Nonrelativistic theory of the radiation reaction interaction

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A nonrelativistic theory of the radiation reaction interaction is formulated, which is based on the strict implementation of the uncertainty principle into classical theory. The same idea is used in formulating the semiclassical theory of the radiation reaction interaction. The semiclassical and classical formulations give the same answer for the dynamics of a particle in a plane electromagnetic wave.

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I. INTRODUCTION

The radiation reaction interaction is one of the most fundamental in nature: without it the conservation of energy law would not hold true. For example, in the dynamics of charge the emitted electromagnetic (EM) energy must come at the expense of its translation energy, a process which can only be described if the radiation reaction interaction is included in the dynamics equations. In classical theory this means including an additional force in Newton's equation of motion, whilst in quantum theory this means an additional coupling term in the Schrodinger equation. The attempts to formulate the interaction properly are nearly a century old (for a good review, see [1]) and the most successful, in the classical domain, is the Abraham-Lorentz theory. Although having great problems it has not been superseded by a better one. With the advent of quantum theory a new era began in understanding the radiation reaction interaction. Formulations went in various directions, in all sorts of combinations of classical, semiclassical, and quantum descriptions (for a good review, see [2]). Reviewing them is a considerable effort, but a general consensus is that the most successful of all theories is quantum electrodynamics (QED) [3], which owes its reputation primarily to its excellent agreement with experiment rather than to its theoretical foundation.

In this work we would like to discuss an approach to this problem which is based on some recently made observations concerning the relationship between classical and quantum theory [4–6]. The essential idea is to understand the consequences of trying to incorporate the uncertainty principle into the basic postulates of classical theory. The immediate consequence is that the concept of trajectory must be replaced by the concept of the probability distribution. Once this is done then one should also modify the way the EM field is incorporated into the dynamics equations. The notion of a pointlike charge must be abandoned, and instead one should work with the probability distribution and the probability current [7]. They play the role of the charge density and the charge current, respectively. In other words, the probability distribution $P(\mathbf{r},t)$ and the probability current $\mathbf{j}(\mathbf{r},t)$ for electrons produce the EM field which is calculated from the equation for the scalar and the vector potentials. In the Lorentz gauge they are

$$\Delta\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi e P(\mathbf{r},t)\ , \tag{1}$$

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}e\mathbf{j}(\mathbf{r},t)\ .$$

The dynamics which produces $P(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ can either be classical or quantum; however, it has been found that in both cases the results are virtually the same [8–10]. Based on these results we can now try to formulate the radiation reaction interaction by incorporating the resulting fields back into the dynamics equations, either classical or quantum. In classical theory this would mean adding an additional force [7], of the Lorentz type, into the equation for the particle. This force has its origin in the field which is obtained from the solution of Eqs. (1). In quantum theory, on the other hand, this would mean writing the (nonrelativistic) equation as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{c\hbar}\mathbf{A}\right)^2\psi + e\Phi\psi + V\psi\ , \tag{2}$$

where V is some external potential in which the particle moves, whilst Φ and $\mathbf{A}$ are the solutions of (1). The sets (1) and (2), in principle, answers the problem of the radiation reaction interaction; however, it can easily be shown to be nonphysical. For example, in the simplest case of the stationary states of the hydrogen atom Eq. (2) is

$$E\psi = -\frac{\hbar^2}{2m}\Delta\psi + e^2\int d^3r'\frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}\psi(\mathbf{r}) - \frac{e^2}{r}\psi\ , \tag{3}$$

where the vector potential is zero because the current $\mathbf{j}$ is zero for stationary states. For large r the additional term is of the Coulomb repulsion type for equal pointlike charges e, and cancels the Coulomb attraction term between electron and proton. This means that in this region the effective potential in which the electron moves is zero, which is definitely not the case.

The nonphysical character of the sets (1) and (2) can

be traced to the formal assumption that $P(\mathbf{r},t)$ is the charge density, which also implies that it interacts with itself through the electrostatic force. The same thing happens if we consider interaction of the current $\mathbf{j}$ with itself due to the magnetostatic field. Interactions of this kind are not possible and, therefore, should be omitted in the dynamics equations. However, the interaction which results from the retardation effect should be retained, because at time t the distributions can interact with the ones from some earlier time.

One way of separating the instantaneous and the retarded components of the field is by using the Coulomb gauge for the potentials. In this gauge $\nabla\mathbf{A} = 0$ and Eqs. (1) become

$$\Delta\Phi = -4\pi e P(\mathbf{r},t) , \tag{4}$$

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}e\mathbf{j}_T(\mathbf{r},t) ,$$

where

$$\mathbf{j}_T = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3r'\frac{\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} . \tag{5}$$

The solution for Φ is simple:

$$\Phi = e\int d^3r'\frac{P(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} , \tag{6}$$

whilst for the vector potential [11]

$$\mathbf{A} = e\int d^3r'\left[\frac{\mathbf{j}(\mathbf{r}',t-R/c)}{cR} - \frac{\mathbf{R}}{R^2}P(\mathbf{r}',t-R/c) - \frac{c\mathbf{R}}{R^3}\left(\int_0^{t-R/c}dt'\,P(\mathbf{r}',t') - \int_0^t dt'\,P(\mathbf{r}',t')\right)\right] , \tag{7}$$

where $\mathbf{R} = \mathbf{r}-\mathbf{r}'$. Strictly speaking the current $\mathbf{j}$ should have the form

$$\mathbf{j} = \frac{\hbar}{m}Im(\psi^*\nabla\psi) - \frac{e}{c}\mathbf{A}\psi^*\psi \tag{8}$$

and the relationship for $\mathbf{A}$ in (7) represents, in fact, the integral equation. In further discussion, however, we will consider the simplifying assumption where $\mathbf{A}$ in the current is neglected.

When the typical time variation of P and $\mathbf{j}$ is much longer than the time it takes the EM field to cross the interaction region, the vector potential can be expanded in powers of R/c, and the first few terms are

$$\mathbf{A} = \frac{e}{c}\mathbf{A}_0 + \frac{e}{c}\nabla\delta + \frac{e}{c}\mathbf{A}_{\rm ret} , \tag{9}$$

where

$$\mathbf{A}_0 = \int d^3r'\frac{\mathbf{j}(\mathbf{r}',t)}{R};\quad \delta = \frac{1}{2}\int d^3r' R\dot{P}(\mathbf{r}',t) , \tag{10}$$

$$\mathbf{A}_{\rm ret} = -\frac{2}{3c}\int d^3r'\dot{\mathbf{j}}(\mathbf{r}',t) + \frac{1}{8c^2}\int d^3r' R[3\ddot{\mathbf{j}} - \hat{\mathbf{n}}\left(\hat{\mathbf{n}}\cdot\ddot{\mathbf{j}}\right) + \cdots ,$$

where $\hat{\mathbf{n}} = \mathbf{R}/R$ and the dot designates a derivative with respect to time. The first term represents the instantaneous vector potential. The second term is in the form of the gradient of a certain function δ, whilst the remaining terms are the true contributions from the retardation effects, and are designated by $\mathbf{A}_{\rm ret}$. We now postulate that the equation for the particle is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{ie^2}{c^2\hbar}\nabla\delta - \frac{ie^2}{c^2\hbar}\mathbf{A}_{\rm ret}\right)^2\psi + V\psi \tag{11}$$

and the EM field is given by the usual expressions. The meaning of this equation is revealed by considering the total energy of the system

$$U = {\rm Re}\left\{\int d^3r\,\psi^*\left[-\frac{\hbar^2}{2m}\left(\nabla - \frac{ie^2}{c^2\hbar}\nabla\delta - \frac{ie^2}{c^2\hbar}\mathbf{A}_{\rm ret}\right)^2 + V\right]\psi\right\} + \frac{1}{8\pi}\int d^3r\left(\mathbf{E}^2+\mathbf{H}^2\right) \tag{12}$$

and its rate of change with time. It can be shown that

$$\frac{dU}{dt} = \frac{e^2}{2}\frac{d}{dt}\int d^3r\int d^3r'\frac{P(\mathbf{r},t)P(\mathbf{r}',t)}{R} + \frac{e^2}{2c^2}\frac{d}{dt}\int d^3r\int d^3r'\frac{\mathbf{j}(\mathbf{r},t)\mathbf{j}(\mathbf{r}',t)}{R} + \frac{c}{4\pi}\int d^3r\,\nabla\cdot(\mathbf{H}\times\mathbf{E}) . \tag{13}$$

The first two terms represent the rate of change of the Coulomb and the magnetic energy of the self-interacting charge and current densities, respectively. These terms appear because in the equation for the particle we have neglected interaction which arises from the effect of treating $|\psi|^2$ as the charge density and $\mathbf{j}$ as the current density. The last term in (13) represents the outflow of energy, in the form of EM radiation, from the volume inside which the process of energy exchange between the particle and the EM field occurs. It can be shown that this term is always positive, and therefore the system always loses energy [we assumed that the potential V in (11) is time independent]. The energy conservation law would have been satisfied therefore, except for the presence of the two previously mentioned terms. However, if the total energy of the system is defined as the expression (12) minus the static energies of the electric and the magnetic components of the EM field, then the energy conservation law is strictly satisfied.

The theory of the radiation reaction interaction which was developed is based on the implementation of the idea

that the dynamics of a particle is in fact the dynamics of the probability distribution, in both classical and quantum theory. In the first, as will be seen, this means solving essentially the Liouville equation in the phase space of the particle, whilst in the latter this is achieved by first solving the equation for the wave function, from which it is simple to obtain the probability distribution. As will be shown, both approaches produce the same result, at least for the example which will be treated. The relationship of this approach to the others which are used for solving the radiation reaction interaction problem will be discussed in the section at the end of the paper.

II. SEMICLASSICAL THEORY

The equation which describes the dynamics of a particle, with the radiation reaction interaction included, is given by (11),

$$i\frac{\partial\psi}{\partial t} = -\tfrac{1}{2}\left(\nabla - i\alpha\nabla\delta - i\alpha\mathbf{A}_{\rm ret}\right)^2\psi + \frac{V}{mc^2}\psi\,, \quad (14)$$

where we used the scaling $\kappa\mathbf{r}\to\mathbf{r}$ and $\kappa ct\to t$, where $\kappa = mc/\hbar$. In our study we will use only the leading term in $\mathbf{A}_{\rm ret}$,

$$\mathbf{A}_{\rm ret} = -\frac{2}{3}\int d^3r'\,\mathbf{j}(\mathbf{r}',t) \quad (15)$$

which is coordinate independent. In this case analysis of Eq. (14) simplifies if for the wave function we write $\psi = \exp(i\alpha\delta + i\alpha\mathbf{r}\cdot\mathbf{A}_{\rm ret})\phi$, where ϕ satisfies the equation

$$i\dot\phi = -\tfrac{1}{2}\Delta\phi + \alpha\dot\delta\phi + \alpha\mathbf{r}\cdot\dot{\mathbf{A}}_{\rm ret}\phi + V\phi\,, \quad (16)$$

where V stands for V/mc^2. The two extra terms which appear describe the effects of the radiation reaction interaction, however, they have different properties. A few preliminary studies (not reported here) indicate that the term $\dot\delta$ is not important because it essentially couples the longitudinal vibration modes of the probability density and the probability current. As such it does not contribute to the energy loss of the system due to the EM radiation. The term appears only to affect the frequencies at which $|\psi|^2$ and $\mathbf{j}$ oscillate, i.e., the term causes only frequency shifts (but only slightly). In our study this term will be neglected, but further investigation of its precise effect on the dynamics of particles is required. Here it is worth mentioning that, if it is neglected, like the Coulomb and the magnetic terms were earlier on, then the term

$$\frac{d}{dt}U_\delta = \frac{e^2}{4c^2}\frac{d}{dt}\int d^3r\int d^3r'\dot P(\mathbf{r},t)\dot P(\mathbf{r}',t)\,|\mathbf{r}-\mathbf{r}'| \quad (17)$$

appears in the time rate of the total energy of the system (13). This means that in the definition of the total energy the term U_δ should be subtracted if the energy conservation law is to hold true.

It turns out that all the energy loss due to the EM radiation comes from $\mathbf{A}_{\rm ret}$, and therefore the equation which will be investigated is

$$i\dot\phi = -\tfrac{1}{2}\Delta\phi + \alpha\mathbf{r}\cdot\dot{\mathbf{A}}_{\rm ret}\phi + V\phi\,, \quad (18)$$

where $\mathbf{A}_{\rm ret}$ is given by (15) and

$$\mathbf{j} = \mathrm{Im}\left[\phi^*\nabla\phi\right]\,. \quad (19)$$

In this work we will consider the simple example of a free particle which is interacting with a linearly polarized (along the z direction), time-varying, EM field of the form

$$V = \epsilon z\cos(w_0 t)\,. \quad (20)$$

It is assumed that initially ($t=0$) the wave function ϕ is of the form

$$\phi_0 = \frac{1}{\pi^{3/4}r_0^{3/2}}e^{-\frac{r^2}{2r_0^2}}\,. \quad (21)$$

The problem is now well defined and solving it requires numerical techniques. One way of doing it is by assuming that the whole system is enclosed in a spherically symmetric potential, which is zero everywhere except that it is infinite at large radius R. Therefore, we can expand the solution ϕ in the basis functions which are the stationary states of a free particle inside the sphere of radius R. We can write

$$\phi = \frac{\sqrt{2}}{R^{3/2}}\sum_{n,l,m}\frac{C_{n,l,m}(t)}{|j_{l+1}(\alpha_n^{(l)})|}j_l\left(\frac{\alpha_n^{(l)}r}{R}\right)Y_{l,m}\,, \quad (22)$$

where $j_l(z)$ is the spherical Bessel function and $\alpha_n^{(l)}$ is its nth root.

When the expansion is put into the equation for ϕ we get a set of equations for the coefficients $C_{n,l,m}$. The problem arises because the highest time derivatives (the second order) of these coefficients are in $\dot{\mathbf{A}}_{\rm ret}$, and extracting them is not simple. Furthermore, having the set of the second-order differential equations requires also initial conditions for $\dot C_{n,l,m}$, which are not easily available. The reason why higher than the first derivatives appear in the set of equations for $C_{n,l,m}$ reflects the fact that the retardation effects are taken into account. If the potential (7) was used from the beginning problems of this nature would not have arisen, instead one would have to assume knowledge of the wave function for all times prior to $t=0$.

In order to avoid the difficulties connected with the appearance of the time derivatives in $\dot{\mathbf{A}}_{\rm ret}$ Eq. (18) can be solved by iteration. The zeroth-order solution for ϕ is

$$i\dot\phi^{(0)} = -\tfrac{1}{2}\Delta\phi^{(0)} + V\phi^{(0)}\,. \quad (23)$$

There are now two ways to proceed. One can solve this equation, calculate the current $\mathbf{j}^{(0)}$, and replace it in $\mathbf{A}_{\rm ret}$. The equation which one then needs to solve is

$$i\dot\phi^{(1)} = -\tfrac{1}{2}\Delta\phi^{(1)} + V\phi^{(1)} + \alpha\mathbf{r}\cdot\dot{\mathbf{A}}_{\rm ret}^{(0)}\phi^{(1)}\,, \quad (24)$$

which is the first-order correction to the true solution.

The next iteration is obvious: from the solution $\phi^{(1)}$ one calculates $\mathbf{j}^{(1)}$ and then $\dot{\mathbf{A}}_{\rm ret}^{(1)}$, which is then put into an equation similar to (24), except that the index of iteration is increased by 1.

The other way of solving (18) is to treat (23) as the relationship for the unknown solution and solve the equation

$$i\dot{\phi}^{(0)} = -\tfrac{1}{2}\Delta\phi^{(0)} + V\phi^{(0)} + \alpha\mathbf{r}\cdot\dot{\mathbf{A}}_{\rm ret}^{(0)}\phi^{(0)} , \quad (25)$$

where the second derivative of $\phi^{(0)}$ in $\dot{\mathbf{A}}_{\rm ret}^{(0)}$ is calculated from (23) by noting that

$$i\ddot{\phi}^{(0)} = -\tfrac{1}{2}\Delta\dot{\phi}^{(0)} + V\dot{\phi}^{(0)} + \dot{V}\phi^{(0)} , \quad (26)$$

where $\dot{\phi}^{(0)}$ is given by (23). In the next iteration we use (25) as the relationship for the first derivative and then it is used for calculating $\dot{\mathbf{A}}_{\rm ret}^{(1)}$. The iteration procedure repeats itself in the obvious way.

It is not clear which of the schemes is better in applications: the first requires solving a large number of equations, whilst the second is more algebraically demanding. However, the two schemes should give the same result, but the question is which one is faster convergent. We will use the first scheme in this work because in the example which we consider much of the work can be done analytically. In the second scheme this is not possible.

In our example V is a function of z and hence we need only consider solving (23) for the z coordinate. The equation is

$$i\dot{\phi}_z^{(0)} = -\frac{1}{2}\frac{\partial^2}{\partial z^2}\phi_z^{(0)} + \epsilon z f(t)\,\phi_z^{(0)} , \quad (27)$$

where $f(t)$ is some function of time. The solution of this equation is

$$\phi_z^{(0)} = \frac{(-b-b^*)^{1/4}}{\pi^{1/4}} e^{\frac{1}{8}\frac{(a+a^*)^2}{b+b^*}} e^{az+bz^2} , \quad (28)$$

where

$$b = \frac{b_0}{1-2itb_0} , \quad (29)$$

$$a = b\left[c_0 - \frac{i\epsilon}{b_0}\int_0^t dt'\, f(t') - 2\epsilon\int_0^t dt'\, t' f(t')\right] ,$$

and b_0 and c_0 should be determined from the initial conditions. In our example, i.e., for the initial condition (21), these parameters are $b_0 = -1/(2r_0^2)$ and $c_0 = 0$. The current is now easily calculated and the integral over the current is

$$\int d^3r\,\mathbf{j} = -\hat{z}\,\epsilon\int_0^t dt'\, f(t') , \quad (30)$$

from which we get

$$\dot{\mathbf{A}}_{\rm ret}^{(0)} = \frac{2\epsilon}{3}\dot{f}(t)\hat{\mathbf{z}} \quad (31)$$

and Eq. (24) is

$$i\dot{\phi}_z^{(1)} = -\frac{1}{2}\frac{\partial^2}{\partial z^2}\phi_z^{(1)} + \epsilon z\left[f(t) + \frac{2\alpha}{3}\dot{f}(t)\right]\phi_z^{(1)} . \quad (32)$$

The equation is exactly the same as (27) except that the function $f(t)$ is replaced by $g(t) = f(t) + 2\alpha\dot{f}(t)/3$. In general, it can be shown that the nth iteration produces the same equation as (27), where $f(t)$ is replaced by

$$g(t) = \sum_{i=0}^{n}\left(\frac{2\alpha}{3}\right)^i \frac{d^i}{dt^i}f(t) . \quad (33)$$

If now $f(t)$ is specified as $\cos(w_0t)$ then in the limit $n\to\infty$ the function $g(t)$ is

$$g(t) = \frac{\cos(w_0t) - q\,\sin(w_0t)}{1+q^2} , \quad (34)$$

where $q = 2\alpha w_0/3$. The solution for the wave function is then given by (28), where $f(t)$ is replaced by $g(t)$. We can call this the exact solution for the dynamics of the particle when the radiation reaction interaction is included, of course, for the initial condition (21). From this solution we obtain the time evolution for the probability distribution

$$P(z,t) = \frac{r_0}{\sqrt{\pi\,(r_0^4+t^2)}} e^{-\frac{r_0^2}{r_0^4+t^2}(z+z_0)^2} , \quad (35)$$

where

$$z_0 = \epsilon\frac{1 - qw_0t - \cos(w_0t) + q\,\sin(w_0t)}{w_0^2\,[1+q^2]} . \quad (36)$$

For simplicity we gave only the z component of the probability distribution. The other components are not affected by the radiation reaction interaction and by the external field, and therefore they have the time evolution of a free probability distribution.

III. CLASSICAL THEORY

The classical equation of motion for a particle interacting with the external potential is

$$\ddot{\mathbf{r}} = -\boldsymbol{\nabla}V , \quad (37)$$

where we used the same scaling as in the previous section. If one postulates that the initial conditions are not given precisely, but rather by a distribution $P_0(\mathbf{r})$ in the coordinates and a distribution $Q_0(\mathbf{p})$ in the momentum, then the only answer one can expect to get from classical theory is the distribution $P(\mathbf{r},t)$ at some later time t. The requirement that the uncertainty principle is incorporated into classical theory means that the initial distributions P_0 and Q_0 are related in such a way that $\Delta r\Delta p > \hbar$ is satisfied. This relationship is satisfied if we write

$$P_0(\mathbf{r}) = |\psi|^2 , \quad Q_0(\mathbf{p}) = |\phi|^2 , \quad (38)$$

and

$$\phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3r\, e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r}) . \quad (39)$$

There are various ways to calculate the time evolution of $P(\mathbf{r},t)$. One, and the simplest, is to generate random initial conditions with the distributions $P_0(\mathbf{r})$ and $Q_0(\mathbf{p})$, and sample the trajectories after they have been calculated from (37) [4,7]. The other procedure is to define the phase space probability distribution $\rho(\mathbf{r},\mathbf{p},t)$ with the property

$$P(\mathbf{r},t) = \int d^3p\, \rho(\mathbf{r},\mathbf{p},t) , \quad Q(\mathbf{p},t) = \int d^3r\, \rho(\mathbf{r},\mathbf{p},t) . \quad (40)$$

The density ρ satisfies the Liouville equation, which is very difficult to integrate, but it also satisfies the relationship

$$\rho_0(\mathbf{r}_0,\mathbf{p}_0,t=0)\, d^3r_0\, d^3p_0 = \rho(\mathbf{r},\mathbf{p},t)\, d^3r\, d^3p , \quad (41)$$

which is very useful in the analysis. The coordinate $\mathbf{r}$ and the momentum $\mathbf{p}$ in (41) are related to $\mathbf{r}_0$ and $\mathbf{p}_0$. The latter are the initial conditions at $t = 0$ for the particle, whilst the appropriate coordinate and the momentum at time t are $\mathbf{r}$ and $\mathbf{p}$, respectively. The volume elements are likewise related. From the relationships (41) and (40) we get for the probability distribution

$$P(\mathbf{r},t) = \int d^3p_0 \frac{\rho_0(\mathbf{r}_0,\mathbf{p}_0)}{J} , \quad (42)$$

where $J = d^3r/d^3r_0$ is the Jacobian relating the volume elements at times t and $t = 0$. The appropriate probability current is obtained as

$$\mathbf{j}(\mathbf{r},t) = \int d^3p \frac{\mathbf{p}}{m}\rho(\mathbf{r},\mathbf{p},t) = \int d^3p_0 \frac{\mathbf{p}}{m}\frac{\rho_0(\mathbf{r}_0,\mathbf{p}_0)}{J} , \quad (43)$$

where $\mathbf{p}$ is the momentum of the particle at time t, if the initial conditions were $\mathbf{r}_0$, and $\mathbf{p}_0$. It should be noted that $\mathbf{r}_0$ in (42) and (43) are related to $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{p}_0$.

The initial $\rho_0(\mathbf{r}_0,\mathbf{p}_0)$ is given as a product $P_0(\mathbf{r}_0)Q_0(\mathbf{p}_0)$, but this is not always the case. If the particle is initially bound, and in a stationary state, then the initial phase space density is not product separable. The requirement of stationarity is an additional restriction which prevents such a simple choice for ρ_0. In this work we will analyze a nonbound particle and the problem of that nature will not arise.

We can now put forward the argument, as we did in the previous section, that $P(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ play the role of the charge and current densities of the EM field. The arguments which support this assertion were given elsewhere, but if this is assumed then the EM field is calculated from the potentials (1). This field also interacts with the distribution $P(\mathbf{r},t)$, which is the essence of the radiation reaction interaction. We can now use the same arguments as in the previous section in order to select the proper form of the interaction. In short, only the part of the vector potential due to the retardation should be considered [we also neglect the part of the vector potential which is related to δ in (9)], from which the EM field is given by

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}_{\rm ret}}{\partial t} ; \quad \mathbf{H} = \nabla \times \mathbf{A}_{\rm ret} . \quad (44)$$

This field excerpts the force on the particle so that the classical equation of motion, with the radiation reaction force included, is

$$\ddot{\mathbf{r}} = -\nabla V + \alpha(\mathbf{E} + \mathbf{v}\times\mathbf{H}) . \quad (45)$$

The appearance of this equation is deceptively simple. For the simplest case when $\mathbf{A}_{\rm ret}$ is given by (15) the equation is

$$\ddot{\mathbf{r}} = -\nabla V + \frac{2\alpha}{3}\int d^3r_0\, d^3p_0\, \rho_0(\mathbf{r}_0,\mathbf{p}_0)\, \ddot{\mathbf{v}} , \quad (46)$$

where we used the property (41). The complexity of the equation is now obvious. The integral on the right hand side is over the whole space of the initial conditions, whilst on the left the derivative $\ddot{\mathbf{r}}$ refers to a trajectory with particular initial conditions. If the integral is approximated by a sum then the result is a set of equations for $\mathbf{r}$ with various initial conditions. This is, in fact, one way of solving Eq. (46); however, the problems which we encounter are analogous to the ones which we met in solving the quantum equation. One of these problems is the appearance of the highest time derivative of $\mathbf{r}$ in the integral of Eq. (46). The alternative is, as in the semiclassical treatment, to use an iterative procedure for solving (46), and the way it is done is analogous to the sequence (23) and (24). The only difference is that the trajectory $\mathbf{r}$ replaces the wave function ϕ.

For the example in the previous section Eq. (46) is

$$\ddot{\mathbf{r}} = -\hat{\mathbf{z}}\epsilon\cos(w_0t) + \frac{2\alpha}{3}\int d^3r_0\, d^3p_0\, \rho_0(\mathbf{r}_0,\mathbf{p}_0)\, \ddot{\mathbf{v}} , \quad (47)$$

where

$$\rho_0(\mathbf{r},\mathbf{v}) = \frac{1}{\pi^3} e^{-\frac{r^2}{r_0^2} - r_0^2 v^2} . \quad (48)$$

In the zeroth order the solution of (47) is obtained from

$$\ddot{\mathbf{r}}^{(0)} = -\hat{\mathbf{z}}\epsilon\cos(w_0t) \quad (49)$$

and the equation for the first-order correction is

$$\ddot{\mathbf{r}}^{(1)} = -\hat{\mathbf{z}}\epsilon\cos(w_0t) + \epsilon q\hat{\mathbf{z}}\sin(w_0t) , \quad (50)$$

where we used the property that $\ddot{\mathbf{r}}^{(0)}$ is independent of the initial conditions so that

$$\int d^3r_0\, d^3p_0\, \rho_0(\mathbf{r}_0,\mathbf{p}_0)\, \ddot{\mathbf{v}}^{(0)} = \frac{d}{dt}\ddot{\mathbf{r}}^{(0)} . \quad (51)$$

The iterations can be carried on (similarly as in the previous section) and the final equation is

$$\ddot{\mathbf{r}} = -\hat{\mathbf{z}}\epsilon\frac{\cos(w_0t) - q\sin(w_0t)}{1+q^2} , \quad (52)$$

which has the solution (for the z coordinate only)

$$z = -z' + z_0 + \dot{z}_0 t \ , \tag{53}$$

where

$$z' = \frac{\epsilon}{w_0^2} \frac{1 - \cos(w_0 t) - q w_0 t + q \sin(w_0 t)}{1 + q^2} \tag{54}$$

and z_0 and $\dot{z}_0$ are the initial conditions.

From the solution (53) we can calculate the probability distribution $P(z,t)$ (only the z component, for simplicity) from

$$P(z,t) = \frac{1}{\pi^3} \int d\dot{z}_0 \, e^{-\frac{z_0^2}{r_0^2} - r_0^2 \dot{z}_0^2} \ , \tag{55}$$

where for our example the Jacobian is unity. If z_0 is calculated from (53) and replaced in the integral, the probability distribution is

$$P(z,t) = \frac{r_0}{\sqrt{\pi\,(t^2 + r_0^4)}} e^{-\frac{r_0^2}{t^2 + r_0^4}(z + z')^2} \ , \tag{56}$$

which is exactly equal to the quantum solution.

The current is calculated from (43). Only the z component is nonzero and it is given by

$$j_z = P(z,t) \left[\frac{t(z + z')}{t^2 + r0^4} - \dot{z}' \right] . \tag{57}$$

It is interesting to note that from the knowledge of the current we get information about the phase of the function defined in (38) (we recognize it as the wave function). The two are related by (19), and if the wave function is parametrized as $\psi = \sqrt{P(z,t)} \exp(i\eta)$ then

$$\frac{\partial \eta}{\partial z} = \frac{j_z(z,t)}{P(z,t)} \ , \tag{58}$$

from where we can calculate the phase η by integration. If we use (57) the resulting phase is exactly the same as the one obtained from the semiclassical theory when solving (18). Therefore the classical theory and the semiclassical theory give the same result for the dynamics of a particle when the radiation reaction interaction is included.

IV. DISCUSSION

We have shown how to formulate the radiation reaction interaction within semiclassical and classical theory. At least for the initial condition (21) the two give the same answer. It is believed, but not investigated here, that similar agreement is expected for other initial conditions, of the sort which are found in the dynamics of a particle when no radiation reaction interaction is included [9].

The classical formulation of the radiation reaction interaction, that is described here is only distantly related to the various other classical formulations, and in particular to the Abraham-Lorentz theory. All the formulations, however, agree on one thing: the EM field, which has its source in the charge, interacts back with the charge. This is the only common ground, but the rest is different. First, in the present formulation the concept of the pointlike charge is not used (and hence the Abraham-Lorentz equations are not used). Second, instead of the charge distribution one works with the probability distribution and through that various interaction terms had to be omitted from the dynamics equations. And thirdly, even if one associates the probability distribution with the charge distribution, the latter is not rigid, as is assumed in the classical theory of the electron. Therefore a number of conclusions in the Abraham-Lorentz theory, which are based on the assumption of a rigid charge distribution, are not applicable in the formulation which is derived here.

As for the semiclassical approach there are a number of objections which are always used when this theory is discussed, and they can be summarized in the statement: the theory has limited use since QED is the complete theory of radiation [3,12]. Despite these objections the semiclassical theory of radiation is quite successful in many circumstances. The theory has its merits: it goes beyond the perturbative schemes, which are the basis of QED, and it allows the calculation of the time evolution of the systems. These are only a few of the advantages, but perhaps the most important reason why one should investigate the semiclassical theory is to learn its true limitations. In the spirit of the last remark we showed in this work how the theory can be used to describe the radiation reaction interaction.

The semiclassical formulation of the radiation reaction interaction is based on the set of equations (1) and (2), and its relativistic version (see, e.g., [13], Eqs. (17.6) and (17.7))

$$\left(\gamma \frac{\partial}{\partial x} + m \right) \psi = ie\gamma \, A(x) \, \psi \ , \tag{59}$$

$$\left(\Delta - \frac{\partial^2}{\partial t^2} \right) A_\mu(x) = -\frac{ie}{2} \left[\overline{\psi}(x), \gamma_\mu \psi(x) \right]$$

forms the basis for QED. The only specific feature of the set (59) is that the "fields" are quantized. More recently the equation for the EM field in (59) was solved in its nonquantized form and then the solution was incorporated into the Hamiltonian of the system (with quantized "fields"), and again without any constraint on the form of the potentials [14–16]. Therefore the idea that the set (1) and (2) solves the problem of the radiation reaction interaction is not new, but the use of this set in the present work differs in some crucial points from its use elsewhere. One of them is that the set (59) is being formally implemented without making distinctions among the terms which are involved. For example, the importance of the Coulomb term is recognized as a contribution to the self-energy of charge [17], but it is not dismissed on the grounds that it is not physical. Sometimes the Coulomb term is just neglected with the argument that one only considers the transversal fields [18].

If, however, the Coulomb term is retained in the equations (as is the case in a number of applications in QED) then there should be consequences of its presence, and one of them will be briefly discussed. We will study the simplest problem: the self-interaction of a free particle, the one which is usually used to demonstrate the self-energy of a particle. Classical theory will be used to calculate the time evolution of the probability distribution, and from that the quantum solution will be inferred.

The reason for using classical theory is that it is simpler in application than quantum theory, but it also gives insight into the nature of the problem. As was shown in the previous section the quantum solution is easily obtained from the classical solution by using the relationship between the classical probability and current distributions and the wave function, summarized in (58).

For simplicity we will consider spherically symmetric probability distributions, in which case the classical equation of motion for a free particle is

$$\ddot{\mathbf{r}} = -\alpha\nabla\int d^3r'\frac{P(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi\alpha\mathbf{r}}{r^3}\int_0^r dr'\,r'^2\,P(r',t)\,, \tag{60}$$

where in Eq. (45) we have considered only the static electric field. For the initial probability distribution we have used (48), and Eq. (60) was solved by the following procedure. N sets of random initial conditions are generated for $\mathbf{r}$ and $\mathbf{v}$, based on the distributions $P_0(\mathbf{r})$ and $Q_0(\mathbf{v})$, respectively. At each time step the distribution $P(r,t)$ is recovered (the method is described in various places), and from that the integral (60) is calculated. In this way the force on each trajectory is calculated, which allows integration of the set of $6N$ coupled differential equations. In our application it was necessary to propagate $N = 1000$ trajectories to achieve reasonable accuracy of the results. More trajectories did not substantially improve the results.

Two examples were calculated in order to illustrate the typical time dependence of the probability distribution, when the Coulomb interaction is included, one for a relatively narrow initial probability distribution, and the other for a very broad one. The last one is of particular interest because it mimics the plane wave limit, which is so much used in quantum theory. Figure 1 shows results of these calculations for the radial probability distribution in those two cases. For a narrow initial distribution ($r_0 = 10$), and after a relatively long time ($t = 250$), the probability distribution (solid, steplike line) is more or less indistinguishable from the distribution when the Coulomb interaction is not included (broken line). On the other hand, for the broad initial distribution ($r_0 = 10^5$) the deviation between the two is enormous. The distribution without the Coulomb interaction included has the time evolution which is expected of it: it slowly dissipates, being nearly stationary (broken line). When the Coulomb interaction is included, the distribution "explodes" producing a "pulse" which travels radially with a very well defined shape (solid, steplike line). In Fig. 1 this "pulse" is shown after $t = 10^9$ units of time. Of course, such a behavior is expected from a charge distribution [which it is assumed that $P(\mathbf{r},t)$ represents], but what is less obvious is that the Coulomb repulsion has a so much greater effect on the broad distributions than on the narrow ones.

Those results are perhaps of interest but they are of no relevance, because they are not physical. However, since in quantum theory the Coulomb term is often retained, it is of interest to learn how it manifests itself in calculations. In particular, we will analyze the Green's function,

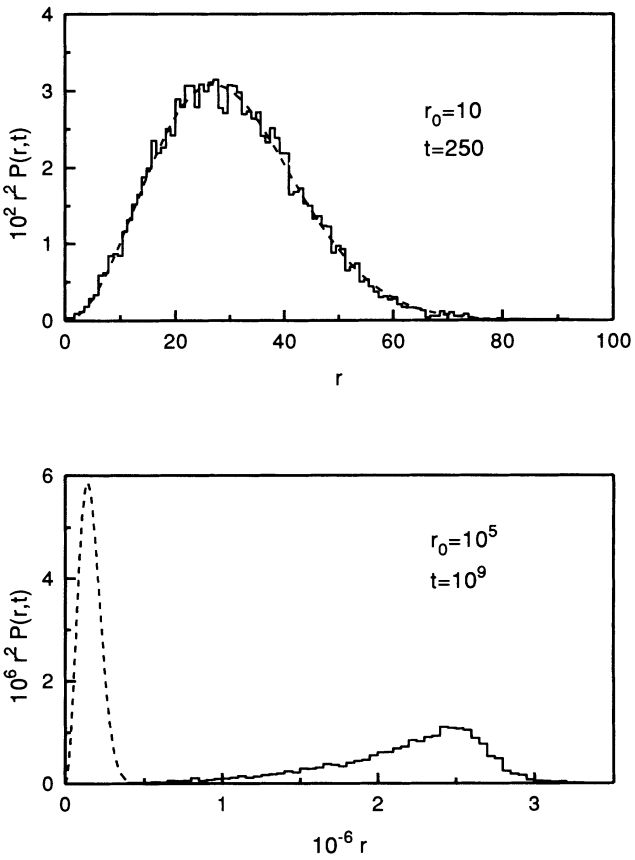

FIG. 1. Impact of the Coulomb interaction on the dynamics of the probability distribution, for a narrow ($r_0 = 10$) and a broad ($r_0 = 10^5$) probability distribution. Probability distribution without the Coulomb interaction included is designated by the broken line.

and its Fourier transform, both in the coordinate and the time variable. Namely, the Green's function plays the central role in solving the problems of QED, and in particular it is used in its Fourier-transformed form [19]. The function is defined as [20]

$$\psi(\mathbf{r},t) = \begin{cases} i\int d^3r'\,G(\mathbf{r}-\mathbf{r}',t)\,\psi(\mathbf{r}',0)\,, & t \geq 0\,, \\ 0, & t<0\,, \end{cases} \tag{61}$$

where $G(\mathbf{r},t)$ is the Green's function. Its Fourier transform is then given by

$$\begin{aligned} g(\mathbf{k},w) &= \frac{1}{(2\pi)^4}\int d^3r\,dt\,G(\mathbf{r},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+iwt} \\ &= \frac{1}{i(2\pi)^3}\frac{\phi(\mathbf{k},w)}{\phi_0(\mathbf{k})}\,, \end{aligned} \tag{62}$$

where

$$\phi(\mathbf{k},w) = \frac{1}{(2\pi)^4}\int d^3r\,dt\,\psi(\mathbf{r},t)\,e^{-i\mathbf{k}\cdot\mathbf{r}+iwt} \tag{63}$$

and

$$\phi_0(\mathbf{k}) = \frac{1}{(2\pi)^3}\int d^3r\,\psi(\mathbf{r},0)\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,. \tag{64}$$

In QED it is the Fourier transform of the Green's function which is usually analyzed, and the Feynman diagrams represent a way of visualizing various contributions in the expansion of $g(\mathbf{k}, w)$ in powers of α [19].

For the analysis of the Green's function, and its expansion in a power series in the fine structure constant, we can use the procedure which was described in the previous section [see Eq. (58)]. The basic ingredients of the wave function are its amplitude, given by the square root of $P(\mathbf{r}, t)$ and its phase, given by (58). Therefore expansion of the Green's function (62) in powers of α is produced by first expanding those two quantities in the series

$$P(\mathbf{r}, t) = P_0(\mathbf{r}, t) + \alpha P_1(\mathbf{r}, t) + \cdots , \quad (65)$$
$$\eta(\mathbf{r}, t) = \eta_0(\mathbf{r}, t) + \alpha \eta_1(\mathbf{r}, t) + \cdots ,$$

and then calculating the Fourier transform (63). However, one is immediately confronted with a problem: since the integration limit in the time variable is infinite, i.e., an integral of the form

$$\phi(\mathbf{k}, w) \approx \int_0^\infty dt \sqrt{P(r,t)}\, e^{iwt + i\eta(r,t)} \quad (66)$$

should be calculated, then both $P(\mathbf{r}, t)$ and $\eta(\mathbf{r}, t)$ must be known at all times. This would not perhaps be such a problem, if we were not to use the expansion (65). The series is useful if it converges rapidly, but this is obviously not the case for all times, and especially for very broad distributions. As can be seen from Fig. 1 the probability distribution with the radiation reaction interaction included is not a perturbation of the distribution when this interaction is not included. Therefore the correction of the order α in the series (65) is enormous for large times, and in fact it is infinite for infinite time. One would, therefore, expect that the expansion of the Green's function in the power series of α has no meaning. This is indeed the case, and it will be shown more specifically in the analysis of the phase of the wave function.

Based on the expansion of the phase according to (65) we write

$$e^{i\eta(r,t)} = e^{i\eta_0(r,t)} + i\alpha\eta_1(r,t)\, e^{i\eta_0(r,t)} + \cdots , \quad (67)$$

where $\eta_0 = tr^2/(t^2 + r_0^4)$ is the phase of the unperturbed wave function (Gaussian) [21]. The phase η is calculated from (58), and the typical dependence of $d\eta/dr$ on r, for a fixed and large t, is shown in Fig. 2. It has a very good linear fit $d'(r,t) \approx f(t)\, r$, where the function has the asymptotic behavior $f(t) \approx \epsilon/t$ for large t (which was calculated numerically). Therefore the leading correction, of order α, to the function $\phi(\mathbf{k}, w)$ is now

$$\phi(\mathbf{k}, w) \approx \alpha \int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}} \int^\infty dt\, \psi_0(\mathbf{r}, t)\, e^{iwt}\, r^2\, \frac{\epsilon - 1}{t} , \quad (68)$$

where $\psi_0(\mathbf{r}, t)$ represents the unperturbed wave function [21]. $\eta_1(r,t)$ was replaced by $\eta(r,t) - \eta_0(r,t) \approx (\epsilon - 1)/t$,

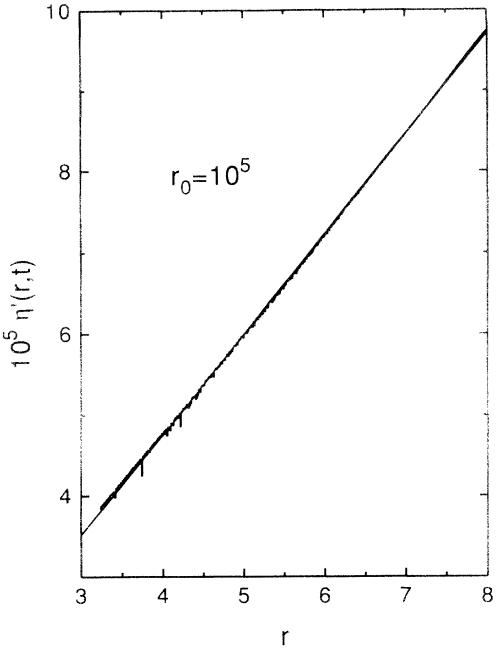

FIG. 2. Typical value of the derivative $\eta'(r,t)$ as a function of r (steplike line), and its best linear fit (solid line).

because the dominant contribution to the integral comes from large values of t. If $\psi_0(r,t)$ is replaced by its Fourier transform [21], the function $\phi(\mathbf{k}, w)$ is

$$\phi(\mathbf{k}, w) \approx \alpha(\epsilon - 1)\Delta_k \int^\infty \frac{dt}{t}\, e^{iwt - a^2k^2/2 - ik^2t/2} , \quad (69)$$

which is either finite ($\epsilon = 1$) or infinite ($\epsilon \neq 1$), with a logarithmic singularity. Detailed numerical analysis showed that $\epsilon \neq 1$ and hence the first-order correction to the momentum space Green's function is infinite. This is the same kind of singularity in Green's function which is also found in QED, and one can, therefore, talk about the classical explanation of the infinities which appear there.

As we have seen, the singularities which appear in the calculations of the Green's function are primarily caused by the nonphysical terms in the dynamics equations. In fact, it is not only the Coulomb term which causes problems, but this is also the case with the magnetic term. A few numerical studies (not reported here) showed that the impact of the magnetic term on the time evolution of the wave function is of a similar nature to that of the Coulomb term. Therefore, it is physically reasonable to omit those two terms from the dynamics equations, and work only with equations of the form (18) or (45). These equations are now being used for calculating various processes where the radiation reaction interaction plays a crucial role (e.g., spontaneous emission), for systems such as the harmonic oscillator and the hydrogen atom. The results so far are quite intriguing, and show that indeed those equations might be a good description of the radiation reaction interaction.

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