omega_2)^2 \approx 0.68 \times 10^{-50}\,\mathrm{cm}^4$. For the length of solitons of 0.5 cm the pump intensities should be about 10^9–10^{10} W/cm^2. In this scheme $u = 1$, i.e., $\zeta_1 = \zeta_2$.

Some experimental data of Ref. [23] devoted to investigations of Raman scattering may be interpreted in the framework of NSW dynamics. Indeed, it has been found that injection of a long intense pump pulse with a sharp leading edge leads to a highly oscillatory structure of Stokes pulse. For a slow sloping leading edge of pump field the Stokes pulse has a more smoothed structure. The nonlinear oscillations had the form shape typical for solitons. The distances between solitons, when they were in a dense packet, increased much slower than when they were in isolation. Such behavior is verified in numerical simulations as well. These facts are in qualitative agreement with the theory presented above.

The practical importance of analytical investigation of modulated wave trains is corroborated by consideration of experimental conditions of observing such phenomena

TABLE I. The relations between the polynomial $P^{(1-3)}$ and $P^{(A,B,C)}$, $i = 1, 2$.

	(A)	(B)	(C)
(1)	$(a)\,\alpha^2 > \delta\epsilon$, $\mathrm{Im}\xi_{1,2} = 0$. $(b)\,\alpha^2 < \delta\epsilon$, $\xi_i = \alpha + i\lambda_i$, $\mathrm{Im}\lambda_i = 0$, $\mathrm{Re}(\mu + \alpha) = 0$.	$\alpha^2 < \delta\epsilon$ $\xi_i = \xi + i\eta$, $\mathrm{Im}\xi = \mathrm{Im}\eta = 0$.	$(a)\,\alpha^2 > \delta\epsilon$, $\xi_i = \xi + i\eta$, $\mathrm{Im}\xi = \mathrm{Im}\eta = 0$. $(b)\,\lambda^2 < \delta\epsilon$, $\mathrm{Im}\xi_i = 0$.
(2)	$(a)\,\mathrm{Im}\lambda_i = 0$, $\mathrm{Im}\mu = 0$. $(b)\,\mathrm{Re}(\mu + \alpha) = 0$, $\mathrm{Re}(\lambda_i) = 0$.	$\mathrm{Re}\lambda_i = 0$.	$\mathrm{Im}\lambda_i^2 = 0$.
(3)	$\lambda_i^2 > \delta\epsilon$, $\lambda_1 \neq \lambda_2$, $\mathrm{Im}\lambda_i^2 = 0$.	$\lambda_i^2 < \delta\epsilon$	$\lambda_1^2 > \delta\epsilon$, $\lambda_2^2 < \delta\epsilon$.

as stimulated RS, TPP, and FWM. These phenomena are often studied experimentally by using nonlinear cells placed in cavities (see, for example, [24]). If changing of the field amplitude after one pass of the cavity is slow and the length of the region of oscillations is about that of cavity, then quasiperiodic behavior of pulses should dominate. The periodic solutions and solutions of Whitham equations may be used for analysis (after suitable modification of theory) of transient decay-type regimes of FWM, which are associated with a continuous spectrum of the spectral problem, see Refs. [5,6]. As is demonstrated in [25], an asymptotic infinite sequence of solitons may arise out of a continuous spectrum. The periodic solutions with time dependent parameters may be used for a theoretical description of more common cases than are considered in Ref. [25].

The diagonal form of Eqs. (4.7) has important physical consequences. Generally, the speeds V_i are distinct and complex. In this situation they replace the single group velocity in the theory of propagating waves. If these characteristic speeds are complex (for complex roots $\xi_{1,2}$ of polynomial $P^{(1)}$), then the modulation equations (4.7) resemble elliptic equations and the wave train is modulationally unstable.

The theory of integrable waves together with numerical simulations has shown that the large time behavior of a wave is frequently dominated by solitary waves. Spatially coherent structure can persist even in temporally chaotic states [13]. The solitons or solitonic trains are natural candidates for these coherent structures. The exact approach used here is very convenient for constructions of perturbation theory, which can be used for analysis of dense solitonic train evolution in near-integrable models.

ACKNOWLEDGMENTS

The author is pleased to thank Tanya Bandmann for most useful discussions and Paul Kolinko for help in preparation of the manuscript.

APPENDIX

For practical purposes, to solve Eq. (3.13) using handbooks (for example, Ref. [20]), it is convenient to classify the polynomials by using the following forms of polynomial depending on only real μ and real roots χ_k:

$$P^{(A)}(\mu) = \prod_{k=1}^{4} (\mu - \chi_k),$$

$$P^{(B)}(\mu) = [(\mu + \chi_1)^2 + \chi_2^2][(\mu + \chi_3)^2 + \chi_4^2],$$

$$P^{(C)}(\mu) = \prod_{j=1}^{2} (\mu + \chi_j)[(\mu + \chi_3)^2 + \chi_4^2].$$

The relationship between these forms of polynomial $P^{(1-3)}$ and $P^{(A,B,C)}$ is presented in Table I.

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