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Phase statistics and phase-space measurements

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The statistics of quantum optical phase observables are recovered as patterns from both number as well as phase-space measurement statistics. A model for the measurement of generalized Q functions studied recently by Leonhardt and Paul [Phys. Rev. A **47**, R2460 (1993); **48**, 3265 (1993)] is shown to yield measurements of arbitrary phase-space observables.

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I. INTRODUCTION

The phase of a single-mode electromagnetic field is an observable which is covariant under the shifts generated by the number observable associated with that mode. Since the advent of quantum mechanics it has been known that there exists no self-adjoint "phase" operator, and a lot of effort has gone into finding an appropriate formal representation of such an observable [1]. At the same time, the question of operationally defining and measuring the phase has remained an issue of intense investigations [2]. The formal part of the "phase problem" is resolved in a surprisingly simple and satisfactory way as soon as the proper formulation of observables as (normalized) positive-operator-valued (POV) measures is taken into account [3]; it is then evident that there is a whole family of phase observables, observables conjugate to the number [4]. In this paper we present an operational method of constructing a class of phase space, and thus phase observables of a single-mode field from some photocounting statistics obtained under various conditions. We then analyze a recent measurement model [5] for determining generalized Q functions to show that it yields, in fact, measurements of arbitrary phase-space observables. Our results confirm the view [6] that there is not just one single phase observable but a class of them corresponding to different measurement schemes.

II. PHASE OBSERVABLES

We consider a single-mode field, the signal, with the annihilation and creation operators a and a^* and associated number observable $N=a^*a=\sum n|n\rangle\langle n|$. An observable $E\colon Y\mapsto E(Y)$ is a *phase observable* of the mode a if it is covariant under the shifts generated by N:

$$e^{i\phi N}E(Y)e^{-i\phi N}=E(Y+\phi)\ , \tag{1}$$

with Y a (Borel) subset of $[0,2\pi)$ and $\phi\in[0,2\pi)$.

There is a *canonical phase observable* M associated with the polar decomposition of a, $a=V\sqrt{N}$, with the partial isometry $V=\sum_{n=0}^{\infty}|n\rangle\langle n+1|$. Indeed, since V is a contraction, there is a unique POV measure M such that $V^n=\int_0^{2\pi}e^{in\phi}M(d\phi)$ and $(V^*)^n=\int_0^{2\pi}e^{-in\phi}M(d\phi)$ [7]. This measure has the form [4]

$$M(Y):=\sum_{m,n=0}^{\infty}(2\pi)^{-1}\int_Y d\phi\, e^{i(n-m)\phi}|n\rangle\langle m| \tag{2}$$

and it fulfills the covariance condition (1).

We shall consider here a class of other phase observables which arise from particular *phase-space observables* and which can be viewed as "noisy" versions of M. To introduce them let $|z\rangle=D_z|0\rangle$ be the coherent state generated from the vacuum state $|0\rangle$ by application of the displacement operator $D_z=\exp(za^*-\bar{z}a)$, $z\in\mathbb{C}$. It is well known that the mapping

$$A\colon Z\mapsto A(Z):=\frac{1}{\pi}\int_Z d^2z|z\rangle\langle z| =\frac{1}{\pi}\int_Z d^2z\, D_z|0\rangle\langle 0|D_z^* \tag{3}$$

defines a (normalized) POV measure on the complex plane $\mathbb{C}$, the phase space. Using the Cartesian coordinates (q,p), $q,p\in\mathbb{R}$, or the polar coordinates (r,ϕ), $r\geq 0$, $\phi\in[0,2\pi)$ to represent the complex plane leads to two sets of marginal observables of A, the first pair being related to the quadrature components of the fields, the second to its number and phase. In particular, with the polar decomposition $z=re^{i\phi}$ one is dealing with sets of the form $Z_R=R\times[0,2\pi)$ and $Z_Y=[0,\infty)\times Y$, which lead to the number and phase marginal observables

$$A_N(R):=A(Z_R)=\int_R dr^2(2\pi)^{-1}\int_0^{2\pi}d\phi\, D_z|0\rangle\langle 0|D_z^* =\sum_{n=0}^{\infty}\int_R dr^2 p_n(r^2)|n\rangle\langle n|\ ,$$

$$p_n(r^2)\equiv e^{-r^2}r^{2n}/n!\ , \tag{4}$$

$$A_{\mathrm{ph}}(Y) := A(Z_Y) = \int_0^\infty dr^2 (2\pi)^{-1} \int_Y d\phi |z\rangle\langle z|$$
$$= \int_Y dr^2 T_r M(Y) T_r^* ,$$
$$T_r \equiv \sum \frac{r^n}{\sqrt{n!}} e^{-(1/2)r^2} |n\rangle\langle n| . \quad (5)$$

Indeed, A_N is a smeared number observable, whereas A_{ph}, satisfying Eq. (1), is a phase observable, being, in fact, a noisy version of the phase observable M [8].

Instead of the vacuum state $|0\rangle$, one may equally well take any other state to generate other phase-space observables. If one takes a number state $|n\rangle$ then one obtains

$$A^{(n)}: Z \mapsto A^{(n)}(Z) = \frac{1}{\pi} \int_Z d^2 z \, D_z |n\rangle\langle n| D_z^* . \quad (6)$$

In this case the marginals associated with the polar coordinates are again number and phase observables $A_N^{(n)}$ and $A_{\mathrm{ph}}^{(n)}$ (with the appropriate shift covariance). Such phase-space observables can be accessed operationally in at least two ways. First of all, they can be obtained from an analysis of the count statistics obtained in certain measurements performed on the signal (Sec. III); and, secondly, these observables result from a particular measurement scheme (Sec. IV). Consequently, the ensuing phase observables $A_{\mathrm{ph}}^{(n)}$ are operationally justified as well.

To close these introductory considerations, we note that any phase observable E gives rise to a self-adjoining "phase operator" $\Phi^E \equiv E^{(1)} := \int \phi E(d\phi)$. However, the higher moments of E do not coincide with the corresponding powers of Φ^E so that these operators cannot be used to determine the moments of the probability distributions given by the POV measure and the states. Furthermore, even the first-moment operator carries with itself a nonuniqueness in view of the phase-shift group: applying a shift operation to Φ^E yields a new operator which does not even commute with the original one. Thus the mere choice of the origin of the phase scale can lead to mutually incommensurable "phase operators." Finally, it should also be emphasized that the POV measure structure of an observable is operationally determined by the totality of measurement outcome statistics, that is, the probability distributions of the measurement results associated with the states of the system [4]. These remarks demonstrate the priority of the POV measure point of view over the operator representation of observables.

III. PHASE DISTRIBUTIONS FROM NUMBER STATISTICS

We consider an experiment in which the signal mode is mixed with a local oscillator (LO) (with the mode operator b and b^*) by means of a beam splitter with transparency ε (Fig. 1). The action of a beam splitter is given by the two-mode mixer [9]

$$U_\alpha = \exp(\bar{\alpha} a \otimes b^* - \alpha a^* \otimes b) , \quad (7)$$

where $\alpha = |\alpha| e^{i\vartheta}$, $\cos|\alpha| = \sqrt{\varepsilon}$, $0 \le |\alpha| \le \pi/2$, and $-\pi/2 < \vartheta \le \pi/2$. If T is the input state of the signal mode and if the local oscillator is in a coherent state $|z\rangle$,

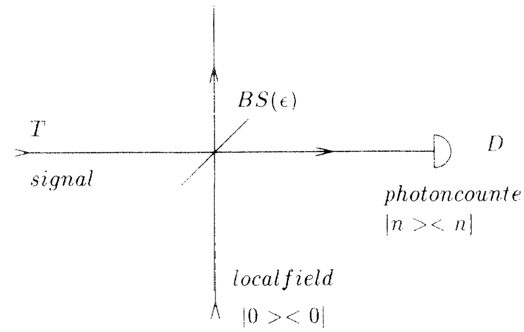

FIG. 1. Photodetection of a signal mixed with a local oscillator.

then $W \equiv U_\alpha (T \otimes |z\rangle\langle z|) U_\alpha^*$ is the state of the two-mode field after the mixing, and the probability of detecting n photons in the detector D (of unit quantum efficiency) is $\mathrm{tr}[W |n\rangle\langle n| \otimes I]$. This counting statistics defines an observable $E^{z,\varepsilon}$ of the signal mode (depending on the beam-splitter parameter ε and on the state $|z\rangle$ of the local oscillator) such that for each input state T and for all n

$$\mathrm{tr}[T E_n^{z,\varepsilon}] = \mathrm{tr}[U_\alpha (T \otimes |z\rangle\langle z|) U_\alpha^* |n\rangle\langle n| \otimes I] . \quad (8)$$

The explicit structure of the observable $E^{z,\varepsilon}$ can be extracted from this equation, and one gets [10]

$$E_n^{z,\varepsilon} = D_{xz} E_n^{0,\varepsilon} D_{xz}^* , \quad (9)$$

where D_{xz} is the signal mode displacement operator, with $x = -(\tan|\alpha|) e^{-i\vartheta}$, and $E^{0,\varepsilon}$ is the observable resulting from mixing the signal with an idle mode,

$$n \mapsto E_n^{0,\varepsilon} = \sum_{m=n}^{\infty} \binom{m}{n} \varepsilon^n (1-\varepsilon)^{m-n} |m\rangle\langle m| . \quad (10)$$

The measurement outcome statistics of the observable (10) is the Bernoulli distribution (with parameter ε) which is known to coincide with the counting statistics of a photodetector with quantum efficiency ε [11]. Similarly, the outcome statistics $n \mapsto \mathrm{tr}[T E_n^{z,\varepsilon}]$ of the observable $E^{z,\varepsilon}$ is just the counting statistics obtained when the input signal is first mixed with coherent (single-mode) light and then detected with a counter of quantum efficiency ε [11].

The first moment of the observable $E^{z,\varepsilon}$ is found by straightforward computation:

$$N_\varepsilon := \sum_{n=0}^{\infty} n E_n^{z,\varepsilon}$$
$$= \varepsilon N + (1-\varepsilon)|z|^2 I + \sqrt{\varepsilon(1-\varepsilon)} (\bar{z} a + z a^*) . \quad (11)$$

The expectation value of this observable in the signal input state T depends, in particular, on the strength $|z|^2$ of the LO field, and it contains an interference term $\sqrt{\varepsilon(1-\varepsilon)}\,\mathrm{tr}[T(\bar{z}a + z a^*)]$ resulting from mixing the signal with the LO field. This suggests that the interference pattern could be used to gain information on the phase of the input signal. To get a high resolution, the strength of the local field should be strong. However, in the limit $|z| \to \infty$, all the effects $E_n^{z,\varepsilon}$ tend (weakly) to the null operator, so that one has to consider a more refined limit allowed by the apparatus parameter ε. Indeed, letting $\varepsilon \to 1$ together with $|z| \to \infty$ such that $xz = u$ is a fixed complex number, one has

$$(1-\varepsilon)|z|^2 = \varepsilon |x|^2 |z|^2 = \varepsilon |u|^2 ,$$

and

$$E_n^{z,\varepsilon} \to D_u |n\rangle\langle n| D_u^* =: E_n^u . \tag{12}$$

Thus photodetection with an almost ideal photocounter ($\varepsilon \simeq 1$) of a single-mode field mixed in the active part of the detector with a strong local oscillator ($|z|^2 \gg 1$) defines a signal observable $n \mapsto E_n^u$, where $|u| = |xz|$ describes the percentage of energy which the signal gains from the coherent pulse $|z\rangle$. The first moment of this limiting observable is

$$N^u := \sum n E_n^u = D_u N D_u^* = N + |u|^2 I + (ua + \bar{u} a^*) , \tag{13}$$

showing still a similar interference pattern to Eq. (11).

In order to obtain phase information on the signal, we consider *several* such photodetection schemes with the ensuing number statistics $n \mapsto \mathrm{tr}[T E_n^u]$, labeled by the apparatus parameter $u = xz \in \mathbb{C}$. Now, for any *fixed* n we may add up all the measurement outcome probabilities $\mathrm{tr}[T E_n^u]$, $u \in \mathbb{C}$, with suitable weights so that we obtain a single probability distribution on the phase space $\mathbb{C}$,

$$Z \mapsto \frac{1}{\pi} \int_Z \mathrm{tr}[T E_n^u] d^2 u . \tag{14}$$

These probability measures define the POV measure

$$A^{(n)}: Z \mapsto A^{(n)}(Z) := \frac{1}{\pi} \int_Z d^2 u \, D_u |n\rangle\langle n| D_u^* , \tag{15}$$

which is just the phase-space observable (6) generated by the number state $|n\rangle$. The phase marginal $A_{\mathrm{ph}}^{(n)}$ of this observable is a phase observable of the signal mode. Therefore we conclude that, by collecting the number statistics $n \mapsto \mathrm{tr}[T E_n^{z,\varepsilon}]$ from different photodetection schemes, with the apparatus parameters $\varepsilon \simeq 1, |z|^2 \gg 1$, one gets the phase distribution

$$X \mapsto \mathrm{tr}[T A_{\mathrm{ph}}^{(n)}(X)] \tag{16}$$

for each possible number outcome n. This type of reasoning is widely practiced also in parameter estimation theory which offers an alternative approach to, and use of, POV measurements [12].

It should be emphasized that the probability measures (14), as they are constituted here, do not correspond to a measurement of the observable (15). They should rather be seen as a pattern that is hidden in the totality of statistics collected in the manifold of E^u measurements labeled with the apparatus parameter u. However, the method followed here is an operational procedure which leads to a definition of the phase-space observables $A^{(n)}$ and the corresponding phase observables $A_{\mathrm{ph}}^{(n)}$ of a single-mode radiation field. A slightly more direct approach to these observables is obtained in the unsharp joint measurement scheme of the quadrature components of the field to be discussed next.

IV. JOINT MEASUREMENT OF THE QUADRATURE COMPONENTS OF A SINGLE-MODE FIELD

We consider again a beam-splitter experiment where now on the two output ports there are detectors which are sensitive to the quadrature components $a^q = (1/\sqrt{2})(a^* + a)$ and $b^p = (i/\sqrt{2})(a^* - a)$ of the modes (Fig. 2). If the incoming field modes are prepared independently in the states T and T', then the detection statistics determines, again, a signal observable $\tilde{A}$,

$$\mathrm{tr}[T \tilde{A}(X \times Y)] := \mathrm{tr}[U_\alpha (T \otimes T') U_\alpha^* E^q(X) \otimes E^p(Y)] , \tag{17}$$

with $E^q(X)$ and $E^p(Y)$ being the spectral projections of a^q and b^p associated with the real (Borel) sets X and Y. In Ref. [5] the corresponding probability distributions were determined for coherent-state and squeezed-state inputs for the b mode. We shall not show that for beam splitters of arbitrary transparency and for any input state of the second mode the ensuing observable is just the phase-space observable (3) (modulo a rescaling).

The beam-splitter coupling (7) can be written in terms of the respective quadrature components

$$U_r = \exp[ir(a^q \otimes b^p - a^p \otimes b^q)] . \tag{18}$$

Here we have put $\alpha = r \in (0, \pi/2)$, that is, we consider a beam splitter with transparency $\varepsilon = \cos^2 r$ and with no phase shift. The operator in the exponent of (17) is formally identical to the angular momentum component L_3. This observation makes it straightforward to evaluate the probability reproducibility condition (17) for the measured observable $\tilde{A}$ [5]. For simplicity we shall consider vector state inputs $T = |\varphi\rangle\langle\varphi|$ and $T' = |\psi\rangle\langle\psi|$ for the two modes. Using the Schrödinger representation one obtains

$$\begin{aligned}
\langle \varphi | \tilde{A}(X \times Y) \varphi \rangle &= \langle \varphi \otimes \psi | U_r^* E^q(X) \otimes E^p(Y) U_r \varphi \otimes \psi \rangle \\
&= \int_{X \times Y} dq \, dp \left| U_F^{(b)} U_r \varphi \psi(q,p) \right|^2 \\
&= (2\pi)^{-1} \int_{X_r \times Y_r} dq \, dp \langle \varphi | \xi_{qp} \rangle \langle \xi_{qp} | \varphi \rangle \\
&=: \langle \varphi | A(X_r \times Y_r) \varphi \rangle .
\end{aligned} \tag{19}$$

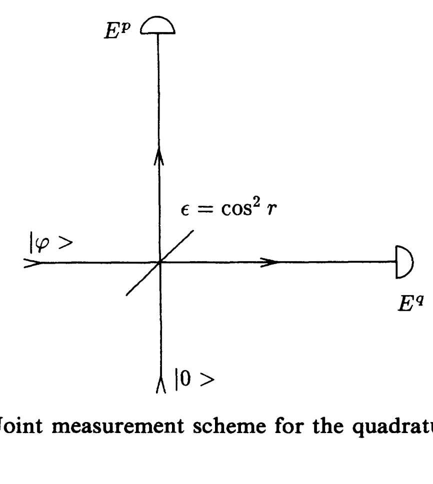

FIG. 2. Joint measurement scheme for the quadrature components.

Here $U_F^{(b)}$ denotes the Fourier-Plancherel operator with respect to the second degree of freedom. We have introduced the scaled sets $X_r = X/\cos r$ and $Y_r = -Y/\sin r$. Furthermore, $\xi_{qp} = W_{qp}\xi$, $W_{qp} = \exp(ipQ - iqP)$, denotes the phase-space translation of the (normalized) state function

$$\xi(y) := (\tan r)^{-1/2}\overline{\psi\left(-\frac{y}{\tan r}\right)} . \quad (20)$$

The accordingly rescaled observable is thus

$$Z \mapsto A_\xi(Z) = \frac{1}{\pi}\int_Z d^2z\, D_z|\xi\rangle\langle\xi|D_z^* , \quad (21)$$

which is a phase-space observable. The coupling (18) with the detection observables a^q and b^p constitutes, therefore, a measurement of the phase-space observable A_ξ and thus, in particular, a joint measurement of its Cartesian as well as its polar marginal observables.

For the case of a vacuum input in the b mode one obtains the Q distribution associated with the observable

$$Z \mapsto A(Z) = \frac{1}{\pi}\int_Z |z\rangle\langle x| d^2z . \quad (22)$$

Now it is interesting to note that one can choose ϕ so as to have ξ be a number state $|n\rangle$. One has thus found measurement schemes yielding the statistics for all the phase-space and phase observables $A^{(n)}$ and $A_{\text{ph}}^{(n)}$ described in Sec. III.

V. CONCLUSION

In our analysis of the "phase problem" we have indicated two basic approached towards understanding POV measures in quantum mechanics. On one hand, a (phase) POV measure may be inferred in the context of quantum estimation theory as a pattern emerging from the statistics of several series of measurements. On the other hand, a more direct way of deriving a phase observable consists of performing a joint measurement of the quadrature observables. The outcome of such a measurement can be interpreted as a pair of Cartesian coordinates in phase space which gives rise to a unique set of polar-coordinate values. That such a scheme amounts to measuring simultaneously though unsharply number and phase observables is statistically confirmed by writing down the corresponding marginal observables. Moreover, on the individual level one may conceive of carrying out a calibration procedure as follows. Consider a coherent state with large amplitude, so that the phase is fairly well defined. For this state it is possible to specify a circular slice in phase space such that the probability of finding an outcome within that region, upon measuring the underlying phase-space observable, is close to unity. It is thus possible to calibrate this measurement so that its outcomes give direct information as to the average number and phase values of the input state. It should be noted finally that the individual measurement outcomes carry with themselves an intrinsic inaccuracy compatible with the Heisenberg uncertainty relation for the quadrature components. In the present model, this unsharpness is determined by the characteristics of the input state of the second mode. A more detailed discussion of these phase-space measurement inaccuracies and their relevance for the state inference problem has been given elsewhere in the context of a joint position-momentum measurement model [13].

[1] There is a vast body of literature on this subject, some of which can be traced from [2,6,8].

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