Quantum states for a Paul-trapped particle in an intense laser field

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(Received 15 March 1994)

The eigenstates are constructed for a Paul-trapped particle in an intense laser field. This is done by using a time-dependent unitary transformation. The application of these results is discussed.

PACS number(s): 32.80.Pj, 03.65.Ge, 32.90.+a

With the development of high-power lasers, the study of physical problems in the presence of an intense laser field has become a subject of considerable interest. It has also stimulated considerable theoretical activity in the development of nonperturbative methods and in the research of exact solutions for time-dependent quantum systems [1–3].

The Paul trap is a device used to confine the motion of charged particles. Many interesting physical features of the system consisting of the charged particles confined by the Paul trap can be investigated [4,5]. As is well known, the "geonium atom" is a single charged particle system in which the atomic nucleus is replaced by an external trapping field such as an electron in the Penning trap. The physics of a single charged particle in an external trapping field is an interesting subject [6].

In this Brief Report we shall be concerned with the problem of a single point-charged particle in a Paul trap subjected to an intense laser field. Brown [7] has recently shown how to get the exact solution for a particle in a Paul trap. Here we discuss the exact solution for a Paul-trapped point-charged particle in the presence of an intense laser field.

For a point-charged particle in the simultaneous presence of a Paul trap and a laser field, the Hamiltonian is

$$H=\frac{1}{2m}(\mathbf{P}+q\,\mathbf{A}/c)^2+q\phi(r,z)\ ,$$

where $\phi(r,z)$ is the potential of a Paul trap [8]. For simplicity, let us discuss the problem of one dimension (three-dimensional results can be obtained by considering the separated variables of the equation). The one-dimensional Hamiltonian can be written as

$$H=\frac{1}{2m}(p+qA/c)^2+\tfrac{1}{2}K(t)x^2\ , \tag{1}$$

where

$$K(t)=a+b\cos\Omega t\ . \tag{2}$$

We now perform a unitary transformation on the wave function, namely,

$$\psi(x,t)=U_1(t)\varphi(x,t) \tag{3}$$

and take $U_1(t)$ as

$$U_1(t)=\exp[i\alpha(t)p]\exp[\beta(t)x]\exp[i\gamma(t)]\ , \tag{4}$$

where α,β,γ are arbitrary functions of time. In order to keep the form of the Schrödinger equation, the Hamiltonian in the new representation is given by

$$H_1=U_1^\dagger HU_1-iU_1^\dagger\partial U_1/\partial t\ . \tag{5}$$

The function $\alpha(t)$ in Eq. (4) produces a translation in space and the function $\beta(t)$ produces a translation in momentum. Since the functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary, we can use them to cancel the unwanted terms in the modified Schrödinger equation to transform the laser-assisted problem into a problem of a particle in the absence of the laser field.

If the new Hamiltonian is taken in the form

$$H_1=\frac{1}{2m}p^2+\tfrac{1}{2}K(t)x^2\ , \tag{6}$$

we obtain that the arbitrary functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy the following equations:

$$\dot\beta(t)-mK(t)\alpha(t)=0\ , \tag{7}$$

$$\dot\alpha(t)+[\beta(t)+qA(t)/c]/m=0\ , \tag{8}$$

$$\dot\gamma(t)+\dot\alpha(t)\beta(t)+[\beta(t)+qA(t)/c]^2/2m+K(t)\alpha^2(t)/2=0\ . \tag{9}$$

We now perform another unitary transformation on the wave function $\varphi(x,t)$ by Brown's method [7], namely, by taking

$$U_2=\exp[iG(t)x^2]\exp[-i(px+xp)\ln|f|^2/4]\ , \tag{10}$$

where $G(t)=(m/4)(\dot f/f+\dot f^*/f^*)$, and $f(t)$ is the solution of the classical equation

$$m\ddot x+K(t)x=0\ . \tag{11}$$

Under the transformation U_2, we have a new Hamiltonian from Eq. (6):

$$H_2=(P^2/2m+mwx^2/2)/|f|^2\ . \tag{12}$$

The solution of the new Schrödinger equation corresponding with the new Hamiltonian H_2 can be obtained immediately:

$$\varphi_2(x,t)=\exp\left[-iE_n\int^t dt'/|f|^2\right](mw/\pi 2^n n!)^{1/2}\times H_n(\sqrt{mw}\,x)\exp(-mwx^2)\ ,$$

where w is the Wronskian of two solutions f and f^* of Eq. (11), H_n is a Hermite polynomial, and E_n is the energy of the one-dimensional ordinary harmonic oscillator, i.e., $E_n = \hbar w(n+\frac{1}{2})$.

By performing the time derivative in Eq. (8) again and then substituting Eq. (7) into it, we have

$$\ddot{\alpha}(t) - K(t)\alpha(t) + q\dot{A}(t)/mc = 0 \ .$$

It is a second-order, linear, inhomogeneous differential equation with variable coefficients. We can then get the solution

$$\alpha(t) = \alpha_0(t)\int dt\, \alpha_0^{-2}(t)\int(-q/mc)\dot{A}(t')\alpha_0(t')dt' \ ,$$

where $\alpha_0(t)$ is the solution of the homogeneous equation

$$\ddot{\alpha}_0(t) - K(t)\alpha_0(t) = 0 \ . \tag{13}$$

Both Eqs. (11) and (13) are Mathieu's equations for $K(t)$, defined by the expression of Eq. (2).

For the Paul trap, it is often of practical interest to consider the case in which the parameters a and $b/(m\Omega^2)$ in Eq. (2) are small $[|a|, b/(m\Omega^2) \ll 0.5]$. In this case, we have a good approximation to the solution [7,9]

$$f = \exp(i\omega_1 t)\{1 + [b/(m\Omega^2)]\cos\Omega t\} \ ,$$

$$\alpha_0 = [1 - (b/\Omega^2)\cos\Omega t]\cos\omega_2 t \ ,$$

where $\omega_i = \beta_i\Omega/2$ and β_i $(i=1,2)$ are Paul-trap parameters [9]. If we note that $(\omega_1/\Omega)^2 \ll 1$, we have the Wronskian $w \simeq \omega_1$.

From the above results, we can say that all quantum mechanics for a Paul-trapped point-charged particle in the presence of a laser field can be obtained from the general solutions of the classical equation of motion. This is similar to Brown's results in the absence of a laser field.

As an example for the application of the obtained wave function, we consider the probability amplitude a_{fi} for the transition from the initial state i to the final state f by the Coulomb potential V as a perturbation. It is

$$a_{fi} = (-i/\hbar)\int dt \langle \psi_f | V | \psi_i \rangle \ . \tag{14}$$

For the one-dimensional problem, by substituting $\psi = U_1 U_2 \varphi_2$ into Eq. (14) and by taking

$$\ln|f| \simeq (d^2/4)(1+\cos 2\Omega t) \ ,$$

where $d = b/(m\Omega^2) \ll 1$, we can obtain

$$a_{fi} = (-i/\hbar)\sum_{l=-\infty}^{\infty}(-i)^l F(n',n,l) \times \int dt |f| \exp\Big[i\alpha(t)\Delta p - i\Delta E_n \int^t dt'/|f|^2 - i2l\Omega t\Big] \ ,$$

where

$$F(n',n,l) = N_{n'}N_n \int dx\, J_l(hx) H_{n'}(\sqrt{mw}\,x) V(x) \times H_n(\sqrt{mw}\,x)\exp(-ihx - mwx^2) \ ,$$

$$\Delta p = p_i - p_f \ ,$$

$$\Delta E_n = E_n - E_{n'} \ ,$$

$$h \equiv d^2 \Delta p/4 \ .$$

We can see that the integrand can be separated and the integral is easy to perform. The time integration corresponds to the energy exchange processes occurring at various harmonic or overtone frequencies.

This work is supported by Grant No. LWTZ-1298 of the Chinese Academy of Science and is partially supported by The National Nature Science Foundation of China.

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