Early-time properties of quantum evolution

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Approximate formulas are given for the effective Hamiltonian $H_{\parallel}(t)$ governing the time evolution in a subspace $\mathcal{H}_{\parallel}$ of the state space $\mathcal{H}$. It is proved that this approximation is correct for any Hamiltonian H of the system under consideration at the early-time period. The approximate form of the survival amplitude for a given state improving Fleming's estimation in the short-time region is found and the properties of a decay rate for small, intermediate, and long times are discussed.

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I. INTRODUCTION

If we are searching for some specific properties of a physical system, it is not always convenient to study the time evolution in the total Hilbert space $\mathcal{H}$ of states $|\psi;t\rangle$, $|\psi\rangle\in\mathcal{H}$ described by solutions of the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi;t\rangle=H|\psi;t\rangle\ , \tag{1a}$$

for the initial conditions

$$|\psi;t=t_0=0\rangle\equiv|\psi\rangle\ , \tag{1b}$$

where H is the total self-adjoint Hamiltonian of the system considered, i.e., to search for the properties of a total unitary evolution operator $U(t)\equiv\exp(-itH)$ acting in $\mathcal{H}$: $|\psi;t\rangle\equiv U(t)|\psi\rangle$. Instead it may be more convenient to study the time evolution in some closed subspace $\mathcal{H}_{\parallel}$ of $\mathcal{H}$ [1–10] and the properties of the effective Hamiltonian $H_{\parallel}(t)$ governing this time evolution. In particular, such an approach seems to be effective in the most general description of the early-time behavior of a given nonstationary state $|\psi;t\rangle$, a problem which has recently been more and more frequently studied [1–18]. Moreover the meaning of such investigations has recently taken on a new significance with the progress of experimental possibilities [18].

In this case the total state space $\mathcal{H}$ splits into two orthogonal subspaces $\mathcal{H}_{\parallel}$ and $\mathcal{H}_{\perp}\equiv\mathcal{H}\ominus\mathcal{H}_{\parallel}$ and thus the Schrödinger equation (1) can be replaced by two coupled equations for subspaces $\mathcal{H}_{\parallel}$ and $\mathcal{H}_{\perp}$. Using a solution of the evolution equation for subspace $\mathcal{H}_{\perp}$, one can obtain the evolution equation in the subspace $\mathcal{H}_{\parallel}$ of vectors $|\psi;t\rangle_{\parallel}$, defined by a projector $P=P^{+}=P^{2}$: $\mathcal{H}_{\parallel}=P\mathcal{H}\ni|\psi;t\rangle_{\parallel}\equiv P|\psi;t\rangle$, which has the following form [1–3] for $t\geq0$:

$$\left[i\frac{\partial}{\partial t}-PHP\right]|\psi;t\rangle_{\parallel}=|\chi;t\rangle-i\int_0^{\infty}K(t-\tau)|\psi;\tau\rangle_{\parallel}d\tau\ , \tag{2a}$$

where the initial condition (1b) is replaced by

$$|\psi;t=t_0\equiv0\rangle_{\parallel}\equiv|\psi\rangle_{\parallel},\quad |\psi;t=t_0\equiv0\rangle_{\perp}\equiv|\psi\rangle_{\perp}\ , \tag{2b}$$

$$|\psi;t\rangle_{\perp}=Q|\psi;t\rangle\in\mathcal{H}_{\perp}\equiv Q\mathcal{H},\quad Q=1-P\ , \tag{3}$$

and

$$K(t)=\Theta(t)PHQe^{-itQHQ}QHP\ , \tag{4}$$

$$|\chi;t\rangle=PHQe^{-itQHQ}|\psi\rangle_{\perp}\ , \tag{5}$$

and $\Theta(t)$ is a step function: $\Theta(t)=1$ for $t\geq0$ and 0 for $t<0$.

If states in the subspace $\mathcal{H}_{\perp}$ are not occupied at the initial instant $t_0=0$, i.e., if

$$|\psi;t_0=0\rangle_{\perp}\equiv|\psi\rangle_{\perp}\equiv Q|\psi\rangle=0\ , \tag{6}$$

then $|\chi;t\rangle\equiv0$ in (2) and $|\psi\rangle\equiv P|\psi\rangle\equiv|\psi\rangle_{\parallel}$. Therefore $|\psi;t\rangle_{\parallel}\equiv P|\psi;t\rangle\equiv PU(t)|\psi\rangle\equiv PU(t)P\ \ P|\psi\rangle=U_{\parallel}(t)|\psi\rangle_{\parallel}$. Therefore Eq. (2) transforms into [1–3]

$$\left[i\frac{\partial}{\partial t}-PHP\right]U_{\parallel}(t)|\psi\rangle_{\parallel}=-i\int_0^{\infty}K(t-\tau)U_{\parallel}(\tau)|\psi\rangle_{\parallel}d\tau\ ,\qquad t\geq0,\ U_{\parallel}(t=0)\equiv P\ , \tag{7}$$

where $U_{\parallel}(t)$ is the (usually nonunitary) evolution operator for the subspace $\mathcal{H}_{\parallel}$.

By studying and applying equations of the form (2) and (7), Krolikowski and Rzewuski have found that sometimes it is convenient to replace these equations by the equivalent, only differential, Schrödinger-like equation, which is the case of initial conditions (6), i.e., for Eq. (7), is written

$$\left[i\frac{\partial}{\partial t}-H_{\parallel}(t)\right]U_{\parallel}(t)=0,\quad t\geq0,\ U_{\parallel}(t=0)=P\ . \tag{8}$$

The equivalence of Krolikowski-Rzewuski Eqs. (8) and (7) follows, e.g., from the identity

$$H_{\parallel}(t)\equiv i\frac{\partial U_{\parallel}(t)}{\partial t}U_{\parallel}^{-1}(t) \tag{9}$$

for $U_{\parallel}(t)$ fulfilling (7). The effective Hamiltonian $H_{\parallel}(t)$ has the form [1–3]

$$H_{\parallel}(t)\equiv PHP+V_{\parallel}(t)\ . \tag{10}$$

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The concrete formulas for $V_{\|}(t)$ can be found in Ref. [3]. The result of the action of the "quasipotential" $V_{\|}(t)$ on $U_{\|}(t)$ depends on the properties of the kernel $K(t)$:

$$V_{\|}(t)U_{\|}(t) \equiv -i\int_0^\infty K(t-\tau)U_{\|}(\tau)d\tau, \quad t \geq 0 . \tag{11}$$

Generally, early-time properties of $V_{\|}(t)$ follow directly from (9) and (11). We have [3,7–9]

$$V_{\|}(t=0) \equiv 0 \tag{12}$$

and

$$V_{\|}(t) \simeq -itPHQHP \equiv -it[PH^2P-(PHP)^2], \quad t \to 0$$
$$\equiv -it(\delta H_P)^2 , \tag{13}$$

quite independently of the properties of H.

II. AN APPROXIMATE FORMULA FOR $V_{\|}(t)$

The use of a retarded solution G of the nonhomogeneous equation

$$\left[i\frac{\partial}{\partial t}-PHP\right]G(t)=P\delta(t), \quad t \geq 0 , \tag{14}$$

i.e., the retarded Green's operator

$$G \equiv G(t) = -i\Theta(t)e^{-itPHP}P \tag{15}$$

enables us to replace the integro-differential equation (7) by the equivalent, purely integral one, and then following the ideas of Ref. [2] and applying the iteration procedure to solve this integral equation for $U_{\|}(t)$, we find that [9]

$$U_{\|}(t)=U_{\|}^0(t)+\sum_{n=1}^{\infty}(-i)^n L\circ L\circ\cdots\circ L\circ U_{\|}^0(t) , \tag{16}$$

where $U_{\|}^0(t)$ is the solution of the "free" equation

$$\left[i\frac{\partial}{\partial t}-PHP\right]U_{\|}^0(t)=0, \quad U_{\|}^0(0)=P , \tag{17}$$

the symbol $\circ$ denotes the convolution $f\circ g(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau$, L is convoluted n times, and

$$L \equiv L(t) = G\circ K(t) \equiv \int_0^\infty G(t-\tau)K(\tau)d\tau . \tag{18}$$

From (16) and (11) one obtains

$$V_{\|}(t)U_{\|}(t) = -iK\circ U_{\|}^0(t)$$
$$-i\sum_{n=1}^{\infty}(-i)^n K\circ L\circ L\cdots\circ L\circ U_{\|}^0(t) . \tag{19}$$

Of course, the formal series (16) and (19) are convergent if $\|L(t)\|<1$ [the existence of $\|L(t)\|$ is assumed]. These series are not the standard perturbation series, i.e., if one considers the Hamiltonian of the general form $H=H_{(0)}+H_I$, then in order that $\|L(t)\|<1$, it is not necessary for the perturbation H_I to be small with respect to the free part $H_{(0)}$. Therefore the approach leading to Eqs. (2), (7), or (8) has some advantage in relation to the standard perturbation methods because it enables us to describe processes generated not only by relatively weak interactions.

So, if for every $t \geq 0$

$$\|L(t)\| \ll 1 , \tag{20}$$

then, to the lowest order of $L(t)$, one finds, for $V_{\|}(t)$,

$$V_{\|}(t) \simeq V_{\|}^1(t) = -i\int_0^\infty K(t-\tau)e^{i(t-\tau)PHP}P\,d\tau, \quad t \geq 0 . \tag{21}$$

This approximate $V_{\|}(t)$, especially in the case of $\dim\mathcal{H}_{\|}=1$, is close to the Weisskopf-Wigner approximation in the long-time region, where one can replace $V_{\|}(t)$ by $V_{\|}(t) \simeq V_{\|}^1(t\to\infty)$. The advantage of formula (21) is that it can be applied for the study of time evolution both in the very-short-time period $t\to 0$, where the Weisskopf-Wigner formula does not work, and in the long-time period $t\to\infty$. However, the most important and useful property of approximation (21) is that at the early-time period it can sufficiently accurately describe not only weak but even very strong processes. This last conclusion follows from the properties of the operator $L(t)$. Indeed, the integral defining $L(t)$ is not taken between the limits $\tau=0$ and ∞ but, in fact, it is taken between $\tau=0$ and t. This follows from the definitions $G(t)$ and $K(t)$ and it is due to the presence of the step function $\Theta(t)$ in $G(t)$ and $K(t)$. Hence from the definition of $L(t)$ (18), we have

$$L(t) \underset{t\to 0}{\longrightarrow} 0 \tag{22}$$

and therefore one can conclude that for any P and H such that $[P,H]\neq 0$, $T_L>0$ always exists such that

$$\|L(t)\| \ll 1 \quad \text{if } 0 \leq t < T_l . \tag{23}$$

From this it follows that for every H, the effective Hamiltonian $H_{\|}(t) \simeq PHP + V_{\|}^1(t)$, where $V_{\|}^1(t)$ is given by formula (21), can describe the dynamics in the subspace $\mathcal{H}_{\|} \equiv P\mathcal{H}$ at the early-time period $0 \leq t < T_l$ to a very good approximation. Therefore the approach based on Eq. (8) seems to be especially effective in searching for the early-time behavior of physical systems. The maximal value T_l of times t for which the approximation (21) is still valid generally depends on a given H and P.

III. TIME EVOLUTION IN THE ONE-DIMENSIONAL SUBSPACE $\mathcal{H}_{\|}$

Let us consider the projector P defined by a normalized vector $|\alpha\rangle \in \mathcal{H}$. Then

$$P \equiv P_\alpha = |\alpha\rangle\langle\alpha| , \tag{24}$$

and, if $|\langle\alpha|H|\alpha\rangle| < \infty$, Eqs. (7) and (8) transform into, respectively,

$$\left[i\frac{\partial}{\partial t}-E_\alpha\right]u_\alpha(t) = -i\int_0^\infty k_\alpha(t-\tau)u_\alpha(\tau)d\tau ,$$
$$t \geq 0, \ u_\alpha(0)=1 \tag{25}$$

and

$$\left[i\frac{\partial}{\partial t}-E_\alpha-v_\alpha(t)\right]u_\alpha(t)=0,\quad t\geq 0\ , \tag{26}$$

where $u_\alpha(t)$, E_α, $k_\alpha(t)$, and $v_\alpha(t)$ replace $U_\parallel(t)$, PHP, $K(t)$, and $V_\parallel(t)$ in (7) and (8):

$$U_\parallel(t)\equiv\langle\alpha|U_\parallel(t)|\alpha\rangle P_\alpha\equiv\langle\alpha|U(t)|\alpha\rangle P_\alpha\equiv u_\alpha(t)P_\alpha\ , \tag{27}$$

$$P_\alpha HP_\alpha\equiv E_\alpha P_\alpha,\quad E_\alpha\equiv\langle\alpha|H|\alpha\rangle\ , \tag{28}$$

$$K(t)\equiv\langle\alpha|K(t)|\alpha\rangle P_\alpha\equiv k_\alpha(t)P_\alpha\ , \tag{29}$$

$$k_\alpha(t)=\Theta(t)\langle\alpha|HQe^{-itQHQ}QH|\alpha\rangle\ ,$$

$$V_\parallel(t)=v_\alpha(t)P_\alpha\ , \tag{30}$$

$$H_\parallel(t)\equiv[E_\alpha+v_\alpha(t)]P_\alpha\ . \tag{31}$$

From (21) we immediately find

$$v_\alpha(t)\simeq v_\alpha^1(t)=\left\langle\alpha\left|HQ\frac{e^{-it(QHQ-E_\alpha)}-1}{QHQ-E_\alpha}QH\right|\alpha\right\rangle$$
$$\equiv-\Delta_\alpha^1(t)-\frac{i}{2}\gamma_\alpha^1(t) \tag{32}$$

with $\Delta_\alpha^1(t)$ and $\gamma_\alpha^1(t)$ real, and, if E_α belongs to the continuous part $\sigma_c(QHQ)$ of the spectrum $\sigma(QHQ)$ of the operator QHQ, i.e., if $E_\alpha\geq\varepsilon_M$, where ε_M denotes the lower bound for the $\sigma_c(QHQ)$, we have

$$\lim_{t\to\infty}v_\alpha^1(t)\equiv-\Sigma_\alpha(E_\alpha)=-\left\langle\alpha\left|HQ\frac{1}{QHQ-E_\alpha-i0}QH\right|\alpha\right\rangle$$
$$\equiv-\Delta_\alpha^1-\frac{i}{2}\gamma_\alpha^1\ , \tag{33}$$

where $\Sigma_\alpha(\varepsilon)$ is the self-energy for the state $|\alpha\rangle$ and

$$\Delta_\alpha^1\equiv\left\langle\alpha\left|HQ\ \mathrm{P}\frac{1}{QHQ-E_\alpha}QH\right|\alpha\right\rangle\ , \tag{34}$$

$$\gamma_\alpha^1\equiv2\pi\langle\alpha|HQ\delta(QHQ-E_\alpha)QH|\alpha\rangle>0\quad\text{if } E_\alpha\geq\varepsilon_M\ , \tag{35}$$

which coincide with the Weisskopf-Wigner results, and

$$\gamma_\alpha^1\equiv0\quad\text{if } E_\alpha<\varepsilon_M\ . \tag{36}$$

In Eq. (34), P denotes principal value.

The imaginary part $\gamma_\alpha(t)$ of the quasipotential $v_\alpha(t)$ corresponds to the decay rate of a given state $|\alpha\rangle$: having the nondecay probability

$$p(t;|\alpha\rangle)\equiv|u_\alpha(t)|^2 \tag{37}$$

for this state, the decay rate Γ can be defined as [10]

$$\Gamma=-\frac{1}{p(t;|\alpha\rangle)}\frac{\partial p(t;|\alpha\rangle)}{\partial t}\ , \tag{38}$$

which, together with the solution

$$u_\alpha(t)=e^{-it[E_\alpha+\overline{v_\alpha(t)}]}\ , \tag{39}$$

of Eq. (26), where

$$\overline{v_\alpha(t)}\equiv\frac{1}{t}\int_0^t v_\alpha(\tau)d\tau\ , \tag{40}$$

leads to

$$\Gamma\equiv-2\,\mathrm{Im}v_\alpha(t)=\gamma_\alpha(t)\ . \tag{41}$$

The decay rate $\gamma_\alpha(t)$ possesses the following properties at the early time region [10]:

$$\gamma_\alpha(0)\equiv0\ , \tag{42}$$

$$\gamma_\alpha(t)=2(\delta H_\alpha)^2t,\quad t\to0\ , \tag{43}$$

where $(\delta H_\alpha)^2=\langle\alpha|H^2|\alpha\rangle-\langle\alpha|H|\alpha\rangle^2$. These properties follow from (12) and (13) and do not depend on a concrete form of H.

In the long time region $t\to\infty$, if $E_\alpha\geq\varepsilon_M$, the approximate expression for the decay rate $\gamma_\alpha^1(t)$ has the nonzero limit (35): $\gamma_\alpha^1(t=\infty)\equiv2\,\mathrm{Im}\Sigma_\alpha(E_\alpha)>0$, and states $|\alpha\rangle\in\mathcal{H}$, for which this property takes place, correspond exactly to the standard quasistationary (unstable) states of the system under consideration. If $E_\alpha<\varepsilon_M$, $\gamma_\alpha^1(t=\infty)\equiv0$ (36) [though, in this case, $\gamma_\alpha^1(t)>0$ for $t\to0$], and therefore those states $|\alpha\rangle\in\mathcal{H}$, for which this result (36) occurs, cannot be identified with the quasistationary states.

The properties of the exact $\gamma_\alpha(t)\equiv-2\,\mathrm{Im}v_\alpha(t)$ in the long-time region can be deduced from Khalfin's theorem [19], which is due to Paley and Wienier's theorem, and states that if the spectrum of the total Hamiltonian H of the system is bounded from below, then [19]

$$p_\alpha(t)\equiv|u_\alpha(t)|^2\underset{t\to+\infty}{\sim}\exp(-bt^q),\quad b>0,\ q<1\ . \tag{44}$$

Therefore the following conclusion should hold: If for the nondecay probability $p_\alpha(t)$, the asymptotic representation (44) is valid, then the decay rate $\gamma_\alpha(t)$ [i.e., the imaginary part of the quasipotential $v_\alpha(t)$, $\mathrm{Im}v_\alpha(t)\equiv-\frac{1}{2}\gamma_\alpha(t)$] behaves in the long-time region $t\to\infty$ as

$$\gamma_\alpha(t)\equiv-2\,\mathrm{Im}v_\alpha(t)\underset{t\to\infty}{\sim}bq\,t^{-\lambda},\quad\lambda\equiv1-q>0 \tag{45}$$

and thus in the exact theory

$$\lim_{t\to\infty}\gamma_\alpha(t)\equiv0\ . \tag{46}$$

This conclusion agrees with the intuitive solution of the problem of which is the decay rate of the completely decayed state (at the moment $t=\infty$, by definition, every unstable state $|\alpha\rangle\in\mathcal{H}$ is not occupied, i.e., it is completely decayed). The relations (45) and (46) can be derived from (38) and (41) using the property (44).

IV. EARLY-TIME BEHAVIOR OF SOLUTIONS OF THE EQUATION FOR THE PROJECTION OF A STATE VECTOR ONTO ONE-DIMENSIONAL SUBSPACE

A. Two-level system

Let us consider, to begin with, the case of a two-state system. Such a system can be solved exactly and thus, in some sense, is trivial, but has a pedagogical meaning and allows one to draw some general conclusions from the form of the solution of Eqs. (25) and (26) on the early-time behavior of more complicated systems. Thus let the vectors [5]

$$|1\rangle=\begin{bmatrix}1\\0\end{bmatrix},\quad |2\rangle=\begin{bmatrix}0\\1\end{bmatrix} \tag{47}$$

form the basis in a state space $\mathcal{H}$ of such a system and let the subspace $\mathcal{H}_{\|}$ be defined by a projector

$$P_\alpha \equiv P_1 = |1\rangle\langle 1| \ . \tag{48}$$

In this case, the self-adjoint Hamiltonian H is 2×2 matrix with the matrix elements $H_{j,k}$ $(j,k=1,2)$ and the solution of Eq. (25), where $u_\alpha(t)\rightarrow u_1(t)$ with the initial condition $u_1(0)=1$, is [5]

$$u_1(t)=e^{-(i/2)tH_+}\left[\cos\frac{\eta t}{2}-i\frac{H_-}{\eta}\sin\frac{\eta t}{2}\right] , \tag{49}$$

where $H_{+(-)}=H_{11}+(-)H_{22}$ and

$$\eta=(H_-^2+4|H_{12}|^2)^{1/2} \ . \tag{50}$$

Let us note that the following relation takes place:

$$|H_{12}|^2\equiv\langle 1|HQH|1\rangle\equiv\langle 1|(H^2-H_{11}^2)|1\rangle\equiv(\delta H_1)^2 , \tag{51}$$

which can be useful in discussing properties of $u_1(t)$.

The "nondecay" (survival) probability for the state $|1\rangle$ equals

$$p(t;|1\rangle)\equiv|u_1(t)|^2=1-4\frac{|H_{12}|^2}{\eta^2}\sin^2\frac{\eta t}{2} , \tag{52}$$

and, in the small time region, determined by the condition

$$\tfrac{1}{2}\eta t \ll 1 , \tag{53}$$

it quadratically decreases with the growth of time t

$$p(t;|1\rangle)=|u_1(t)|^2\simeq 1-|H_{12}|^2t^2, \quad \frac{\eta t}{2}\ll 1 \ . \tag{54}$$

Expressions (49) and (52) are exact and will be helpful in discussing early-time properties of a physical system with infinite degrees of freedom.

B. Infinite-level system

Now let us consider the case of the infinite-dimensional state space $\mathcal{H}$ and one-dimensional subspace $\mathcal{H}_{\|}$. In this case the projector P defining $\mathcal{H}_{\|}$ has the form (24). The early-time, model-independent, solution of Eq. (25) for the amplitude $u_\alpha(t)$ can be relatively easily found (see Appendix B). Namely, for very small times $t\rightarrow 0$, the quantity $k_\alpha(t)$ (29) can be approximated by [see (A6) and (A7)]

$$k_\alpha(t\rightarrow 0)\simeq\Theta(t)\langle\alpha|HQH|\alpha\rangle\equiv\Theta(t)(\delta H_\alpha)^2 \ . \tag{55}$$

Replacing $k_\alpha(t)$ in Eq. (25) by $k_\alpha(t\rightarrow 0)$ (55) leads to the solution [see (B4)–(B6)]

$$u_\alpha(t)\simeq e^{-(i/2)tE_\alpha}\left[\cos\frac{\eta_\alpha t}{2}-i\frac{E_\alpha}{\eta}\sin\frac{\eta_\alpha t}{2}\right] , \quad t\rightarrow 0 \tag{56}$$

where

$$\eta_\alpha\equiv[E_\alpha^2+4(\delta H_\alpha)^2]^{1/2} \ . \tag{57}$$

This solution is valid for time t limited by relation (A8).

One should stress that this solution is quite independent of the properties of the Hamiltonian H of the system under consideration. Strictly speaking, there is one (only) restriction on the state $|\alpha\rangle$: the appropriate solution $u_\alpha(t\rightarrow 0)$ (56) of Eq. (25) exists provided that $(\delta H_\alpha)^2$ (55) exists.

In the case of small times considered, from (56) one finds, for the survival probability,

$$p(t;|\alpha\rangle)\equiv|u_\alpha(t)|^2 \underset{t\rightarrow 0}{\simeq} \cos^2\frac{\eta_\alpha t}{2}+\frac{E_\alpha^2}{\eta_\alpha^2}\sin^2\frac{\eta_\alpha t}{2} \equiv 1-\frac{4(\delta H_\alpha)^2}{\eta_\alpha^2}\sin^2\frac{\eta_\alpha t}{2} \ . \tag{58}$$

The above estimations of small-time behavior for the amplitude $u_\alpha(t)$, and thus for the survival probability $|u_\alpha(t)|^2$, are consistent with the Fleming estimation [13], known as "Fleming's rule" or "Fleming's unitary limit," and correct and improve his result.

The comparison of amplitude $u_1(t)$ (49) and the probability $p(t;|1\rangle)$ (52), obtained for the case of a two-state system, with the approximate $u_\alpha(t\rightarrow 0)$ (56) and corresponding to it the probability $p(t;|\alpha\rangle)$ (58), respectively, leads to a conclusion concerning the general early-time properties raised by quantum dynamics. Namely, comparing these results mentioned one finds that if to prepare a physical system at the initial instant $t_0=0$ in such a way that the initial state $|\psi;t_0=0\rangle$ of this system belongs to the subspace $\mathcal{H}_{\|}$, i.e., if $|\psi;t_0=0\rangle\equiv|\psi\rangle_{\|}\in\mathcal{H}_{\|}$ and $|\psi;t_0=0\rangle_\perp\equiv|\psi\rangle_\perp\equiv 0$ (6), then, at the early-time period $t\rightarrow 0$, a system with an infinite degree of freedom ($\dim\mathcal{H}=\infty$) behaves like a system with two degrees of freedom described by state space $\mathcal{H}$ of $\dim\mathcal{H}=2$.

In other words, at the early-time period $t\rightarrow 0$, the transition probability from a subspace $\mathcal{H}_{\|}$ of $\dim\mathcal{H}_{\|}=1$ into subspace of states $\mathcal{H}_\perp$

$$\|QU(t)|\alpha\rangle\|^2=1-\|U_{\|}(t)|\alpha\rangle\|^2\equiv 1-p(t;|\alpha\rangle) \tag{59}$$

behaves like

$$\|QU(t)|\alpha\rangle\|^2\cong\frac{4(\delta H_\alpha)^2}{\eta_0^2}\sin^2\frac{\eta_0 t}{2}, \quad t\rightarrow 0 \tag{60}$$

[where η_0 may denote η (50) or η_α (57) depending on the problem considered] quite independently of $\dim\mathcal{H}_\perp$ (i.e., of whether $\dim\mathcal{H}_\perp=1$ or $\dim\mathcal{H}_\perp>1$, e.g., $\dim\mathcal{H}_\perp=\infty$).

V. FINAL REMARKS

From (58) one sees that if, analogously to the case of $\dim\mathcal{H}=2$ [see (54)],

$$\tfrac{1}{2}\eta_\alpha t \ll 1 \ , \tag{61}$$

then

$$p(t;|\alpha\rangle)=|u_\alpha(t)|^2 \simeq 1-t^2(\delta H_\alpha)^2 \ . \tag{62}$$

The properties (54) and (62) and, especially, the conditions (53) and (61) under which they take place seem to be important because some authors [20] believe that the smallness of (δH_α) [2], i.e., the condition $t^2(\delta H_\alpha)^2 \ll 1$, guarantees that the survival probability has the form (54) and (62) and thus defines the so-called "Zeno time region" in the case of many successive, so-called ideal measurements of a given state. In other words, one interpretation of this is that if $t^2(\delta H_\alpha)^2 \ll 1$, then the joint probability $\mathcal{P}(\Delta_n,\dots,\Delta_2,\Delta_1;|\alpha\rangle)$ of finding the system in a given state $|\alpha\rangle$, in any of n measurements separated by time interval $\Delta_n,\dots,\Delta_2,\Delta_1$ (see, e.g., Refs. [5,6,18,20–22])

$$\mathcal{P}(\Delta_n,\dots,\Delta_2,\Delta_1;|\alpha\rangle) \equiv \prod_{k=1}^{n} p_\alpha(\Delta_k;|\alpha\rangle) \equiv \prod_{k=1}^{n} |u(\Delta_k)|^2 \ , \tag{63}$$

for $\Delta_n=\cdots=\Delta_2=\Delta_1\equiv\Delta\le t$, takes the form

$$\mathcal{P}(\Delta_n,\dots,\Delta_2,\Delta_1;|\alpha\rangle)|_{\Delta_k=\Delta,\ k=1,\dots,n} \equiv \mathcal{P}(n,\Delta;|\alpha\rangle) \cong [1-\Delta^2(\delta H_\alpha)^2]^n, \tag{64}$$

which, for $\Delta\equiv t/n$ and suitable, very large n, transforms into

$$\mathcal{P}\left[n,\frac{t}{n};|\alpha\rangle\right] \cong 1-\frac{t^2}{n}(\delta H_\alpha)^2$$

and, as a consequence, leads to the Zeno paradox [6,21]. From (53) and (61), and (54) and (62), it follows that such an interpretation is wrong; it may happen that $t^2(\delta H_\alpha)^2 \ll 1$, but $t^2E_\alpha^2$ [or, in the case of $\dim\mathcal{H}=2$, $t^2H_-^2$; see (50)] is so large that condition (53) or (61), which are necessary and sufficient for probability $p(t;|\alpha\rangle)$ to be of the form (54) or (62) required, cannot be fulfilled. Only the smallness of both $t^2(\delta H_\alpha)^2$ and $t^2E_\alpha^2$ (or, if $\dim\mathcal{H}=2$, $t^2H_-^2$) together results in the approximate expressions (54) and (62) for $p(t;|\alpha\rangle)$ and thus, in formula (64) for the probability $\mathcal{P}(n,\Delta;|\alpha\rangle)$ appearing in the case of many successive, quasicontinuous measurements, i.e., leads to the Zeno paradox. Taking into account these properties of short-time evolution seems to be useful in designing experiments for confirming and searching for such occurrences as the Zeno paradox, especially if one considers the case of transitions from a single, isolated state into a continuum.

In the case of a two-level system, the expressions for the joint probability $\mathcal{P}$ of finding the system in the state $|\alpha\rangle\equiv|1\rangle$ in each of the n intermediate measurements (63) and for the probability $p(t;|1\rangle)$ (52) suggest not only an experiment for searching for the quantum Zeno effect [23,18], but also another one for Δ's much longer than those proper by defining the so-called Zeno time region [24], i.e., much longer than those determined by relation (53). Namely, one can choose, for instance, $\Delta_n,\dots,\Delta_2,\Delta_1$ so as to be $p(T_1\equiv\sum_{l=1}^{n}\Delta_l;|1\rangle)=1$ (here $T_1=2\pi j/\eta$, $j=1,2,\dots$) and $\Delta_k\equiv\Delta\equiv T_0$ for $k\neq l$ (where T_0 is defined by the property $\sin[(\eta/2)T_0]\equiv 1$—see (52)—i.e., $T_0=[(2r+1)/\eta]\pi$, $r=1,2,\dots$), $T_0 \ll T_1$, and $\Delta_l\equiv T_1-(n-1)T_0$. Then it should be [24]

$$\mathcal{P}(\Delta_n,\dots,\Delta_2,\Delta_1;|\alpha\rangle)|_{\Delta_k=\Delta\equiv T_0;\Delta_l=T_1-(n-1)\Delta;k=1,\dots,(l-1),(l+1),\dots,n} \ll p\left[T_1\equiv\sum\Delta_k;|1\rangle\right]\equiv 1 \ , \tag{65}$$

etc., which can be verified experimentally, and thus the reduction postulate leading to formula (63) for the probability $\mathcal{P}$ can be verified.

The general properties of the decay rate $\gamma_\alpha(t)$ (41) and, strictly speaking, of its approximate form $\gamma_\alpha^1(t)$ (32) and (A5) are determined by the properties of the imaginary part of the self-energy $\Sigma_\alpha(\varepsilon)$, i.e., density [6,10,15] $\rho_\alpha(\varepsilon)$ (A4); see (A5). The spectral density $\rho_\alpha(\varepsilon)$ must obey the following basic requirements [25]: positively, threshold behavior $[\rho_\alpha(\varepsilon)\sim(\varepsilon-\varepsilon_M)^{1/2}$ as $\varepsilon\to\varepsilon_M$ for two-particle decays], and vanishing for $\varepsilon\to\infty$. Its concrete form is determined by the spectrum of the decay channel space $\mathcal{H}_\perp$ and the transition operator QHP. The simplest model satisfying these minimal physical requirements has been studied in Refs. [6] and [10]: the calculations have been performed there for

$$\rho_\alpha(\varepsilon)\equiv\rho_\alpha(\varepsilon;k)=a_k(\lambda)\frac{\varepsilon^{1/2}}{(\varepsilon+\lambda)^k} \ , \tag{66}$$

with two parameters characterizing the behavior $\rho_\alpha(\varepsilon)$ at $\varepsilon\to\infty$: the cutoff λ and the power k, and assuming $\varepsilon_M=0$. The coefficient $a_k(\lambda)$ is related to the rate γ_α^1 (35), which has been assumed to be very small, by the relation $\rho_\alpha(E_\alpha)\equiv\frac{1}{2}\gamma_\alpha^1$ (33) and (A4):

$$a_k(\lambda)\equiv\frac{(E_\alpha+\lambda)^k}{2\gamma_\alpha^1(E_\alpha)^{1/2}} \ .$$

The results obtained in Ref. [10] are presented in Fig. 1. All curves in this figure start from a straight line, according to the result (43), and then, later, $\gamma_\alpha^1(t)$ begin to oscillate. The amplitude of these oscillations decreases relatively quickly with the growth of time coming up to the asymptotic value γ_α^1 from $t \geq (50-100)E_\alpha^{-1}$ practically, and the oscillations amplitude increases strongly for increasing k and decreasing λ.

Also the behavior of $\gamma_\alpha^1(t)$ at the earliest and at the subsequent stages of time evolution seems to have some meaning for an understanding and a proper description of the earliest instants of the existence of the Universe. Namely, it seems that this behavior can be considered as a candidate for a possible mechanism producing the mass density fluctuations at the initial instant of the time evolution of the Universe [26].

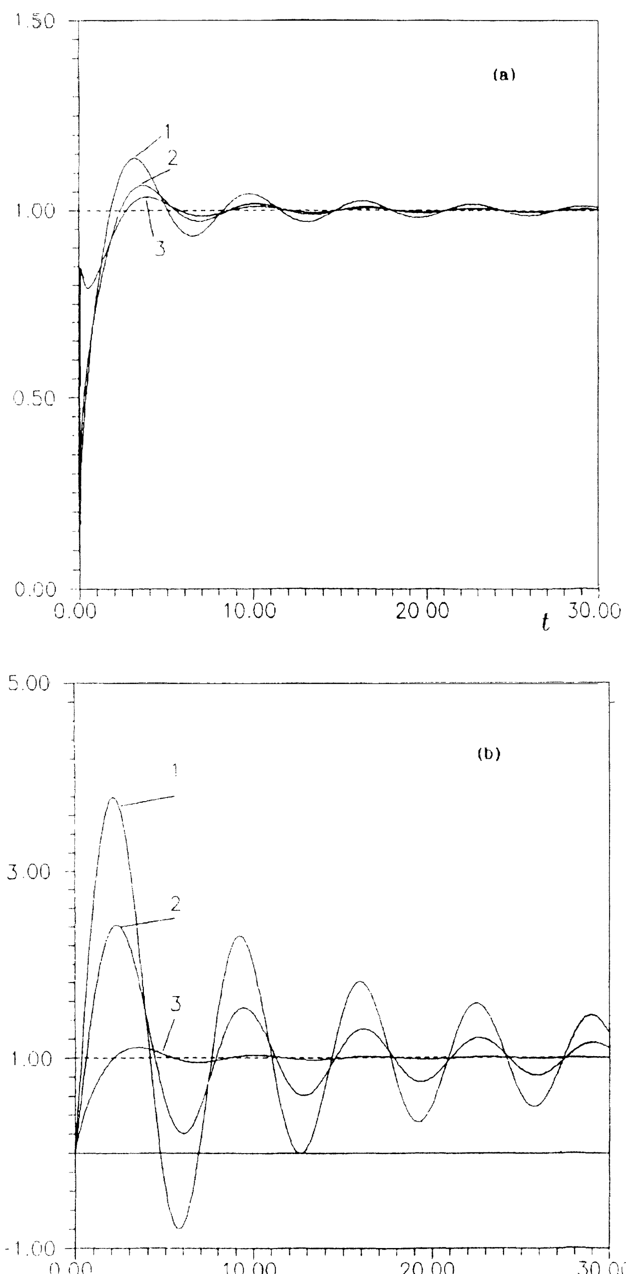

FIG. 1. Dependence of $\gamma_\alpha^1(t)/\gamma_\alpha^1$ on t for different λ and k: (a) $k=1$ and (1) $\lambda=0{,}2E_\alpha$, (2) $\lambda=1{,}0E_\alpha$, and (3) $\lambda=10E_\alpha$; (b) $k=4$ and (1) $\lambda=0{,}2E_\alpha$, (2) $\lambda=0{,}5E_\alpha$, and (3) $\lambda=3{,}0E_\alpha$. Time t is measured in units of E_α^{-1}.

APPENDIX A

Using the complete set of eigenvectors $|F\rangle$ of the operator QHQ for $F \in \sigma_c(QHQ)$: $QHQ|F\rangle = F|F\rangle$, $F \geq \varepsilon_M$, normalized as usual $\langle F|F'\rangle = \delta(F-F')$, with the completeness relation

$$\int_{\varepsilon_M}^{\infty} |F\rangle\langle F| dF = Q \ , \tag{A1}$$

leads to the following representation for $k_\alpha(t)$ and $v_\alpha^1(t)$:

$$k_\alpha(t) \equiv \frac{1}{\pi}\Theta(t)\int_{\varepsilon_M}^{\infty} e^{-itF}\rho_\alpha(F)dF \ , \tag{A2}$$

$$v_\alpha^1(t) = \frac{1}{\pi}\int_{\varepsilon_M}^{\infty} \frac{\exp[-it(F-E_\alpha)]-1}{F-E_\alpha}\rho_\alpha(F)dF, \quad t \geq 0 \ , \tag{A3}$$

where

$$\rho_\alpha(F) = \pi|\langle F|H|\alpha\rangle|^2 \equiv \mathrm{Im}\Sigma_\alpha(F) \ . \tag{A4}$$

One finds

$$\gamma_\alpha^1(t) = \frac{2}{\pi}\int_{\varepsilon_M}^{\infty} \frac{\sin[t(F-E_\alpha)]}{F-E_\alpha}\rho_\alpha(F)dF, \quad t \geq 0 \ . \tag{A5}$$

It is easy to see that for $E_\alpha \geq \varepsilon_M$ this $\gamma_\alpha^1(t)$ tends to $\gamma_\alpha^1 > 0$, given by formula (35) as $t \to \infty$, and, if $E_\alpha < \varepsilon_M$, by Riemann's lemma, according to (36), to the zero value as $t \to \infty$.

From (A2) one obtains

$$k_\alpha(t) \equiv \frac{1}{\pi}\Theta(t)\sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\int_{\varepsilon_M}^{\infty} F^n\rho_\alpha(F)dF \tag{A6a}$$

$$\underset{t\to 0}{\simeq} \frac{1}{\pi}\Theta(t)\int_{\varepsilon_M}^{\infty}\rho_\alpha(F)dF + \cdots \ . \tag{A6b}$$

The density $\rho_\alpha(F)$ can be related to the dispersion δH_α as follows:

$$\frac{1}{\pi}\int_{\varepsilon_M}^{\infty}\rho_\alpha(F)dF \equiv \langle\alpha|HQH|\alpha\rangle \equiv \langle\alpha|H^2|\alpha\rangle - \langle\alpha|H|\alpha\rangle^2 = (\delta H_\alpha)^2 \ . \tag{A7}$$

The approximation (A6b) for $k_\alpha(t)$ and thus the solution $u_\alpha(t)$ (56) of Eq. (25) are valid provided that times t fulfill the following inequality;

$$\left| t\int_{\varepsilon_M}^{\infty} F\rho_\alpha(F)dF \right| \ll \int_{\varepsilon_M}^{\infty}\rho_\alpha(F)dF \equiv \pi(\delta H_\alpha)^2 \ . \tag{A8}$$

APPENDIX B

In terms of Laplace transforms, defined as

$$f(z) = \int_0^{\infty} f(t)e^{-zt}dt \ , \tag{B1}$$

the solution $u_\alpha(t)$ of Eq. (25) for $t \geq 0$ is written

$$u_\alpha(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{zt}}{z+iE_\alpha+\mathscr{k}_\alpha(z)}dz, \quad \sigma > 0 \ . \tag{B2}$$

Inserting into this formula the Laplace transform of the approximate expression for $k_\alpha(t)$ (A6), valid in the short-time region $t \to 0$, i.e.,

$$\not{k}_\alpha(z) \simeq \frac{1}{z} \langle \alpha | HQH | \alpha \rangle \equiv \frac{1}{z} (\delta H_\alpha)^2 , \quad \text{(B3)}$$

yields

$$u_\alpha(t) \simeq \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{z e^{zt}}{z^2 + izE_\alpha + (\delta H_\alpha)^2} dz, \quad t \to 0 . \quad \text{(B4)}$$

The integration in (B4), after rewriting the integrand in the form

$$u_\alpha(t) \simeq \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{z e^{zt}}{(z - z_1)(z - z_2)} dz, \quad t \to 0 , \quad \text{(B5)}$$

where

$$z_{1,2} = -\frac{i}{2} (E_\alpha \mp \eta_\alpha) , \quad \text{(B6)}$$

can easily be performed and, as a result, leads to the approximate formula (56) for $u_\alpha(t \to 0)$.

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