Generalized antinormally ordered quantum phase-space distribution functions

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A class of positive quantum phase-space distributions based on antinormal ordering of the squeezed-photon annihilation and creation operators is introduced. This class of distributions, referred to as the generalized antinormally ordered distributions, is shown to be generated when the Wigner distribution is smoothed with a squeezed Gaussian wave packet. The distribution function associated with generalized antinormal ordering can thus be identified as the Husimi distribution function. The usefulness of the distributions in studies of the quantum particle dynamics is illustrated with simple examples.

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The phase-space formulation of quantum mechanics, which dates back to the classic work of Wigner [1], allows one to calculate expectation values of quantum-mechanical observables in the classical manner using only constant number equations [2]. It is well known, however, that there is no unique way of defining the quantum phase-space distribution function due to the noncommutability of quantum-mechanical operators. Perhaps the best-known scheme of classifying quantum phase-space distribution functions is due to Cohen [3], according to which a wide class of the distribution functions can be defined as

$$F(q,p,t)=\frac{1}{4\pi^2}\int d\xi\int d\eta \mathrm{Tr}\{\hat{\rho}e^{i\xi\hat{q}+i\eta\hat{p}}f(\xi,\eta)\}\times e^{-i\xi q-i\eta p}, \tag{1}$$

where $\hat{\rho}$ is the density operator and $f(\xi,\eta)$ is an arbitrary function which completely specifies ordering of the noncommuting operators $\hat{q}$ and $\hat{p}$ and thus the distribution function itself. The simplest choice of $f(\xi,\eta)=1$ yields the celebrated Wigner distribution function.

In this paper we consider a class of distribution functions characterized by

$$f(\xi,\eta)=e^{-\frac{\hbar}{4\kappa}\xi^2-\frac{\hbar\kappa}{4}\eta^2}, \tag{2}$$

where κ is assumed to be a real positive constant. With f given by Eq. (2), one can show that Eq. (1) can be written as

$$A(q,p,t)=\frac{1}{4\pi^2}\int d\xi\int d\eta\ \mathrm{Tr}\{\hat{\rho}e^{-z^*\hat{b}}e^{z\hat{b}^\dagger}\}\times e^{-i\xi q-i\eta p}, \tag{3}$$

where $A(q,p,t)$ refers to the distribution function $F(q,p,t)$ that results from choosing the function f according to Eq. (2), and the operator $\hat{b}$ and the constant z are given by

$$\hat{b}=\frac{1}{\sqrt{2\hbar\kappa}}(\kappa\hat{q}+i\hat{p}), \tag{4}$$

$$z=i\xi\sqrt{\frac{\hbar}{2\kappa}}-\eta\sqrt{\frac{\hbar\kappa}{2}}. \tag{5}$$

Of course, a different value of the parameter κ leads to a different distribution function $A(q,p,t)$. Hence the distribution functions $A(q,p,t)$ defined as in Eq. (3) represent a general class of "antinormally ordered" distribution functions.

The physical significance of the generalized antinormally ordered distribution function $A(q,p,t)$ introduced above becomes clear when it is used to describe quantum states of a radiation field. The operator $\hat{b}$ can be written in terms of the usual photon annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{b}=\mu\hat{a}+\nu\hat{a}^\dagger, \tag{6}$$

where

$$\mu=\frac{1}{2}\left(\sqrt{\frac{\kappa}{\omega}}+\sqrt{\frac{\omega}{\kappa}}\right),\quad \nu=\frac{1}{2}\left(\sqrt{\frac{\kappa}{\omega}}-\sqrt{\frac{\omega}{\kappa}}\right), \tag{7}$$

and ω is the frequency of the field. It can be easily seen that $[\hat{b},\hat{b}^\dagger]=1$ and $\mu^2-\nu^2=1$. One thus sees that the operators $\hat{b}$ and $\hat{b}^\dagger$ are the annihilation and creation operators of a "squeezed photon." The eigenstate $|\beta\rangle_s$ of the operator $\hat{b}$

$$\hat{b}|\beta\rangle_s=\beta|\beta\rangle_s \tag{8}$$

is the well-known minimum uncertainty squeezed state [4, 5] with one of the quadrature amplitudes squeezed by a factor of $|\mu-\nu|=\sqrt{\frac{\omega}{\kappa}}$ (if $\kappa<\omega$) or $|\mu+\nu|=\sqrt{\frac{\kappa}{\omega}}$ (if $\kappa>\omega$). It then is immediately clear that our generalized antinormally ordered distribution function can simply be written as

$$A(\beta,\beta^*,t)={}_s\langle\beta|\hat{\rho}|\beta\rangle_s/\pi, \tag{9}$$

where $A(\beta,\beta^*,t)$ is the distribution function defined in the complex β plane (squeezed state phase space) as

$$A(\beta,\beta^*,t)=\frac{1}{\pi^2}\int d^2z\ Tr\{\hat{\rho}e^{-z^*\hat{b}}e^{z\hat{b}^\dagger}\}e^{z^*\beta-z\beta^*}, \tag{10}$$

$$\beta=\frac{1}{\sqrt{2\hbar\kappa}}(\kappa q+ip), \tag{11}$$

and $d^2z = d(\text{Re } z)d(\text{Im } z)$. Equation (9) indicates that our generalized antinormally ordered distribution function is the density operator expressed in the squeezed state representation. Obviously if the parameter κ is set equal to the frequency ω of the radiation field, the distribution function A is reduced to the usual antinormally ordered Q function[6–9], $A(\alpha, \alpha^*, t) = \langle\alpha|\hat{\rho}|\alpha\rangle/\pi$, where $|\alpha\rangle$ refers to the usual coherent state. Equation (9) indicates also that the distribution function $A(q, p, t)$ is nowhere negative, an important property of the generalized antinormally ordered distribution function.

We now wish to show that the generalized antinormally ordered distribution function $A(q, p, t)$ is obtained by smoothing the Wigner distribution function $W(q, p, t)$ with a Gaussian wave packet. The simplest way to show this is to make use of the formula [10] that relates two distribution functions $F_1(q, p, t)$ and $F_2(q, p, t)$ associated with $f_1(\xi, \eta)$ and $f_2(\xi, \eta)$,

$$F_1(q,p,t) = \int dq' \int dp'\; g(q'-q, p'-p) F_2(q', p', t), \quad (12)$$

where

$$g(q,p) = \frac{1}{4\pi^2} \int d\xi \int d\eta\; \frac{f_1(\xi,\eta)}{f_2(\xi,\eta)} e^{i\xi q + i\eta p}. \quad (13)$$

Substituting Eq. (2) for f_1 and $f_2 = 1$ and performing integrations in ξ and η, we immediately obtain

$$A(q,p,t) = \int dq' \int dp' e^{-\frac{\kappa}{\hbar}(q'-q)^2 - \frac{1}{\kappa\hbar}(p'-p)^2} \times W(q', p', t). \quad (14)$$

Thus, the distribution function A is obtained when the Wigner distribution function W is smoothed by a Gaussian function representing a minimum-uncertainty squeezed state. Of course, when κ is equal to the frequency ω of the field, the smoothing Gaussian function becomes the coherent state wave packet yielding the Q function.

The distribution function given by Eq. (14) is widely known as the Husimi distribution function [11]. Since a positive distribution results when the Wigner distribution is smoothed with a Gaussian function whose product of variances $\Delta q \Delta p$, is at least that of the minimum-uncertainty wave packet [12], Eq. (14) provides another proof that our generalized antinormally ordered distribution function $A(q, p, t)$ is nowhere negative. More importantly, the identification of the Husimi distribution function as a class of Cohen's signifies that, despite Gaussian smoothing, the Husimi distribution function contains as much information about the system under consideration as the Wigner distribution function or, for that matter, any other distribution function of Cohen associated with different $f(\xi, \eta)$, because it allows evaluation of the expectation value of any combination of moments $\hat{q}^n\hat{p}^m$; one only needs to rearrange the given operator in the antinormal order of $\hat{b}$ and $\hat{b}^\dagger$. In principle, when any of the phase-space distribution functions as defined by Eq. (1) is given, one can obtain the density operator for the system [13].

The usefulness of the generalized antinormally ordered distribution function outside the realm of quantum optics comes mainly from its identification as a Gaussian smoothed Wigner distribution function as indicated by Eq. (14). In fact, it was realized earlier [14, 15] that, since Gaussian smoothing is necessarily involved in an observational process, the Husimi distribution function is usually more useful than the Wigner distribution function in studies of the correspondence (or noncorrespondence) between quantum and classical dynamics. In order to illustrate the usefulness of the Husimi distribution function in the study of the quantum particle dynamics, we consider a particle in an infinite square well potential. Figure 1 shows contour plots of the distribution function $A(q, p, t)$ computed using Eq. (14) for three different values of the "coarse-graining" parameter κ [16,

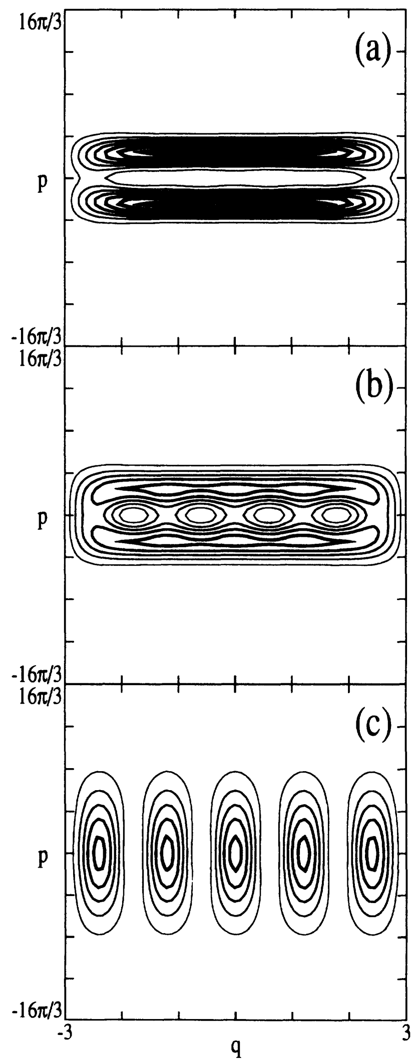

FIG. 1. Contour plots of the Husimi distribution function for a particle of mass $m = 1$ in a symmetric infinite square well potential of width $D = 6$ in a unit system in which $\hbar = 1$. The particle is assumed to be in its fifth eigenstate. Contours are drawn in steps of 0.005. The coarse-graining parameter κ is (a) 1.56, (b) 3.12, and (c) 31.2.

17]; $\kappa = 1.56$, 3.12, and 31.2, assuming that the particle is in its fifth eigenstate and taking the mass of the particle $m = 1$, the width of the square well $D = 6$ and $\hbar = 1$. One sees that the distribution function takes a drastically different shape depending upon the coarse-graining parameter κ, even if the same system in the same eigenstate is represented. At a small (large) value of κ, the Wigner distribution function is smoothed with a Gaussian function of large Δq (large Δp) and thus fast variations along the q direction (p direction) tend to get wiped out, i.e., the coarse-graining parameter determines ultimately the relative resolution in q space as compared to p space. Thus, one sees in Fig. 1 that the two-peak structure along the p axis which is obvious at $\kappa = 1.56$ disappears at $\kappa = 31.2$, while the five-peak structure along the q axis becomes evident only when κ is increased to 31.2. Depending upon what one wants to see, the Husimi distribution function furnishes a wide spectrum of positive quantum distributions from which one can choose. Despite the differences, however, all distributions associated with different values of κ are equal in the sense that they all contain the same information. Full information on the structure along the q direction [furnished directly by $A(q,p,t)$ with large κ] automatically ensures full information along the p direction [furnished directly by $A(q,p,t)$ with small κ] and vice versa, because $\Psi(q,t)$ and $\Psi(p,t)$, wave functions in the coordinate and momentum spaces, are related via Fourier transforms.

Another point worth noting about the Husimi distribution function $A(q,p,t)$ is that, since it is an averaged version of the Wigner distribution function, it generally is simpler in structure and therefore more readily amenable to physical interpretation than the Wigner distribution function. In fact, it was realized [9, 18] that a transition from the Wigner to the Husimi distribution function is analogous to a time evolution of the temperature distribution in a heat conducting system; any variation in temperature gets smoother and broader as time goes by. This property of the Husimi distribution function bears importance especially in studies of the quantum dynamics of classically chaotic systems for which the Wigner distribution often exhibits extremely complex patterns. As an illustration, we show in Fig. 2 "quantum Poincaré maps" of the Wigner and Husimi distribution functions for a particle of mass m in an infinite square well potential of width D driven by an external force $F = F_0 \cos \Omega t$, where the parameter values are chosen to be $m = 1$, $D = 6$, $F_0 = 4$, $\Omega = 5$, and $\hbar = 1$. The quantum Poincaré map of the Wigner distribution function shown in Figs. 2(a) and 2(b) is defined by

$$\bar{W}(q,p) = \frac{1}{N} \sum_{n=0}^{N-1} W(q,p,nT), \tag{15}$$

where T is the period of the driving force, $T = 2\pi/\Omega$, while that of the Husimi distribution function shown in Figs. 2(c) and 2(d) is given by the same formula except that the Wigner distribution function $W(q,p,t)$ is replaced by the Husimi distribution function $A(q,p,t)$. Figures 2(a) and 2(c) are drawn for the case when the particle is given initially as a wave packet localized in the neighborhood of the center ($q = -3, p \cong \pm 9.6$) of the

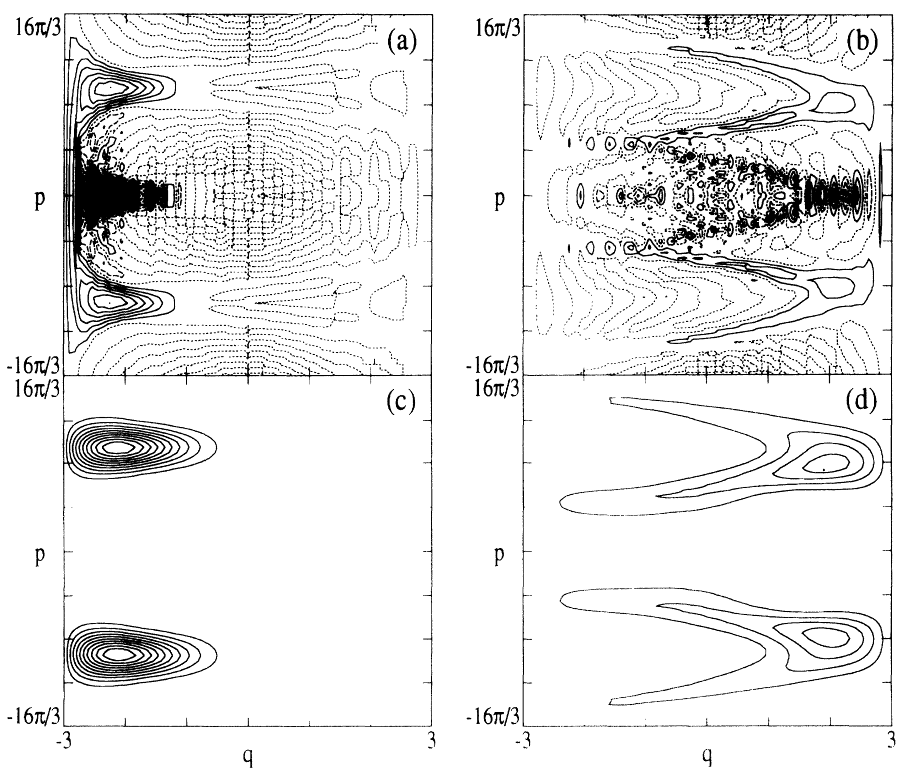

FIG. 2. Quantum Poincaré maps of the Wigner distribution function [(a) and (b)] and the Husimi distribution function [(c) and (d)]. The particle of mass $m = 1$ in a symmetric infinite potential well of width $D = 6$ is driven by a sinusoidal force $F_0 \cos \Omega t$ with $F_0 = 4$ and $\Omega = 5$ in a unit system in which $\hbar = 1$. The Wigner plots are drawn in steps of 0.02, where the solid, dotted, and dashed curves, respectively, represent positive, zero, and negative contours. The Husimi plots are drawn in steps of 0.006 and the coarse-graining parameter is chosen to be $\kappa = 1.56$. The particle is assumed to be given as a wave packet centered at the elliptic fixed point, $q \cong -3, p \cong \pm 9.6$, of the period-1 resonance for (a) and (c), and at a point, $q \cong 3, p \cong \pm 8.4$, for (b) and (d).

period-1 primary resonance, whereas Figs. 2(b) and 2(d) assume that the initial wave packet is centered at the point ($q \cong 3, p \cong \pm 8.4$) that lies in a classically chaotic region. In all cases we choose $N = 110$, and for the Husimi plots the coarse-graining parameter $\kappa = 1.56$ is used. One sees immediately from Figs. 2(a) and 2(c) that a wave packet prepared well within the period-1 resonance tends to remain localized near the initial distribution. On the other hand, Figs. 2(b) and 2(d) indicate that, if a wave packet is prepared near the separatrix of the resonance, there exists some spreading of the probability. One notes, however, that the spreading is limited in that there is very little penetration into the central region of the period-1 resonance or into the period-3 resonance which is located at $p \cong \pm 3.2$. This last observation, in particular, is significant because, at $F_0 = 4$ at which Fig. 2 is drawn, the period-3 resonance is completely destroyed and is dominated by a chaotic sea in the classical phase-space map [19]. One thus sees that the important phenomenon of the quantum suppression of classical chaos is suggested by both the Wigner plot of Fig. 2(b) and the Husimi plot of Fig. 2(d). In general, all observations that can be made from the Wigner plots concerning the nature of the quantum motion involved can also be derived from the corresponding Husimi plots. It is clear, therefore, that the Husimi plots exhibit all essential structures necessary to obtain meaningful information, whereas the Wigner plots contain extremely complex patterns especially in the neighborhood of nodal curves that make it difficult to extract information out of it. The complex structure showing rapid oscillations of the Wigner distribution in the region $p \cong 0$ that exists in Figs. 2(a) and 2(b) has no physical significance when an actual observation is considered. It is thus no surprise that the Husimi distribution function has found a growing popularity recently, as witnessed by its frequent employment in many studies of the quantum-classical correspondence in the regular and chaotic dynamics of a variety of systems[15–17, 20–23].

In conclusion we have introduced a general class of antinormally ordered quantum phase-space distribution functions which can be identified as the density operator in the squeezed state representation. We have shown that this class of distribution functions is generated when the Wigner distribution function is smoothed with a squeezed Gaussian wave packet. As the process of Gaussian smoothing tends to average out fast variations without loss of any essential information, the smoothed distribution seems well suited to studies of the quantum dynamics of complex systems. Furthermore, it offers an added advantage as one is free to choose the coarse-graining parameter which determines the relative resolution in q space versus p space.

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