Generalized coherent state for multimode bosonic realization of the su(2) Lie algebra

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We have constructed the multimode bosonic realization of the su(2) Lie algebra, on the basis of which the SU(2) generalized coherent state in the multimode Fock space is derived. It is shown that in the multimode coherent state each bosonic mode has the sub-Poissonian statistics, and that the unitary displacement operator can be identified as the generalized multimode rotation operator.

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In recent years the Lie algebras su(1,1) and su(2) have been used by many researchers in the study of the nonclassical properties of light in quantum-optical systems. It has been shown that the single- and two-mode bosonic realizations of the su(1,1) Lie algebra have immediate relevance to the nonclassical squeezing properties of light, and that the associated generalized coherent states are, in fact, special cases of the so-called single- and two-mode squeezed states [1]. Recently, Lo and Liu [2] constructed the multimode bosonic realization of the su(1,1) Lie algebra, on the basis of which the SU(1,1) generalized coherent state in the multimode Fock space was derived. They showed that the multimode coherent state was actually the generalized squeezed vacuum state discussed by Lo and Sollie [3], and that the unitary displacement operator could be identified as the generalized multimode squeezing operator. Similar to the single- and two-mode cases, the multimode bosonic realization of the su(1,1) Lie algebra has immediate relevance to the squeezing properties of boson fields. On the other hand, another type of squeezing, namely, the SU(2) squeezing for the su(2) generators, associated with the SU(2) generalized coherent states in the so-called (two-mode) Schwinger bosonic representation of the su(2) Lie algebra has been found in the study of interferometers [4] and in other applications in quantum optics [1]. These SU(2) generalized coherent states generated as a result of a model interaction involving angular-momentum algebra are called the SU(2) squeezed states. Furthermore, the bosonic realization of the su(2) Lie algebra has been receiving extensive attention in nuclear physics recently [5–8]. In the present work we are interested in generalizing the two-mode bosonic realization of the su(2) Lie algebra to the multimode case, on the basis of which the SU(2) generalized coherent state in the multimode Fock space is derived and its properties are discussed.

We shall begin by briefly reviewing the main properties of the su(2) Lie algebra [1,9]. The su(2) Lie algebra consists of three generators K_0, K_+, and K_- satisfying the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \quad , \quad [K_+, K_-] = 2K_0 \quad . \tag{1}$$

The corresponding Casimir operator $\mathcal{C}$ is given by

$$\mathcal{C} = K_0^2 + \frac{1}{2}(K_+K_- + K_-K_+) \ , \tag{2}$$

which satisfies $[\mathcal{C}, K_\pm] = [\mathcal{C}, K_0] = 0$. The discrete representation of the su(2) Lie algebra is characterized by [9]

$$\begin{aligned} \mathcal{C}|m,j\rangle &= j(j+1)|m,j\rangle \ , \\ K_0|m,j\rangle &= m|m,j\rangle \ , \\ K_\pm|m,j\rangle &= \sqrt{(j\mp m)(j\pm m+1)}|m\pm 1,j\rangle \ , \end{aligned} \tag{3}$$

where $K_-|-j,j\rangle = K_+|j,j\rangle = 0$. In this case $j = 0, 1/2, 1, 3/2, 2, \ldots$ and $m = -j, -j+1, \ldots, j-1, j$. The set of states $\{|m,j\rangle : m = -j, -j+1, \ldots, j-1, j; j = \text{const}\}$ forms a complete orthonormal basis:

$$\langle j,m|n,j\rangle = \delta_{m,n} \ , \qquad \sum_{m=-j}^{j} |m,j\rangle\langle j,m| = 1 \ . \tag{4}$$

Following Perelomov [9], the SU(2) generalized coherent states $|\theta,\phi\rangle$ are defined as

$$|\theta,\phi\rangle = \exp(\alpha K_+ - \alpha^* K_-)|-j,j\rangle \ , \tag{5}$$

where $\alpha = (\theta/2)\exp(-i\phi)$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. The ladder operators $K_\pm$ select the vacuum state $|\text{vac}\rangle$ from the states $|m,j\rangle$ in the usual way, namely, $K_-|\text{vac}\rangle = K_-|-j,j\rangle = 0$. Using the disentangling theorem for the su(2) generators [1], we can rewrite Eq. (5) in the following form:

$$\begin{aligned} |\tau\rangle &= (1+|\tau|^2)^{-j}\exp(\tau K_+)|-j,j\rangle \\ &= (1+|\tau|^2)^{-j}\sum_{m=-j}^{j}\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\tau^{j+m}|m,j\rangle, \end{aligned} \tag{6}$$

where $\tau = \tan(\theta/2)\exp(-i\phi)$. Similar to the ordinary Glauber coherent states, these states $|\tau\rangle$ are not orthonormal:

$$\langle\tau_1|\tau_2\rangle = \frac{(1+\tau_1^*\tau_2)^{2j}}{(1+|\tau_1^2|)^j(1+|\tau_2^2|)^j} \ , \quad (7)$$

and are overcomplete,

$$I = \int d\mu(\tau)|\tau\rangle\langle\tau| \ , \ d\mu(\tau) = \frac{2j+1}{\pi}\frac{d(\mathrm{Re}\tau)d(\mathrm{Im}\tau)}{(1+|\tau^2|)^2} \ . \quad (8)$$

The usual (two-mode) Schwinger bosonic representation of the su(2) Lie algebra is given by [10,11]

$$K_+ = a_1^\dagger a_2 \ , \ K_- = a_2^\dagger a_1 \ , \ K_0 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \ ; \quad (9)$$

in this case the Casimir operator is $\mathcal{C} = \mathcal{J}(\mathcal{J}+1)$ where $\mathcal{J} = (a_1^\dagger a_1 + a_2^\dagger a_2)/2$. In this unitary representation of the Lie algebra the total number $\mathcal{N}$ of bosons in the system is constant as well as the vacuum state is understood as the state with no boson in mode 1 and $\mathcal{N} = 2j$ bosons in mode 2, i.e., $|\mathrm{vac}\rangle = |\mathcal{N} = 2j\rangle_2|0\rangle_1$. The corresponding SU(2) generalized coherent state is given by

$$|\theta,\phi\rangle = \exp(\alpha a_1^\dagger a_2 - \alpha^* a_2^\dagger a_1)|\mathrm{vac}\rangle \ , \quad (10)$$

or

$$|\tau\rangle = (1+|\tau|^2)^{-j}\exp(\tau a_1^\dagger a_2)|\mathrm{vac}\rangle \ . \quad (11)$$

Furthermore, the unitary displacement operator $D(\alpha) = \exp(\alpha K_+ - \alpha^* K_-)$ transforms the annihilation and creation operators as follows [12]:

$$\begin{aligned} D(\alpha)^\dagger a_1 D(\alpha) &= a_1\cos(|\alpha|) + a_2\frac{\alpha}{|\alpha|}\sin(|\alpha|) \ , \\ D(\alpha)^\dagger a_1^\dagger D(\alpha) &= a_1^\dagger\cos(|\alpha|) + a_2^\dagger\frac{\alpha^*}{|\alpha|}\sin(|\alpha|) \ , \\ D(\alpha)^\dagger a_2 D(\alpha) &= a_2\cos(|\alpha|) - a_1\frac{\alpha^*}{|\alpha|}\sin(|\alpha|) \ , \\ D(\alpha)^\dagger a_2^\dagger D(\alpha) &= a_2^\dagger\cos(|\alpha|) - a_1^\dagger\frac{\alpha}{|\alpha|}\sin(|\alpha|) \ . \end{aligned} \quad (12)$$

In the following we shall generalize the bosonic realization of the su(2) Lie algebra to the case of multimodes.

Let us now introduce the multimode bosonic realization of the three su(2) generators as follows:

$$\begin{aligned} K_o &= \sum_{i,j}\Lambda_{ij}a_i^\dagger a_j \ , \quad K_+ = \sum_{i,j}\beta_{ij}a_i^\dagger a_j \ , \\ K_- &= K_+^\dagger = \sum_{i,j}\beta_{ji}^* a_i^\dagger a_j = \sum_{i,j}\beta_{ij}^\dagger a_i^\dagger a_j \ , \end{aligned} \quad (13)$$

with $\Lambda_{ij} = \sum_k(\beta_{ik}\beta_{kj}^\dagger - \beta_{ik}^\dagger\beta_{kj})/2$. Note that the matrix Λ_{ij} is Hermitian. In order to satisfy the commutation relations in Eq. (1), we must require

$$\beta_{ij} = \sum_k(\Lambda_{ik}\beta_{kj} - \beta_{ik}\Lambda_{kj}) \ . \quad (14)$$

In the matrix notation the above condition can be compactly rewritten as

$$\boldsymbol{\beta} = [\boldsymbol{\Lambda},\boldsymbol{\beta}] = \frac{1}{2}[[\boldsymbol{\beta},\boldsymbol{\beta}^\dagger],\boldsymbol{\beta}] \ . \quad (15)$$

Obviously, one particular type of solution to Eq. (15) is that the $N\times N$ matrix $\boldsymbol{\beta}$ is just the matrix representation of the angular momentum operator J_+ of angular momentum $J = (N-1)/2$ (with $\hbar = 1$) [13]. Then the matrices $\boldsymbol{\beta}^\dagger$ and $\boldsymbol{\Lambda}$ correspond to the matrix representations of the angular momentum operators J_- and J_0, respectively. For $N = 2$, we shall recover the usual (two-mode) Schwinger bosonic representation. In this multimode case the vacuum state is identified as the state with no boson in the first $N-1$ modes and $\mathcal{N} = j|\Lambda_{NN}|^{-1}$ bosons in mode N, i.e., $|\mathrm{vac}\rangle = |\mathcal{N} = j|\Lambda_{NN}|^{-1}\rangle_N\prod_{i=1}^{N-1}|0\rangle_i$. Note that the total number $\mathcal{N}$ of bosons in the system remains constant. The corresponding SU(2) generalized coherent state can be obtained by displacing the vacuum state with the unitary displacement operator, namely,

$$|\theta,\phi\rangle = \exp\left(\alpha\sum_{i,j}\beta_{ij}a_i^\dagger a_j - \alpha^*\sum_{i,j}\beta_{ij}^\dagger a_i^\dagger a_j\right)|\mathrm{vac}\rangle \ , \quad (16)$$

or

$$|\tau\rangle = (1+|\tau|^2)^{-j}\exp\left(\tau\sum_{j,k}\beta_{jk}a_j^\dagger a_k\right)|\mathrm{vac}\rangle \ . \quad (17)$$

Besides, the unitary displacement operator induces the following transformation of the creation and annihilation operators:

$$\begin{aligned} D(\alpha)^\dagger a_j D(\alpha) &= \sum_k[\exp(i\boldsymbol{B})]_{jk}\ a_k \\ &\Longrightarrow D(\alpha)^\dagger \boldsymbol{a} D(\alpha) = \exp(i\boldsymbol{B})\ \boldsymbol{a} \ , \\ D(\alpha)^\dagger a_j^\dagger D(\alpha) &= \sum_k[\exp(-i\boldsymbol{B}^T)]_{jk}\ a_k^\dagger \\ &\Longrightarrow D(\alpha)^\dagger \boldsymbol{a}^\dagger D(\alpha) = \exp(-i\boldsymbol{B}^T)\ \boldsymbol{a}^\dagger \ , \end{aligned} \quad (18)$$

where $\boldsymbol{a}$ is the column vector consisting of annihilation operators a_i $(i = 1,2,...,N)$ and $\boldsymbol{a}^\dagger$ is the vector of creation operators. The matrix $\boldsymbol{B} = -i(\alpha\boldsymbol{\beta} - \alpha^*\boldsymbol{\beta}^\dagger)$ is Hermitian and $\boldsymbol{B}^T$ is the transpose of $\boldsymbol{B}$. This transformation is a multimode generalization of the rotation transformation in the single-mode case:

$$\exp(-i\phi a^\dagger a)\ a\ \exp(i\phi a^\dagger a) = \exp(i\phi)\ a \quad (19)$$

with ϕ being real. In fact, the displacement operator can be identified as a special case of the generalized multimode rotation operator $R(\boldsymbol{\xi}) = \exp(\sum_{i,j=1}^N\xi_{ij}a_i^\dagger a_j - \xi_{ij}^\dagger a_i^\dagger a_j)$, which transforms $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ as follows [14]:

$$\begin{aligned} R(\boldsymbol{\xi})^\dagger\ \boldsymbol{a}\ R(\boldsymbol{\xi}) &= \exp(\boldsymbol{\xi} - \boldsymbol{\xi}^\dagger)\ \boldsymbol{a} \ \equiv \ \exp(i\boldsymbol{\Phi})\ \boldsymbol{a} \ , \\ R(\boldsymbol{\xi})^\dagger\ \boldsymbol{a}^\dagger\ R(\boldsymbol{\xi}) &= \exp(-i\boldsymbol{\Phi}^T)\ \boldsymbol{a}^\dagger \ . \end{aligned} \quad (20)$$

The matrix $\boldsymbol{\xi}$ does not necessarily obey Eq. (15).

Next we study the statistical properties of the bosonic fields in the SU(2) generalized coherent states by evaluating the Mandel's Q parameter for each mode [15]. The Q parameter of mode j is defined as

$$Q_j = \frac{\langle\tau|n_j^2|\tau\rangle - \langle\tau|n_j|\tau\rangle^2 - \langle\tau|n_j|\tau\rangle}{\langle\tau|n_j|\tau\rangle} \ , \quad (21)$$

where $n_j = a_j^\dagger a_j$, and is a natural measure of the departure of the variance of the boson number from the variance of a Poisson distribution. For the mean boson number $\langle\tau|n_j|\tau\rangle$, one can find

$$\begin{aligned}\langle\tau|n_j|\tau\rangle &= \mathcal{N}[\exp(-i\boldsymbol{B})]_{Nj}[\exp(i\boldsymbol{B})]_{jN} \\ &= \mathcal{N}|[\exp(i\boldsymbol{B})]_{jN}|^2 \ . \end{aligned} \quad (22)$$

Similarly, the mean value for the squared boson number can also be obtained, and is given by

$$\begin{aligned}\langle\tau|n_j^2|\tau\rangle &= \langle\tau|n_j|\tau\rangle + \langle\tau|a_j^{\dagger 2}a_j^2|\tau\rangle \\ &= \langle\tau|n_j|\tau\rangle + \mathcal{N}(\mathcal{N}-1)|[\exp(i\boldsymbol{B})]_{jN}|^4 \\ &= \langle\tau|n_j|\tau\rangle + \left(1-\frac{1}{\mathcal{N}}\right)\langle\tau|n_j|\tau\rangle^2 \ . \end{aligned} \quad (23)$$

As a result, the parameter Q_j can be evaluated:

$$Q_j = -\frac{1}{\mathcal{N}}\langle\tau|n_j|\tau\rangle = -|[\exp(i\boldsymbol{B})]_{jN}|^2 \le 0 \ . \quad (24)$$

It is obvious that all modes have sub-Poissonian statistics, which means that the variance of the boson number of each mode is less than that of a Poisson distribution.

Finally we shall see how the SU(2) generalized coherent state discussed above can be generated. It is obvious that the dynamic Hamiltonian which can produce these states is given by

$$\begin{aligned}H &= AK_0 + BK_+ + B^*K_- \\ &= \sum_i \omega_i a_i^\dagger a_i + \sum_{i\ne j}(g_{ij}a_i^\dagger a_j + g_{ij}^* a_i a_j^\dagger) \ , \end{aligned} \quad (25)$$

where $\omega_i = A\Lambda_{ii} + B\beta_{ii} + B^*\beta_{ii}^*$ and $g_{ij} = (A\Lambda_{ij} + B\beta_{ij} + B^*\beta_{ji}^*)/2$. A (real) and B are arbitrary parameters. This Hamiltonian describes the process in which the photon in one mode (say mode i) is annihilated simultaneously with the creation of the photon in another mode (mode j), and vice versa (the total number of photons remains constant). It follows that the statistical properties of photons in different modes are changed simultaneously with their transformation from one mode into the other as well. In other words, in this process one photon field with sub-Poissonian statistics can be transformed into another sub-Poissonian photon field. According to the Wei-Norman algebraic procedure [16], the evolution operator corresponding to this Hamiltonian can be written in the form

$$U(t,0) = \exp[b_+(t)K_+]\exp[b_0(t)K_0]\exp[b_-(t)K_-] \ , \quad (26)$$

where $b_0(t)$, $b_+(t)$ and $b_-(t)$ are determined by the associated Schrödinger equation. If the initial state $|\Psi(t=0)\rangle$ is the SU(2) vacuum state $|\text{vac}\rangle$ defined by $K_-|\text{vac}\rangle = 0$, then one can easily show that the state at any time $t > 0$, $U(t,0)|\text{vac}\rangle$, will be a SU(2) generalized coherent state. That is, the time evolution of the system is essentially a SU(2) displacement process to generate the SU(2) generalized coherent state. Furthermore, it should be noticed that this Hamiltonian has exactly the same form as the effective Hamiltonian describing the multiple scattering of light in an inhomogeneous medium (with no photon creation from the medium), discussed in Refs. [17,18].

In summary, we have constructed the multimode bosonic realization of the su(2) Lie algebra, on the basis of which the SU(2) generalized coherent state in the multimode Fock space is derived. It is shown that in the multimode coherent state each bosonic mode has the sub-Poissonian statistics, and that the unitary displacement operator can be identified as the generalized multimode rotation operator. Since the SU(2) group is very useful in many branches of physics, and according to the Levi theorem [19], the su(2) algebra is one of the essential building blocks of every Lie algebra — this means that we can deal with a generic Lie algebra by decomposing it into its fundamental blocks, we believe that the results obtained in the present work should have valuable potential applications. For instance, the SU(2) generalized coherent state can be used in time-dependent variational approaches to multidimentional and time-dependent quantum systems, with possible extensions to many-body systems. [20]

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